Piecewise Circular Approximation of Spirals and Polar Polynomials

Francesca Taponecco
Interactive Graphics System Group
Department of Computer Graphics
Technische Universität Darmstadt
Fraunhofer Str. 5, 64283 Darmstadt
Germany
ftapone@gris.informatik.tu-darmstadt.de

Marc Alexa
Interactive Graphics System Group
Department of Computer Graphics
Technische Universität Darmstadt
Fraunhofer Str. 5, 64283 Darmstadt
Germany
alexa@gris.informatik.tu-darmstadt.de

ABSTRACT

Spirals are surprisingly common in science, nature, physics, astronomy, flora and fauna, and the arts. In Cartesian coordinates they are typically transcendental functions, which makes the evaluation on Cartesian grids an inefficient process. We propose a construction scheme for piecewise circular approximations. The algorithm is convergent and consists of generating center coordinates and radii for quarter circles given an arbitrary monotone polynomial, exponential, or logarithmic function in polar coordinates. Evaluating quarter circles as well as generating the parameters can be done incrementally with few integer operations, thus, the algorithm is fast and stable.

Keywords

Spirals, scan conversion, circular approximation

1. INTRODUCTION

Drawing straight lines and curved primitives is an important part of computer graphics.

Most methods are similar in spirit to Bresenham's approach [Brese65] - this is due to the speed and simplicity of implementation of the algorithm.

More complicated curve primitives, as in our case spirals, could be scan converted using general purpose graphing techniques, and have been less considered as objects for direct scan conversion.

Nevertheless, spiral forms are common in science, physics, biology, and other disciplines; the functions are also used for curve approximation in CAD and CAGD.

As most spirals are transcendental functions in Cartesian coordinates, simple drawing implementations are difficult to realize and incremental algorithms are likely to suffer from drift, as scan conversion of high-degree polynomials is cumbersome and error-prone [VANAK85]. A piecewise approximation using a simple primitive in polar coordinates is more promising. Thus, we propose to use circular arcs with displaced centers.

The main contributions of this work are strategies to efficiently compute the centers and radii of circular segments, and the generalization of the algorithm to any type of spiral.

2. MOTIVATION

It is interesting to study the omnipresence of spiral forms: they are found especially in the fields of

- Mathematics/Geometry
- Physics/Magnetism
- Astronomy
- Self similarity in fractals
- Flora & fauna, growing structures
- Art

Note the figures at the bottom of the pages: every image is dominated by spiraling patterns.

In scientific applications, further examples of spirals are current distribution, radiation patterns, power transmissions, centrifugal power of rotating objects, fluid movements, etc.

A major reason for spirals to be common in nature and the arts is the efficient use of space. Growing processes of several organisms have always been studied and exhibit spiral patterns. The spiral has been named "the curve of life" [COOK79], as exponential functions model reproduction processes in biology, explosions in chemistry, evolution processes in economy, etc.

Furthermore it is also being employed for visual data mining: [RIEGE02], [WEBER01], because it effectively uses screen real estate and allows to identify cyclic behavior in the data.

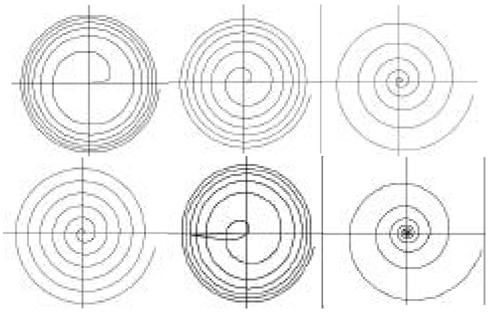


Figure 1: Archimedean spirals ($m = 1/4$, $m = 1/2$ "Fermat", $m = 3$, $m = 1$ "Archimedes"), Logarithmic spiral (singularity in the origin due to the range $j \in [0,1]$), Exponential spiral

3. RELATED WORK: ARCHIMEDES SPIRAL

The basic idea of the algorithm is a piecewise approximation in displaced polar coordinates (i.e. circular arcs with off-zero centers) and has been investigated for linear spirals [TAPON02].
Consider the expression

$$r(j) = m \cdot j + b \qquad (1)$$

where b is the offset. Assuming $d = m \cdot p/4$ and defining four points $C_i = (\pm d, \pm d)$ in the four quadrants of the coordinate system, circular quarter arcs having such centers are successively drawn with constantly (by the quantity $2d$) increasing radius A_n :

$$A_{n+1} = A_n + 2d = A_1 + 2d \cdot n \qquad (2)$$

with $A_1 = 2d + b$, $n \in N^+$. Let $i \in N$ be the cycle number and $q \in [1,2,3,4]$ the index of the first arc radius in each quadrants: in a general step $n = q + 4i$:

$$A_{q+4i} = A_q + 8i \cdot d \qquad (3)$$

Briefly: the arc's center points define the four edges of a square. As the spiral grows, the consecutive center points iteratively move through the squares vertices, shifting every time by a fixed distance (the squares edge length), as the Archimedes spiral linearly increases.
Every arc starting point is simply derived from the previous arc ending point.
It has been demonstrated that the approximation algorithm for an Archimedes spiral converges to the exact formula (1) with an upper bounded error.

4. ARBITRARY NUMBER OF SEGMENTS

We generalize now the algorithm to any number of segments: that is using a regular $n-gon$ (Fig. 2) in place of squares and drawing n circular arcs (Fig. 3) instead of four quarter circles. Thus, the arc length (in radians) is given by $x|_{rad} = \frac{2p}{n}$.

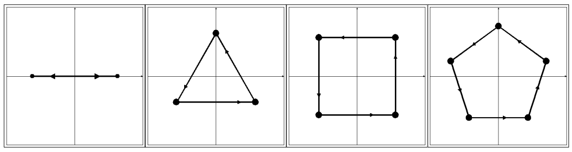


Figure 2 Arc centers' position on regular polygon vertexes ($n = 2$, $n = 3$, $n = 4$, $n = 5$)

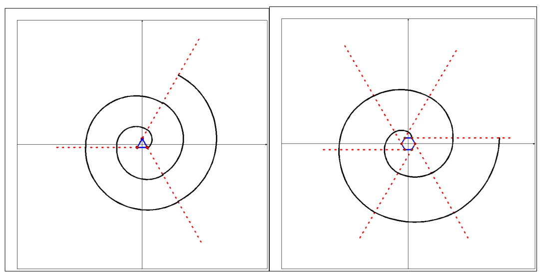
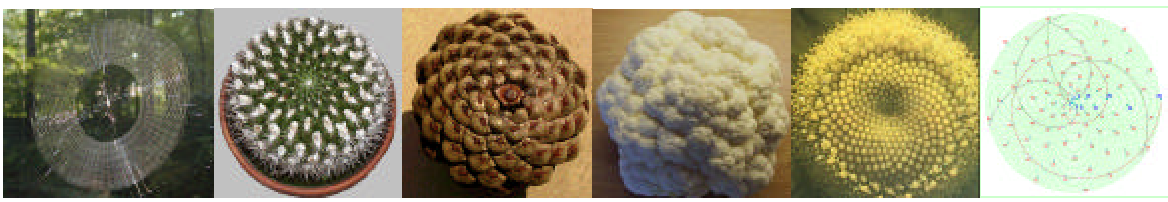


Figure 3 Construction steps ($n = 3$, $n = 6$)

Note that one can trade approximation quality for speed by increasing the number of segments.
However, we proceed here with the quarter approach (four arcs) as we believe it is the most practical: for the implementation it guarantees alignment of arc centers with screen coordinates and only one coordinate has to be changed per step.



Spiral's patterns: spider web; fruits & vegetables: succulenta, pinecone, caulis, flowers´ core (the parastichies of different families can be identified), the last image shows the growth curves according which petals in flowers or leaves in plants are sequentially produced. This study is known in botanic research as "Phyllotaxis".

5. ARBITRARY SPIRAL FUNCTIONS

In general, a spiral is a curve with $t(s)/k(s)$ equal to a constant for all s, where t is the torsion and k is the curvature. We can express the whole class of curves as

$$r(j) = f(j) \tag{4}$$

where f is a monotonic function of the angle variable j, i.e. $\frac{df}{dj} > 0$.

One can distinguish several classes of spirals, i.e. polynomial, exponential, logarithmic functions in polar coordinates. Polynomial spirals are analyzed in Section 6. Exponential/logarithmic spirals are represented by the "expansion function"

$$r(j) = e^{a \cdot j + b} \tag{5}$$

where the relation with the angle j is non-linear. In principal, all spirals could be approximated using same approach: center points for circle quadrants are defined to lie on a set of lines through the origin. One has to compute only the distance to the origin on that line. Note that this distance is easy to determine from the radius of the spiral at the respective angle: Since we know the endpoint from the previous circular segment, the radius of the spiral at that point immediately yields the distance of the next center to the origin.

While this procedure works for all spirals it requires to evaluate the defining equation at regular intervals. As we show next, for polynomial spirals this could be avoided at all.

6. ARBITRARY ARCHIMEDEAN SPIRALS

Polynomial spirals are characterized by the following expression

$$r(j) = m \cdot j^{p} + b \tag{6}$$

where $p \in R$; and are sometimes called Archimedean. The "Spiral of Archimedes" (1) is one of the spirals that belong to this family (if $p = 1$) and we can also say that this class is as a generalization of Archimedes' spiral.

Since the behavior of the spiral curve is dictated by the exponent p we distinguish three cases:

1. Linear case: $p = 1$
2. Power case: $p > 1$
3. Root case: $0 < p < 1$

Case 3. is the complementary of case 2.

Even for $p < 0$ one obtains spiraling functions, e.g. particular cases are the "Hyperbolic" ($p = -1$) or "Lituus" ($p = -1/2$) spirals.

Figure 4 describes the iterative sequential position of the quarter circles' centers: constant movements provide linear growth of the spiral, while an increasing step produces spiraling-out curves, depending on the parameter p. Figure 5 shows the resulting construction steps for different spirals, by means of circular arcs.

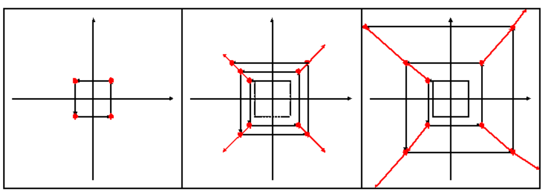


Figure 4 : Quarter circles center points and their sequence for $p = 1$, $p = 2$, $p > 2$

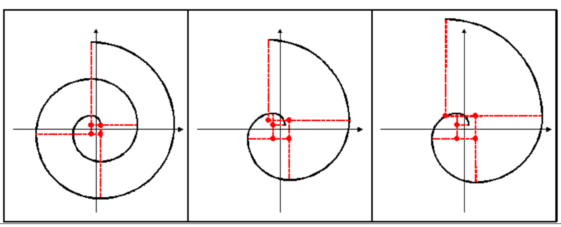
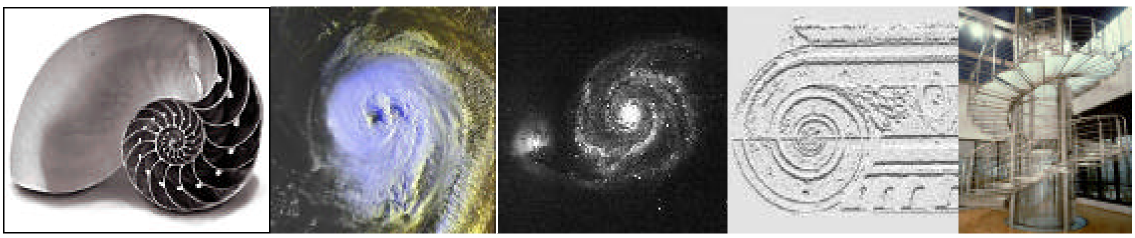


Figure 5 : Spiral construction by means of circle quarter arcs ($p = 1$, $p = 2$, $p > 2$)

Algorithm

We demonstrate here the algorithm for $p = 2$: case of quadratic growth in radius.



Spiral's structures in nature (Nautilus seashell), phenomena (hurricane), astronomy (galaxies M51 "Perfect spiral" and Hubble galaxies classification), art (volute), architecture (spiral stairs).

For the linear spiral (see Section 3) the distance between two successive windings was fixed to be $2p \cdot m$; now, the radius of the Archimedean spiral varies as a non-linear function. Let this time

$$c = m \cdot p^{2}/2 \tag{7}$$

define the new set of quarter circle centers C_i as follows: beside the constant increment of the linear spiral we sum, for each step n, an additional term c. The points C_i, instead of staying on the four positions $(\pm d, \pm d)$, move spiraling out. Note that the radius increases in each step by $2d$ plus a non-constant term.

The expression for the quarter circles radii is:

$$A_n = A_{n-1} + 2d + c \cdot n = 2d \cdot n + c \cdot \frac{n \cdot (n+1)}{2} \tag{8}$$

with $A_1 = 2d + b + c$ and $n \in N^{+}$.

We extended and validated the method also for any exponent p, so that we can freely choose the monotonic polynomial function that models the increment of the quarter circles radius in our spiral. Note that the calculations allow an integer implementation of the approximation algorithm.

7. CONCLUSIONS

We produce spirals by joining quarters of circles of increasing radii: given the radius and the starting point, for every step a quarter of circle is drawn.

The main features of our algorithm are the following:

- The approximation is characterized by a constrained error [TAPON02], which rapidly decreases with increasing cycle number. Therefore, the convergence of our generalized algorithm to the exact function is $\Phi(1/j)$.
- The generated curve is G^1.
- The algorithm construction is based on symmetry, which is not the case of other algorithms (e.g. Paduan or polygonal approximations). Here it is possible to implement an efficient algorithm for just a quarter circle and then to iteratively apply it to every quadrant of the coordinate system.
- The algorithm permits the use of pure integer arithmetic, avoiding the complexity of trigonometric functions; in fact quarter circles should be drawn using the midpoint algorithm.
- The method is valid for the whole family of Archimedean spirals and also for exponential and logarithmic spirals of any base. In general, the algorithm can be applied to any kind of spiral and the code stays simple even if the spiral expression becomes more complicated.

8. FUTURE WORK

The generalization of the algorithm provides the possibility of approximating other functions, just by means of circular or ellipsoidal arcs.

We are interested in extending the method to polar curves of any complexity.

9. REFERENCES

[BRESE65] J. E. Bresenham. Algorithm for computer control of a digital plotter. IBM Systems Journal, 4(1): 25-30, 1965.

[TAPON02] F. Taponecco, M. Alexa. Scan Converting Spirals. Proc. of WSCG 2002, Plzen, Czech. Republic

[STEPH00] P. Stephenson and B. Litow. Why step when you can run? Iterative line digitalisation algorithms based on hierarchies of runs. IEEE Computer Graphics & Applications, 20(6): 76-84, November / December 2000. ISSN 0272-1716.

[STEPH01] Peter Stephenson and Bruce Litow. Running the line: Line drawing using runs and runs of runs. Computers & Graphics, 25(4): 681-690, August 2001. ISSN 0097-8493.

[TUPPE01] Jeff Tupper. Reliable two-dimensional graphing methods for mathematical formulae with two free variables. Proceedings of SIGGRAPH 2001, pages 77-86, August 2001. ISBN 1-58113-292-1.

[VANAK85] J. R. VanAken, M. Novak. Curve drawing algorithms for raster displays ACM Trans. on Graphics, 4, 147-169, 1985.

[RIEGE02] T. Rieger, F. Taponecco. Interactive Information Visualization Of Entity-Relationship-Data. Proc. of WSCG 2002, Plzen, Czech. Republic.

[WEBER01] Marc Weber, Marc Alexa and Wolfgang Müller. Visualizing time-series on spirals. Proceedings of InfoVis 2001.

[COOK79] A. S. Cook. The Curves of Life: Being an Account of Spiral Formations and Their Application to Growth in Nature, to Science, and to Art: With Special Reference to the Manuscripts of Leonardo da Vinci. New York: Dover, 1979.