Annals of **F**uzzy **M**athematics and **I**nformatics
Volume 5, No. 2, (March 2013), pp. 349–360
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

@FMI
© Kyung Moon Sa Co.
http://www.kyungmoon.com

Anti fuzzy ideal of a ring

F. A. Azam, A. A. Mamun, F. Nasrin

Received 3 October 2011; Accepted 1 August 2012

Abstract. An anti fuzzy ideal and lower level ideals of a ring X are defined. The fuzzification of lower level subset of fuzzy set is redefined and some properties are proved. In addition, the set $\frac{X}{A} = \{y + A : y \in X\}$ is shown as a quotient ring induced by the anti fuzzy ideal A.

2010 AMS Classification: 20N25, 03E72

Keywords: Fuzzy ideal, Anti fuzzy ideal, Lower level subset, Isomorphism

Corresponding Author: F. A. Azam (faqruddinaliazam@gmail.com)

1. Introduction

Rosenfeld [13] first studied the fuzzy subgroup of a group and then the fuzzification of algebraic structures started to grow up. Afterward, Liu [11] introduced the notion of fuzzy ideal. This idea of fuzzy ideal motivated Kumbhojkar and Bapat [9], Dixit et al. [5] and Zaid [15] to investigate the concepts of fuzzy coset and fuzzy quotient ring. Furthermore, the idea of anti fuzzy subgroups was introduced by Biswas [3] which ultimately was extended by many researchers, e.g., [1, 6, 7, 8, 10, 12, 16].

Our work is an extension of the Biswas' [3] idea of anti fuzzy subgroup of a group. In our paper we apply this idea to the theory of ring. We introduce a notion of anti fuzzy ideal A of a ring X and some of its properties are discussed. We give a definition of lower level ideal of a ring in this paper. We prove that a fuzzy set A of a ring X is an anti fuzzy ideal of X if and only if the lower level subsets $\bar{A}_t$ [3] of A are ideals of X. By giving an example, we show that Biswas' [3] idea of fuzzification of lower level subsets of fuzzy set is not valid in general. Accordingly, a modified definition is given to fuzzify the lower level subsets $\bar{A}_t$ of the fuzzy set A and it is revealed that if A is an anti fuzzy ideal of X, then so is $\delta_{\bar{A}_t}$, the fuzzification of $\bar{A}_t$. In addition, the set $\frac{X}{A} = \{y + A : y \in X\}$ is proved as a factor ring of the ring X induced by the anti fuzzy ideal A of X and some isomorphism theorems are established.

Unless otherwise stated, X is considered as a ring associated with two binary operations '+' and '.' throughout this paper; negative of x, $x + (-y)$ and $x.y$ are written as $-x$, $x - y$ and xy, respectively; the zero (respectively identity, if exist) element of X is denoted by $\mathbf{0}$ (respectively $\mathbf{1}$). Thus $x + (-x) = \mathbf{0}$ and $x - y = x + (-y)$. The characteristic function of a subset U of X is denoted by $\mathbf{1}_U$.

2. Preliminaries

In this section, some definitions are recalled that have been employed in our analysis.

Definition 2.1 ([14]). A fuzzy set A in a nonempty set X is a mapping $A : X \rightarrow [0, 1]$. If A is a fuzzy set in X and $x \in X$, then $A(x)$ represents the membership value of x. Also by A^c, we denote the complement of A which is defined as $A^c(x) = 1 - A(x)$, $\forall\, x \in X$. For two fuzzy sets A and B in X. We define
(i) $A = B$ if and only if $A(x) = B(x)\ \forall\, x \in X$.
(ii) $A \leq B$ if and only if $A(x) \leq B(x)\ \forall\, x \in X$.
(iii) $(A \vee B)(x) = max\{A(x), B(x)\}\ \forall\, x \in X$.
(iv) $(A \wedge B)(x) = min\{A(x), B(x)\}\ \forall\, x \in X$.

Definition 2.2 ([4]). Let $f : X \rightarrow Y$ be a mapping between sets and A a fuzzy set in X. Then the image $f(A)$ is a fuzzy set in Y which is defined as

$$f(A)(y) = \begin{cases} sup\ \{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 0, & \text{if } f^{-1}(y) = \phi \end{cases}$$

Definition 2.3 ([7]). Let $f : X \rightarrow Y$ be a mapping between sets and A a fuzzy set in X. Then $f_{-}(A)$ is a fuzzy set in Y which is defined as

$$f_{-}(A)(y) = \begin{cases} inf\{A(x) : x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq \phi \\ 1, & \text{if } f^{-1}(y) = \phi \end{cases}$$

Definition 2.4 ([4]). Let $f : X \rightarrow Y$ be a mapping between sets and B a fuzzy set in Y. Then the inverse image $f^{-1}(B)$ is a fuzzy set in X which is defined as

$$f^{-1}(B)(x) = B(f(x)), \forall\ x\ \in X.$$

Definition 2.5 ([11]). A fuzzy set A in X is called a fuzzy left (respectively, right) ideal of X if
(i) $A(x - y) \geq min\{A(x), A(y)\}$
(ii) $A(xy) \geq min\{A(x), A(y)\}$ and
(iii) $A(xy) \geq A(y)$ (respectively, $A(xy) \geq A(x)$).

Definition 2.6 ([3]). For a fuzzy set A in X and for $t \in [0, 1]$, the set $\bar{A}_t = \{x \in X : A(x) \leq t\}$ is called the lower level subset of the fuzzy set A.

Definition 2.7 ([3]). A fuzzy set μ of a group G is an anti fuzzy subgroup of G if and only if $\forall x, y \in G$, $\mu(xy^{-1}) \leq max\{\mu(x), \mu(y)\}$.

Definition 2.8. Let R and S be two rings. A function $f : R \rightarrow S$ such that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)\ \forall\ a, b \in R$ is called a homomorphism; if f is onto, i.e., $f(R) = S$, then f is called an epimorphism.

3. Anti fuzzy ideal

In this section, an anti fuzzy ideal A of X is defined and some results on this are proved.

Definition 3.1. A fuzzy set A of X is called an anti fuzzy left (respectively, right) ideal of X if $\forall\ x,\ y \in X$,
(i) $A(x-y) \leq max\{A(x),\ A(y)\}$,
(ii) $A(xy) \leq max\{A(x),\ A(y)\}$ and
(iii) $A(xy) \leq A(y)$ (respectively, $A(xy) \leq A(x)$).

Definition 3.2. A fuzzy set A of X is called an anti fuzzy ideal of X if it is an anti fuzzy left ideal as well as an anti fuzzy right ideal of X.

Remark 3.3. (i) A fuzzy set A of X is an anti fuzzy left (respectively, right) ideal of X if and only if A^c is a fuzzy left (respectively, right) ideal of X. (ii) Every anti fuzzy (left or right) ideal of X is an additive anti fuzzy subgroup of X.

Remark 3.4. If A is an anti fuzzy ideal of X, then $\forall\ x,\ y \in X$,
(i) $A(x-y) \leq max\{A(x),\ A(y)\}$ and
(ii) $A(xy) \leq min\{A(x),\ A(y)\}$.

For every anti fuzzy (left or right) ideal A of X and $\forall\ x \in X$, we have $A(\mathbf{0}) = A(x-x) \leq max\{A(x), A(x)\} = A(x)$,
$A(-x) = A(\mathbf{0}-x) \leq max\{A(\mathbf{0}), A(x)\} = A(x)$ and $A(x) = A(\mathbf{0}-(-x)) \leq max\{A(\mathbf{0}), A(-x)\} = A(-x)$.

Again if $x, y \in X$ and A is an anti fuzzy (left or right) ideal of X such that $A(x-y) = A(\mathbf{0})$, then $A(y) = A(x-(x-y)) \leq max\{A(x), A(x-y)\} = A(x)$ and $A(x) = A(y-(y-x)) \leq max\{A(y),\ A(y-x)\} = max\{A(y), A(x-y)\} = A(y)$.

Thus we have the following proposition:

Proposition 3.5. *For every anti fuzzy (left or right) ideal A of X,*
(i) $A(\mathbf{0}) \leq A(x),\ \forall\ x \in X$.
(ii) $A(x) = A(-x),\ \forall\ x \in X$.
(iii) $A(x-y) = A(\mathbf{0}) \Rightarrow A(x) = A(y), \forall\ x, y \in X$.

Theorem 3.6. *Let A and B be two anti fuzzy left (respectively, right) ideals of X. Then $A \vee B$ is also an anti fuzzy left (respectively, right) ideal of X.*

Proof. $\forall\ x, y \in X$, we have
(i) $(A \vee B)(x-y) = max\{A(x-y), B(x-y)\} \leq max\{A(x), A(y), B(x), B(y)\}$
$= max\{(A \vee B)(x), (A \vee B)(y)\}$,
(ii) $(A \vee B)(xy) = max\{A(xy), B(xy)\} \leq max\{A(x), A(y), B(x), B(y)\}$
$= max\{(A \vee B)(x), (A \vee B)(y)\}$ and
(iii) $(A \vee B)(xy) = max\{A(xy),\ B(xy)\} \leq max\{A(y),\ B(y)\}$
(respectively, $max\{A(x), B(x)\}) = (A \vee B)(y)$ (respectively, $(A \vee B)(x)$).
Thus we see that $A \vee B$ is an anti fuzzy left (respectively, right) ideal of X. □

Corollary 3.7. *The sup of any set of anti fuzzy left (respectively, right) ideals of X is an anti fuzzy left (respectively, right) ideal of X.*

The intersection of two anti fuzzy ideals is not necessarily an anti fuzzy ideal, which is justified in the following example:

Example 3.8. Let $X = (\mathbf{Z}, +, .)$, where $\mathbf{Z}$ is the set of integers. Define two fuzzy sets A and B in X by

$$A(x) = \left\{ \begin{array}{ll} \frac{1}{2}, & \text{if } x \text{ is a multiple of 3} \\ 1, & \text{otherwise} \end{array} \right\} \text{ and } B(x) = \left\{ \begin{array}{ll} \frac{4}{5}, & \text{if } x \text{ is even} \\ \frac{5}{6}, & \text{otherwise} \end{array} \right\}.$$

It can be verified that A and B are anti fuzzy ideals of X. Now, take $x = 9$ and $y = 4$. We see that $A(x) = \frac{1}{2}$, $A(y) = 1$, $A(x-y) = 1$, $B(x) = \frac{5}{6}$, $B(y) = \frac{4}{5}$ and $B(x-y) = \frac{5}{6}$. Clearly $(A \wedge B)(x) = \frac{1}{2}$, $(A \wedge B)(y) = \frac{4}{5}$ and $(A \wedge B)(x-y) = \frac{5}{6}$. Readily $(A \wedge B)(x-y) > \max\{(A \wedge B)(x), (A \wedge B)(y)\}$.

Thus we see that, the intersection of two anti fuzzy (left or right) ideals of X need not to be an anti fuzzy (left or right) ideal of X.

Example 3.9. Let $X = (\mathbf{Z}, +, .)$, where $\mathbf{Z}$ is the set of integers. Define two fuzzy sets A and B in X by

$$A(x) = \left\{ \begin{array}{ll} \frac{4}{5}, & \text{if } x \text{ is even} \\ \frac{5}{6}, & \text{otherwise} \end{array} \right\} \text{ and } B(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \text{ is even} \\ 1, & \text{otherwise} \end{array} \right\}.$$

Now, $A \wedge B(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \text{ is even} \\ 1, & \text{otherwise} \end{array} \right\}$. It can be verified that A, B and $A \wedge B$ are anti fuzzy ideals of X.

Theorem 3.10. *Let X be a skew field. Then for every anti fuzzy (left or right) ideal A of X and $\forall\ x \in X,\ x \neq \mathbf{0}$, $A(x) = A(\mathbf{1})$.*

Proof. Let $x \in X, x \neq \mathbf{0}$. Suppose A is an anti fuzzy left ideal of X. Now $A(x) = A(x.\mathbf{1}) \leq A(\mathbf{1}) = A(x^{-1}.x) \leq A(x) \Rightarrow A(x) = A(\mathbf{1})$.

Again, let A be an anti fuzzy right ideal of X. Now $A(x) = A(\mathbf{1}.x) \leq A(\mathbf{1}) = A(x.x^{-1}) \leq A(x) \Rightarrow A(x) = A(\mathbf{1})$. □

Theorem 3.11. *Let A be a fuzzy set in X such that $\forall\ x \in X,\ x \neq \mathbf{0}$, $A(x) = A(x_0)$, where x_0 is a fixed element of X. Then A is an anti fuzzy ideal of X.*

Proof. Let $x,\ y \in X$. Now consider the following cases:

Case-1: ($x = \mathbf{0}$ and $y \neq \mathbf{0}$) or ($x \neq \mathbf{0}$ and $y = \mathbf{0}$). Clearly $A(x-y) = A(x_0) \geq A(\mathbf{0})$ and $A(xy) = A(\mathbf{0})$, and so (i)$A(x-y) = max\{A(x), A(y)\}$ and (ii)$A(xy) = min\{A(x), A(y)\}$.

Case-2: $x = y = \mathbf{0}$. The proof is trivial.

Case-3: $x \neq \mathbf{0},\ y \neq \mathbf{0}$. Clearly $A(x) = A(y) = A(x_0) \geq A(\mathbf{0})$. Now

(i) $A(x-y) = \left\{ \begin{array}{ll} A(\mathbf{0}), & \text{if } x = y \\ A(x_0), & \text{if } x \neq y \end{array} \right\} \leq A(x_0) = max\{A(x), A(y)\}$

and

(ii) $A(xy) = \left\{ \begin{array}{ll} A(\mathbf{0}), & \text{if } xy = \mathbf{0} \\ A(x_0), & \text{if } xy \neq \mathbf{0} \end{array} \right\} \leq A(x_0) = min\{A(x), A(y)\}$.

Thus we see that A is an anti fuzzy ideal of X. □

Theorem 3.12. *U is a left (respectively, right) ideal of X if and only if $\mathbf{1}_{U^c}$ is an anti fuzzy left (respectively, right) ideal of X.*

Proof. First, let U be a left (respectively, right) ideal of X and $x,\ y \in X$. Now consider the following cases:

Case-1: $\{x,\ y\} \subseteq U$. Clearly $x-y,\ xy \in U$. Now $\mathbf{1}_{U^c}(x) = \mathbf{1}_{U^c}(y) = \mathbf{1}_{U^c}(x-y) = \mathbf{1}_{U^c}(xy) = 0$, and therefore, $\mathbf{1}_{U^c}$ is an anti fuzzy ideal of X.

Case-2: $\{x,\ y\} \cap\ U = \varnothing$. Clearly $max\{\mathbf{1}_{U^c}(x), \mathbf{1}_{U^c}(y)\} = 1$, and so, (i) $\mathbf{1}_{U^c}(x - y) \leq max\{\mathbf{1}_{U^c}(x), \mathbf{1}_{U^c}(y)\}$ and (ii) $\mathbf{1}_{U^c}(xy) \leq max\{\mathbf{1}_{U^c}(x),\ \mathbf{1}_{U^c}(y)\}$. Moreover $y \in U \Rightarrow xy \in U$ (respectively, $yx \in U$) $\Rightarrow \mathbf{1}_{U^c}(xy)$ (respectively, $\mathbf{1}_{U^c}(yx)) = \mathbf{1}_{U^c}(y) = 0$. On the other hand, $y \notin U \Rightarrow \mathbf{1}_{U^c}(xy) \leq \mathbf{1}_{U^c}(y) = 1$ and $\mathbf{1}_{U^c}(yx) \leq \mathbf{1}_{U^c}(y) = 1$. Therefore $\mathbf{1}_{U^c}$ is an anti fuzzy left (respectively, right) ideal of X.

Conversely, let $\mathbf{1}_{U^c}$ be an anti fuzzy left (respectively, right) ideal of X, $x, y \in U$ and $z \in X$. Now $\mathbf{1}_{U^c}(x) = \mathbf{1}_{U^c}(y) = 0$. Consequently $\mathbf{1}_{U^c}(x - y) = 0$, $\mathbf{1}_{U^c}(xy) = 0$ and $\mathbf{1}_{U^c}(zx)$ (respectively, $\mathbf{1}_{U^c}(xz)) = 0$, and so $x - y, xy$ and xz (respectively, $zx) \in U$. Thus we see that U is a left (respectively, right) ideal of X. Hence the theorem is proved. □

Theorem 3.13. *Let X be a commutative ring with $\mathbf{1}$ such that for each anti fuzzy ideal A of X, $A(x) = A(\mathbf{1})$, $\forall\ x \in X, x \neq \mathbf{0}$. Then X is a field.*

Proof. Let U be a nonzero ideal of X. Now $\mathbf{1}_{U^c}$ is an anti fuzzy ideal of X. Therefore $\mathbf{1}_{U^c}(x) = \mathbf{1}_{U^c}(\mathbf{1})$, $\forall\ x \in X,\ x \neq \mathbf{0}$. In particular, if $x \in U$, then $\mathbf{1}_{U^c}(\mathbf{1}) = \mathbf{1}_{U^c}(x) = 0$. This implies that $\mathbf{1} \in U$, and so $U = X$. Thus we see that X has no non-zero proper ideal. Therefore X is a field. □

Theorem 3.14. *Every homomorphic pre-image of an anti fuzzy left (respectively, right) ideal is also in an anti fuzzy left (respectively, right) ideal.*

Proof. Consider a homomorphism $f : X \to Y$ between rings. Let B be an anti fuzzy left (respectively, right) ideal of Y. Now $\forall\ x_1,\ x_2 \in X$,

(i) $f^{-1}(B)(x_1 - x_2) = B(f(x - x_2)) = B(f(x_1) - f(x_2))$
$\leq max\{B(f(x_1)), B(f(x_2))\} = max\{f^{-1}(B)(x_1), f^{-1}(B)(x_2)\}$,

(ii) $f^{-1}(B)(x_1x_2) = B(f(x_1x_2)) = B(f(x_1)f(x_2))$
$\leq max\{B(f(x_1)), B(f(x_2))\} = max\{f^{-1}(B)(x_1), f^{-1}(B)(x_2)\}$

and

(iii) $f^{-1}(B)(x_1x_2) = B(f(x_1x_2)) = B(f(x_1)f(x_2)) \leq B(f(x_2))$
(respectively, $B(f(x_1))) = f^{-1}(B)(x_2)$ (respectively, $f^{-1}(B)(x_1))$.

Thus we see that, $f^{-1}(B)$ is an anti fuzzy left (respectively, right) ideal of X. □

Theorem 3.15. *Let $f : X \to Y$ be an epimorphism between rings and A an anti fuzzy ideal of X. Then $f_-(A)$ is an anti fuzzy ideal of Y.*

Proof. It can be verified that $(f_-(A))^c = f(A^c)$. Now A^c is a fuzzy ideal of X and so $f(A^c)$ is a fuzzy ideal of Y (cf. [9]). Therefore by the Remark 3.3, $f_-(A)$ is an anti fuzzy ideal of Y. □

Theorem 3.16. *Let A be a fuzzy set in X. Then A is an anti fuzzy left (respectively, right) ideal of X if and only if for each t with $A(\mathbf{0}) \leq t \leq 1$, the lower level subset $\bar{A}_t$ is a left (respectively, right) ideal of X.*

Proof. First, let A be an anti fuzzy left (respectively, right) ideal of X. Suppose $A(\mathbf{0}) \leq t \leq 1$, $\{a, b\} \subseteq \bar{A}_t$ and $x \in X$. Then $A(a) \leq t$ and $A(b) \leq t$. Now $A(a - b) \leq max\{A(a), A(b)\} \leq t$, $A(ab) \leq max\{A(a), A(b)\} \leq t$ and $A(xa) \leq A(a) \leq t$ (respectively, $A(ax) \leq A(a) \leq t$). Therefore, $a - b$, ab, ax (respectively, $xa) \in \bar{A}_t$ and so $\bar{A}_t$ is a left (respectively, right) ideal of X.

Conversely, let $\forall\ t$ with $A(\mathbf{0}) \le t \le 1$, the lower level subset $\bar{A}_t$ is a left (respectively, right) ideal of X. Suppose $x, y \in X$, $t = max\{A(x), A(y)\}$. Now $A(x) \le t$ and $A(y) \le t$, and so $x, y \in \bar{A}_t$. Consequently $x - y$, $xy \in \bar{A}_t$ and therefore, $A(x-y) \le t = max\{A(x), A(y)\}$ and $A(xy) \le t = max\{A(x), A(y)\}$.

Again $y \in \bar{A}_{A(y)}$. Consequently $xy \in \bar{A}_{A(y)}$ (respectively, $yx \in \bar{A}_{A(y)}$). This implies that $A(xy) \le A(y)$ (respectively, $A(yx) \le A(y)$). Hence the theorem is proved. □

Definition 3.17. Let A be an anti fuzzy ideal of X. Then for $A(\mathbf{0}) \le t \le 1$, the lower level subsets $\bar{A}_t$ are called lower level ideals of A. In particular, the set $\bar{A}_{A(\mathbf{0})} = \{x \in X : A(x) = A(\mathbf{0})\}$ is also an ideal of X which will be denoted later on by A_0 .

Theorem 3.18. *Given* $0 \le s < t \le 1$, *and* A *is an anti fuzzy ideal of* X. *Then* $\bar{A}_s = \bar{A}_t \Leftrightarrow\ \exists$ *no* $x \in X$ *such that* $s < A(x) \le t$.

Proof. First, let $\bar{A}_s = \bar{A}_t$. Therefore $x \in \bar{A}_t \Rightarrow x \in \bar{A}_s$. That is, $A(x) \le t \Rightarrow A(x) \le s$. Thus $\exists$ no $x \in X$ such that $s < A(x) \le t$.

Conversely, let $\exists$ no $x \in X$ such that $s < A(x) \le t$. Therefore $A(x) \le t \Rightarrow A(x) \le s$. That is, $x \in \bar{A}_t \Rightarrow x \in \bar{A}_s$. Thus $\bar{A}_t \subseteq \bar{A}_s$. Moreover $\bar{A}_s \subseteq \bar{A}_t$, since $s < t$. Therefore $\bar{A}_s = \bar{A}_t$. □

Theorem 3.19. *For every anti fuzzy ideal* A *of* X, *there exists an anti fuzzy ideal* $\widehat{A}$ *of* $\frac{X}{A_0}$ *such that* $\widehat{A}(x + A_0) = A(x)$. *On the other hand, if* U *is an ideal of* X *and* $\widehat{B}$ *is an anti fuzzy ideal of* $\frac{X}{U}$ *such that* $\widehat{B}(x+U) = \widehat{B}(U) \Leftrightarrow x \in U$, *then there exists an anti fuzzy ideal* A *of* X *such that* $A_0 = U$ *and* $\widehat{A} = \widehat{B}$.

Proof. Let A be an anti fuzzy ideal of X. Define $\widehat{A} : \frac{X}{A_0} \to [0, 1]$ by $\widehat{A}(x+A_0) = A(x)$. $\widehat{A}$ is well defined since, $x + A_0 = y + A_0 \Rightarrow x - y \in A_0 \Rightarrow A(x - y) = A(\mathbf{0}) \Rightarrow A(x) = A(y) \Rightarrow \widehat{A}(x + A_0) = \widehat{A}(y + A_0)$. Also we see that $\widehat{A}$ is an anti fuzzy ideal of $\frac{X}{A_0}$, since

(i) $\widehat{A}((x + A_0) - (y + A_0)) = \widehat{A}((x - y) + A_0) = A(x - y) \le max\{A(x), A(y)\}$
$= max\{\widehat{A}(x + A_0), \widehat{A}(y + A_0)\}$

and

(ii) $\widehat{A}((x + A_0)(y + A_0)) = \widehat{A}((xy) + A_0) = A(xy) \le min\{A(x), A(y)\}$
$= min\{\widehat{A}(x + A_0), \widehat{A}(y + A_0)\}$.

Again, let U be an ideal of X and $\widehat{B}$ an anti fuzzy ideal of $\frac{X}{U}$ such that $\widehat{B}(x+U) = \widehat{B}(U) \Leftrightarrow x \in U$.

Now, define $A : X \to [0, 1]$ by $A(x) = \widehat{B}(x + U)$. We see that A is well defined, since $x = y \Rightarrow x + U = y + U \Rightarrow \widehat{B}(x + U) = \widehat{B}(y + U) \Rightarrow A(x) = A(y)$. Also, A is an anti fuzzy ideal of X as,

(i) $A(x-y) = \widehat{B}((x-y)+U) = \widehat{B}((x+U)-(y+U)) \le max\{\widehat{B}(x+U), \widehat{B}(y+U)\}$
$= max\{A(x), A(y)\}$

and

(ii) $A(xy) = \widehat{B}((xy) + U) = \widehat{B}((x + U)(y + U)) \le min\{\widehat{B}(x + U), \widehat{B}(y + U)\}$
$= min\{A(x), A(y)\}$.

Again $x \in U \Leftrightarrow \widehat{B}(x + U) = \widehat{B}(U) \Leftrightarrow A(x) = A(\mathbf{0}) \Leftrightarrow x \in A_0$, and so $U = A_0$.

Finally, $\widehat{B}(x+U) = A(x) = \widehat{A}(x+A_0) = \widehat{A}(x+U)$. Thus $\widehat{A} = \widehat{B}$. □

4. Fuzzification of a lower level subset

According to Biswas [3], the fuzzification of the lower level set $\bar{\mu}_t$ of the fuzzy set μ is the fuzzy set $A_{\bar{\mu}_t}$ defined by

$$A_{\bar{\mu}_t}(x) = \left\{ \begin{array}{ll} \mu(x) & \text{if } x \in \bar{\mu}_t \\ 0 & \text{otherwise} \end{array} \right\}.$$

Based on this definition it was claimed [in [3], Proposition 5.1] that if μ is an anti fuzzy subgroup of a group G, then $A_{\bar{\mu}_t}$ is also an anti fuzzy subgroup of G. But our analysis proves that this proposition is not valid in general. For, if it is possible to find $x, y \in G$ such that $x \notin \bar{\mu}_t$, $y \notin \bar{\mu}_t$ but $xy^{-1} \in \bar{\mu}_t$, then $A_{\bar{\mu}_t}(xy^{-1}) = \mu(xy^{-1})$, $A_{\bar{\mu}_t}(x) = 0$ and $A_{\bar{\mu}_t}(y) = 0$, and therefore, the condition $A_{\bar{\mu}_t}(xy^{-1}) \leq max\{A_{\bar{u}_t}(x), A_{\bar{u}_t}(y)\}$ is not satisfied, in general, unless $\mu(xy^{-1}) = 0$. As for example, let $\mathbf{Z}$ denotes the set of integers and $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z} \right\}$. Clearly G is a group under matrix addition. Now, consider two subgroups S_1 and S_2 of G such that $S_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $S_2 = \left\{ \begin{pmatrix} 0 & 2m \\ 0 & 2n \end{pmatrix} : \ m, n \in \mathbf{Z} \right\}$. Define $\mu : G \to [0, 1]$ by

$$\mu(x) = \begin{cases} 0, & \text{if } x \in S_1 \\ \frac{1}{2}, & \text{if } x \in S_2 - S_1 \\ 1, & \text{if } x \in G - S_2 \end{cases}$$

It can be verified that μ is an (additive) anti fuzzy subgroup of G and $\bar{\mu}_{\frac{1}{2}} = S_2$. Now, take $x = \begin{pmatrix} 0 & 3 \\ 0 & 5 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$. Then $x \notin \bar{\mu}_{\frac{1}{2}}$, $y \notin \bar{\mu}_{\frac{1}{2}}$ but $x - y \in \bar{\mu}_{\frac{1}{2}}$. Clearly $A_{\bar{\mu}_{\frac{1}{2}}}(x) = 0$, $A_{\bar{\mu}_{\frac{1}{2}}}(y) = 0$ and $A_{\bar{\mu}_{\frac{1}{2}}}(x - y) = \frac{1}{2}$. Thus we see that $A_{\bar{\mu}_{\frac{1}{2}}}$ is not an (additive) anti fuzzy subgroup of G.

Here we give a modified definition of fuzzification of lower level subset:

Definition 4.1. Let A be a fuzzy set in X. For $t \in [0, 1]$, the fuzzification of the lower level subset $\bar{A}_t = \{x \in X : A(x) \leq t\}$ is a fuzzy set $\delta_{\bar{A}_t}$ and is defined by

$$\delta_{\bar{A}_t}(x) = \left\{ \begin{array}{ll} A(x) & \text{if } x \in \bar{A}_t \\ 1 & \text{if } x \notin \bar{A}_t \end{array} \right\}.$$

Theorem 4.2. *For a lower level subset $\bar{A}_t$ of a fuzzy set A in X, $\overline{(\delta_{\bar{A}_t})_t} = \bar{A}_t$.*

Proof. Let $x \in \overline{(\delta_{\bar{A}_t})_t}$. Then $\delta_{\bar{A}_t}(x) \leq t$. Also it is clear that $A \leq \delta_{\bar{A}_t}$. Consequently, $A(x) \leq t$, and so, $x \in \bar{A}_t$. Thus $\overline{(\delta_{\bar{A}_t})_t} \subseteq \bar{A}_t$.

Conversely, let $x \in \bar{A}_t$. Then $\delta_{\bar{A}_t}(x) = A(x) \leq t$. Therefore $x \in \overline{(\delta_{\bar{A}_t})_t}$. Thus $\bar{A}_t \subseteq \overline{(\delta_{\bar{A}_t})_t}$. Hence the theorem is proved. □

Theorem 4.3. *Let A be an anti fuzzy left ideal of X. Then $\delta_{\bar{A}_t}$ is also an anti fuzzy left ideal of X.*

Proof. Let $x, y \in X$. Now consider the following cases:

Case-1: Suppose $x, y \in \bar{A}_t$. Clearly $x - y, xy \in \bar{A}_t$, since $\bar{A}_t$ is a left ideal of X. Now

(i) $\delta_{\bar{A}_t}(x-y) = A(x-y) \leq max\{A(x), A(y)\} = max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\}$

(ii) $\delta_{\bar{A}_t}(xy) = A(xy) \leq max\{A(x), A(y)\} = max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\}$

and

(iii) $\delta_{\bar{A}_t}(xy) = A(xy) \leq A(y) = \delta_{\bar{A}_t}(y)\}$.

Case-2: Suppose $x \in \bar{A}_t, y \notin \bar{A}_t$. Now $x - y \notin \bar{A}_t$, otherwise $y = x - (x - y) \in \bar{A}_t$, a contradiction. Clearly $\delta_{\bar{A}_t}(x-y) = \delta_{\bar{A}_t}(y) = 1$ and $\delta_{\bar{A}_t}(x) = A(x)$. Now

(i) $\delta_{\bar{A}_t}(x-y) = max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\}$

(ii) $\delta_{\bar{A}_t}(xy) \leq max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\}$

and

(iii) $\delta_{\bar{A}_t}(xy) \leq \delta_{\bar{A}_t}(y)$.

Case-3: Suppose $x \notin \bar{A}_t$, $y \in \bar{A}_t$. Now $x - y \notin \bar{A}_t$ and $xy \in \bar{A}_t$. Clearly $\delta_{\bar{A}_t}(x-y) = \delta_{\bar{A}_t}(x) = 1$, $\delta_{\bar{A}_t}(y) = A(y)$ and $\delta_{\bar{A}_t}(xy) = A(xy)$. Now

(i) $\delta_{\bar{A}_t}(x-y) = max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\}$

(ii) $\delta_{\bar{A}_t}(xy) = A(xy) \leq max\{A(x), A(y)\}$
$\leq max\{1, \delta_{\bar{A}_t}(y)\} = max\{\delta_{\bar{A}_t}(x)\}, \delta_{\bar{A}_t}(y)\}$

and

(iii) $\delta_{\bar{A}_t}(xy) = A(xy) \leq A(y) = \delta_{\bar{A}_t}(y)$.

Case-4: Suppose $x \notin \bar{A}_t, y \notin \bar{A}_t$. Clearly $\delta_{\bar{A}_t}(x) = \delta_{\bar{A}_t}(y) = 1$ and therefore

(i) $\delta_{\bar{A}_t}(x-y) \leq max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\} = 1$

(ii) $\delta_{\bar{A}_t}(xy) \leq max\{\delta_{\bar{A}_t}(x), \delta_{\bar{A}_t}(y)\} = 1$

and

(iii) $\delta_{\bar{A}_t}(xy) \leq \delta_{\bar{A}_t}(y) = 1$. Hence the theorem is proved. □

In a similar way we can prove the following theorem:

Theorem 4.4. *Let A be an anti fuzzy right ideal of X. Then $\delta_{\bar{A}_t}$ is also an anti fuzzy right ideal of X.*

Corollary 4.5. *If A is an anti fuzzy ideal of X, then so is $\delta_{\bar{A}_t}$.*

5. Quotient ring

In [2], Bingxue gave a concept of fuzzy quotient ring of the form $\frac{X}{E}$ considering E as a fuzzy semi-ideal of X. In this section, a concept of quotient ring of the form $\frac{X}{A}$, where A is an anti fuzzy ideal of X, is given and some isomorphism theorems are established.

Definition 5.1 ([2, 5, 9, 15])**.** Let $A : X \to [0, 1]$. $\forall\ y \in X$, $y + A$ is a fuzzy set in X which is defined as follows: $(y + A)(x) = A(x - y)$, $\forall\ x \in X$.

Theorem 5.2. *Let A be an anti fuzzy ideal of X. Then $\forall\ y_1,\ y_2\ \in X$, $y_1 + A \leq y_2 + A \Rightarrow A(y_1) = A(y_2)$.*

Proof. We have $y_1 + A \leq y_2 + A \Rightarrow (y_1 + A)(x) \leq (y_2 + A)(x)\ \forall\ x \in X$. Now, $A(y_2 - y_1) = (y_1 + A)(y_2) \leq (y_2 + A)(y_2) = A(\mathbf{0}) \Rightarrow A(y_2 - y_1) = A(\mathbf{0}) \Rightarrow A(y_1) = A(y_2)$. □

Theorem 5.3. *Let A be an anti fuzzy ideal of X. Then $A(y_2 - y_1) = A(\mathbf{0}) \Rightarrow y_1 + A = y_2 + A,\ \forall\ y_1,\ y_2 \in X$.*

Proof. We have $(y_1 + A)(x) = A(x - y_1) = A((x - y_2) - (y_1 - y_2)) \leq max\{A(x - y_2), A(y_1 - y_2)\} = max\{A(x - y_2), A(\mathbf{0})\} = A(x - y_2)(x) = (y_2 + A)(x),\ \forall\ x \in X$. Therefore $y_1 + A \leq y_2 + A$. Similarly we can show that $y_2 + A \leq y_1 + A$. □

Corollary 5.4. *Let A be an anti fuzzy ideal of X. Then $\forall\ y_1,\ y_2 \in X$, $y_1 + A = y_2 + A \Leftrightarrow A(y_1 - y_2) = A(\mathbf{0})$.*

Theorem 5.5. *Let A be an anti fuzzy ideal of X. Then $\forall\ x_1, x_2, x_3, x_4 \in X$,*

$$\left\{ \begin{array}{c} x_1 + A = x_2 + A \\ and \\ x_3 + A = x_4 + A \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} (x_1 + x_3) + A = (x_2 + x_4) + A \\ and \\ (x_1x_3) + A = (x_2x_4) + A \end{array} \right\}$$

Proof. Clearly $A(x_2 - x_1) = A(x_3 - x_4) = A(\mathbf{0})$. Now $A((x_2 + x_4) - (x_1 + x_3)) = A((x_2 - x_1) - (x_3 - x_4)) \leq max\{A(x_2 - x_1),\ A(x_3 - x_4)\} = A(\mathbf{0}) \Rightarrow A((x_2 + x_4) - (x_1 + x_3)) = A(\mathbf{0})$. Therefore $(x_1 + x_3) + A = (x_2 + x_4) + A$.

Again $A(x_1x_3 - x_2x_4) = A((x_1 - x_2)x_3 - x_2(x_4 - x_3))$
$\leq max\{A((x_1 - x_2)x_3), A(x_2(x_4 - x_3))\}$
$\leq max\{A(x_1 - x_2), A(x_4 - x_3)\} = A(\mathbf{0}) \Rightarrow A(x_1x_3 - x_2x_4) = A(\mathbf{0})$.
Therefore $x_1x_3 + A = x_2x_4 + A$. □

The results obtained in Theorem 5.5 lead us to establish the following theorem:

Theorem 5.6. *Let X be a ring and A be an anti fuzzy ideal of X. Then the set $\frac{X}{A} = \{x + A : x \in X\}$ is a quotient ring under the following operations:*

$$(x + A) + (y + A) = (x + y) + A \text{ and } (x + A)(y + A) = xy + A.$$

Proof. $\forall\ x, y, z \in X$, the following are obvious:
(1) $(x + A) + (y + A) = (x + y) + A \in \frac{X}{A}$.
(2) $(x + A)(y + A) = xy + A \in \frac{X}{A}$.
(3) $(x + A) + (y + A) = (y + A) + (x + A) = (x + y) + A$.
(4) $[(x+A)+(y+A)]+(z+A) = (x+A)+[(y+A)+(z+A)] = (x+y+z)+A$.
(5) $A + (x + A) = (x + A) + A = x + A$.
(6) $(x + A) + (-x + A) = A$.
(7) $(x+A)[(y+A)+(z+A)] = (x+A)(y+A)+(x+A)(z+A) = (xy+xz)+A$.
(8) $[(x+A)+(y+A)](z+A) = (x+A)(z+A)+(y+A)(z+A) = (xz+yz)+A$.
Hence the theorem is proved. □

Theorem 5.7. *Let A be an anti fuzzy ideal X. Then $\frac{X}{A} \cong \frac{X}{A_0}$.*

Proof. Define $f : X \to \frac{X}{A}$ by $f(x) = x + A$. Clearly f is an epimorphism. Now ker(f)= $\{x \in X : x + A = A\} = \{x \in X : A(x) = A(\mathbf{0})\} = A_0$, and therefore by the 'fundamental theorem of homomorphism', $\frac{X}{A} \cong \frac{X}{A_0}$. □

Theorem 5.8. *Let A and B be two anti fuzzy ideals X. Then $\frac{B_0}{A}$ is an ideal of $\frac{X}{A}$.*

Proof. Clearly B_0 is an ideal of X. Let $b_1 + A,\ b_2 + A \in \frac{B_0}{A}$ and $x + A \in \frac{X}{A}$. Then $b_1 - b_2$, b_1b_2, xb_1, $b_1x \in B_0$. Therefore $(b_1 + A) - (b_2 + A)$, $(b_1 + A)(b_2 + A)$, $(b_1 + A)(x + A)$, $(x + A)(b_1 + A) \in \frac{B_0}{A}$. Hence $\frac{B_0}{A}$ is an ideal of $\frac{X}{A}$. □

Theorem 5.9. *Consider an epimorphism $f : X \to Y$ between rings and let B be an anti fuzzy ideal of Y. Then* $\frac{X}{f^{-1}(B)} \cong \frac{Y}{B}$.

Proof. Consider a map $g : \frac{X}{f^{-1}(B)} \to \frac{Y}{B}$ defined by $g(x + f^{-1}(B)) = f(x) + B$. Now, $\forall\ x_1,\ x_2 \in X$, $x_1 + f^{-1}(B) = x_2 + f^{-1}(B) \Leftrightarrow f^{-1}(B)(x_1 - x_2) = f^{-1}(B)(\mathbf{0}) \Leftrightarrow B(f(x_1 - x_2)) = B(\mathbf{0}) \Leftrightarrow B(f(x_1)) - B(f(x_2)) = B(\mathbf{0}) \Leftrightarrow f(x_1) + B = f(x_2) + B \Leftrightarrow g(x_1 + f^{-1}(B)) = g(x_2 + f^{-1}(B)) \Rightarrow\ g$ is well defined and one-one.

Clearly g is onto as so is f.

Again $\forall\ x_1,\ x_2 \in X$, $g((x_1+f^{-1}(B))+(x_2+f^{-1}(B))) = g((x_1+x_2)+f^{-1}(B)) = f(x_1+x_2)+B = (f(x_1)+f(x_2))+B = (f(x_1)+B)+(f(x_2)+B) = g(x_1+f^{-1}(B))+g(x_2 + f^{-1}(B))$.

Similarly we can show that,

$$g((x_1 + f^{-1}(B)).(x_2 + f^{-1}(B))) = g(x_1 + f^{-1}(B)).g(x_2 + f^{-1}(B)).$$

Hence g is an isomorphism and so $\frac{X}{f^{-1}(B)} \cong \frac{Y}{B}$. □

Theorem 5.10. *Let A and B be two anti fuzzy ideals of X such that $B \leq A$ and $A(\mathbf{0}) = B(\mathbf{0})$. Then* $\frac{\frac{X}{A}}{\frac{B_0}{A}} \cong \frac{X}{B_0}$.

Proof. Define $f : \frac{X}{A} \to \frac{X}{B_0}$ by $f(x + A) = x + B_0$. Let $x_1,\ x_2\ \in X$. Now $x_1 + A = x_2+A \Rightarrow A(x_1-x_2) = A(\mathbf{0})$. Since $B \leq A$, $B(x_1-x_2) \leq A(x_1-x_2) = A(\mathbf{0}) = B(\mathbf{0})$. Therefore $B(x_1-x_2) = B(\mathbf{0})$, and so, $x_1+B_0 = x_2+B_0 \Rightarrow f$ is well defined. Clearly f is onto. Again

$f((x_1+A)+(x_2+A)) = f((x_1+x_2)+A) = (x_1+x_2)+B_0 = (x_1+B_0)+(x_2+B_0) = f((x_1 + A)) + f((x_2 + A))$

and

$f((x_1 + A)(x_2 + A)) = f((x_1x_2) + A) = (x_1x_2) + B_0 = (x_1 + B_0)(x_2 + B_0) = f(x_1 + A)f(x_2 + A)$.

Thus we see that f is an epimorphism.

Now $ker(f) = \{x + A : x + B_0 = B_0\} = \{x + A : x \in B_0\} = \frac{B_0}{A}$. Therefore $\frac{\frac{X}{A}}{\frac{B_0}{A}} \cong \frac{X}{B_0}$. □

Theorem 5.11. *Let A and B be two anti fuzzy ideals of a ring X such that $A(\mathbf{0}) = B(\mathbf{0})$. Then* $\frac{A_0+B_0}{A} \cong \frac{B_0}{A\vee B}$.

Proof. Define $f : \frac{A_0+B_0}{A} \to \frac{B_0}{A\vee B}$ by $f(a + b + A) = b + A \vee B$, where $a \in A_0$ and $b \in B_0$. Let $(a_1 + b_1) + A = (a_2 + b_2) + A$, where $a_1,\ a_2 \in A_0$ and $b_1,\ b_2 \in B_0$. Then $A(a_1 + b_1 - a_2 - b_2) = A(\mathbf{0}) = A(a_1 - a_2)$. Now $A(b_1 - b_2) = A((a_1 + b_1 - a_2 - b_2) - (a_1-a_2)) \leq max\{A(a_1+b_1-a_2-b_2),\ A(a_1-a_2)\} = A(\mathbf{0})$. Hence $A(b_1-b_2) = A(\mathbf{0})$. Therefore $(A \vee B)(b_1 - b_2) = (A \vee B)(\mathbf{0})$, and so, $b_1 + A \vee B = b_2 + A \vee B \Rightarrow\ f$ is well defined.

Again, let $f((a_1 + b_1) + A) = f((a_2 + b_2) + A)$. Then $b_1 + A \vee B = b_2 + A \vee B$ and therefore, $(A\vee B)(b_2-b_1) = (A\vee B)(\mathbf{0}) = A(\mathbf{0})$. This implies that $A(b_2-b_1) = A(\mathbf{0})$. Now $A(a_1 + b_1 - a_2 - b_2) = A((a_1 - a_2) - (b_2 - b_1)) \leq max\{A(a_1 - a_2), A(b_2 - b_1)\} = max\{A(\mathbf{0}), A(\mathbf{0})\} = A(\mathbf{0})$. Therefore $A(a_1 + b_1 - a_2 - b_2) = A(\mathbf{0})$. Readily $(a_1 + b_1) + A = (a_2 + b_2) + A$. Thus we see that f is one-one.

Moreover, it is clear that f is onto. Again,

$f(((a_1+b_1)+A)+((a_2+b_2)+A)) = f((a_1+b_1)+A)+f((a_2+b_2)+A) = (b_1+b_2)+A\vee B$
and
$f(((a_1+b_1)+A)((a_2+b_2)+A)) = f((a_1+b_1)+A)f((a_2+b_2)+A) = (b_1b_2)+A\vee B$.
Thus we see that f is a homomorphism and so it is an isomorphism. Therefore $\frac{A_0+B_0}{A} \cong \frac{B_0}{A\vee B}$. □

6. Conclusion

Biswas' [3] idea of anti fuzzy ideal of groups is extended and a notion of anti fuzzy ideal of rings is introduced in this paper. For any fuzzy set A of a ring X, it is found that the corresponding lower level subsets $\bar{A}_t$ are ideals of X if and only if A is an anti fuzzy ideal of X. A modified definition of fuzzification of lower level subsets $\bar{A}_t$ is given and it is observed that if A is an anti fuzzy ideal of X, then the fuzzification $\delta_{\bar{A}_t}$ of lower level subsets $\bar{A}_t$ of A is also an anti fuzzy ideal of X. In addition, a concept of qoutient ring of the form $\frac{X}{A}$, where A is an anti fuzzy ideal of a ring X is given and various isomorphism theorems are established.

Acknowledgements. The authors would like to be thankful to the anonymous reviewers for their valuable suggestions.

References

[1] M. Akram, Anti fuzzy Lie ideals of Lie algebras, Quasigroups Related Systems 14 (2006) 123–132.
[2] Y. Bingxue, Fuzzy semi-ideal and generalized fuzzy quotient ring, Iran. J. Fuzzy Syst. 5 (2008) 87–92.
[3] R. Biswas, Fuzzy subgroups and anti fuzzy subgroups, Fuzzy Sets and Systems 44 (1990) 121–124.
[4] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968) 182–190.
[5] V. N. Dixit, R. Kumar and N. Ajmal, On fuzzy rings, Fuzzy Sets and Systems 49 (1992) 205–213.
[6] S. M. Hong and Y. B. Jun, Anti fuzzy ideals in BCK-algebras, Kyungpook Math. J. 38 (1998) 145–150.
[7] K. H. Kim, Y. B. Jun and Y. H. Yon, On anti fuzzy ideals in near-rings, Iran. J. Fuzzy Syst. 2(2) (2005) 71–80.
[8] S. D. Kim and H. S. Kim, On fuzzy ideals of nearrings, Bull. Korean Math. Soc. 33(4) (1996) 593–601.
[9] H. V. Kumbhojkar and M. S. Bapat, Correspondence theorem for fuzzy ideals, Fuzzy Sets and Systems 41 (1991) 213–219.
[10] M. Li, Y. Feng and Y. Han, Existence and uniqueness of anti-fuzzy ideal, Fuzzy Information and Engineering, Advances in Soft Computing 54 (2009) 101–106.
[11] W. Liu, Fuzzy invariant subgroups and fuzzy ideals, Fuzzy Sets and Systems 8 (1982) 133–139.
[12] K. C. Rao and V. Swaminathan, Anti-homomorphism in fuzzy ideals, Int. J. Comput. Math. Sci. 4(7) (2010) 319–322.
[13] A. Rosenfeld, Fuzzy groups, J. Math. Anal. Appl. 35 (1971) 512–517.
[14] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[15] S. A. Zaid, On fuzzy ideals and fuzzy quotient rings of a ring, Fuzzy Sets and Systems 59 (1993) 205–210.
[16] M. Zhou, D. Xiang and J. Zhan, On anti fuzzy ideals of Γ-rings, Ann. Fuzzy Math. Inform. 1(2) (2011) 197–205.

F. A. Azam (faqruddinaliazam@gmail.com)
Institute of Natural Sciences, United International University, Dhaka-1209, Bangladesh.

A. A. Mamun (mamun3213ssh@gmail.com)
Institute of Natural Sciences, United International University, Dhaka-1209, Bangladesh.

F. Nasrin (farzananasrin@ymail.com)
Institute of Natural Sciences, United International University, Dhaka-1209, Bangladesh.