Free Algebras for Gödel-Löb Provability Logic

Sam J. van Gool [1]

Mathematical Institute, University of Bern, Switzerland

Abstract

We give a construction of finitely generated free algebras for Gödel-Löb provability logic, GL. On the semantic side, this construction yields a notion of canonical graded model for GL and a syntactic definition of those normal forms which are consistent with GL. Our two main techniques are incremental constructions of free algebras and finite duality for partial modal algebras. In order to apply these techniques to GL, we use a rule-based formulation of the logic GL by Avron (which we simplify slightly), and the corresponding semantic characterization that was recently obtained by Bezhanishvili and Ghilardi.

Keywords: Stone duality, provability logic, diagonalizable algebra, Magari algebra, partial modal algebra, one-step constructions.

1 Introduction

The provability logic GL is the axiomatic extension of the basic modal logic K by the Gödel-Löb axiom $\Box(\Box p \to p) \to \Box p$. The intended interpretation of the modal operator $\Box$ in GL is "it is provable in T that ...", where T is a sufficiently strong formal theory, such as, for example, Peano arithmetic. A classical theorem of Solovay [19] shows that, indeed, GL is exactly the logic of provability of Peano arithmetic. From a modal logic perspective, the logic GL is interesting because, on the one hand, it has reasonably nice model-theoretic properties (notably, GL is complete with respect to the class of finite irreflexive transitive frames), but on the other hand it fails to be canonical, that is, the canonical model of the logic GL does *not* validate the Gödel-Löb axiom.
The situation with normal forms for GL is similarly subtle: Boolos [7] gave normal forms for the fragment of GL-formulas containing no propositional variables, but in [8] Boolos showed that the same method does not apply to the

[1] The author would like to thank Nick Bezhanishvili, Mai Gehrke, and Silvio Ghilardi for many interesting and helpful discussions on the topic of this paper. The research reported in this paper was performed during a research stay at Université Paris 7 and INRIA, hosted by Paul-André Melliès, to whom the author is also grateful for providing the inspiration for the "graded" view in Section 6 of this paper. Finally, the author would like to thank the anonymous referees for their many useful suggestions.

fragment of GL-formulas in n variables for $n > 0$. This result has sometimes been cited in the literature as saying that *no* normal forms exist for GL on n variables, but this is not what Boolos proved, nor does it seem to be what he intended to claim [2]. Indeed, one of the contributions of this paper is to give an explicit construction of normal forms for GL on an arbitrary finite number of variables.

It has been known since the work of Fine [10] (which was put into algebraic perspective in [1] and [12]) that any modal formula of modal degree [3] n on k variables $\{p_1, \ldots, p_k\}$ is equivalent in K to a finite (possibly empty) disjunction of *normal forms of degree* n. Here, the normal forms of degree 0 are the formulas of the form $\bigwedge_{i \in T} p_i \wedge \bigwedge_{j \notin T} \neg p_j$, for $T \subseteq \{p_1, \ldots, p_k\}$, and normal forms of degree $n+1$ are formulas of the form

$$\phi \wedge \nabla\Psi, \tag{1}$$

where ϕ is a normal form of degree 0, Ψ is a finite set of normal forms of degree n, and $\nabla\Psi$ abbreviates $\Box\left(\bigvee_{\psi\in\Psi} \psi\right) \wedge \left(\bigwedge_{\psi\in\Psi} \Diamond\psi\right)$, a notation from coalgebraic logic [16] (we will make some remarks on the connection with coalgebraic logic in the conclusion of this paper).

In the modal logic K, each of the normal forms (1) is satisfiable. However, this is clearly no longer the case in GL. In this paper we give a bottom-up construction of the free GL-algebra (Section 5), and from the construction we extract a bottom-up definition of the normal forms that are consistent with GL (Definition 6.3). In Section 6, we also develop a notion of *graded model* for GL, and construct a canonical graded model for GL.

Let us now discuss the methods of this paper in some more detail. Recall that a *modal algebra* is a tuple $(A, \vee, \wedge, \neg, 0, 1, \Box)$ where $(A, \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra and $\Box : A \to A$ is a unary operation on A which preserves $\wedge$ and 1. As is well-known, the variety of modal algebras is the algebraization of the basic modal logic K, in the sense that a formula ϕ is a tautology of K if, and only if, the equation $\phi = 1$ is valid in every modal algebra. By definition, a GL-*algebra* [4] is a modal algebra in which the equation $\Box(\Box a \to a) \leq \Box a$ is valid. The variety of GL-algebras algebraizes the logic GL. The *free* GL-*algebra* over variables $p_1, \ldots, p_k$ is the GL-algebra consisting of GL-equivalence classes of modal formulas using variables from $p_1, \ldots, p_k$. We shall denote this algebra by $\mathbb{F}_{\mathsf{GL}}(k)$.

The notion of modal degree allows one to approximate a finitely generated modal algebra by an infinite chain of finite algebras, in each of which the

---

[2] Note in particular that, in the title of [8], the word "certain" is under the scope of a negated existential quantifier.

[3] Recall that the *modal degree* of a modal formula is the maximum length of a string of nested occurrences of the modal operator $\Box$.

[4] These algebras are the same (although axiomatized slightly differently) as the algebras introduced by Magari [15] and are also called *Magari algebras* or *diagonalizable algebras* in the literature.

modal operator is partially defined. Concretely, if A is a modal algebra with generators $a_1, \ldots, a_k$, let

$$A_n := \{\phi^A(a_1, \ldots, a_k) \mid \phi(p_1, \ldots, p_k) \text{ an } k\text{-variable formula of degree} \leq n\}. \tag{2}$$

Then $A_0 \subseteq A_1 \subseteq \cdots$ is an increasing chain of finite Boolean subalgebras of A, and $A = \bigcup_{n \geq 0} A_n$, since A is generated by $a_1, \ldots, a_k$. Moreover, the operation $\Box$ on the modal algebra A restricts, for each $n \geq 0$, to an operation $\Box_n : A_n \to A_{n+1}$. The tuple $(A_{n+1}, A_n, \Box_n)$ is an example of a *partial modal algebra* (cf. Definition 2.1 below). Ghilardi's pioneering idea in [12], the germ of which appeared earlier in unpublished lectures by Abramsky [1], was to describe the increasing chain of Boolean algebras $A_0, A_1, \ldots$ and the operations $\Box_n$ between them *from the bottom up*. That is, for many classes V of modal algebras, it is possible to describe a functor F on partial modal algebras with the property that, if A is a finitely generated modal algebra in V and $(A_1, A_0, \Box_0)$ is the first algebra in the chain described above, then the n^{th} such algebra is (isomorphic to) $(F_V)^n(A_1, A_0, \Box_0)$. In this case, one obtains an incremental, bottom-up construction for algebras in the class V, by starting from a partial modal algebra and taking an appropriate colimit of the chain of algebras $(F_V)^n(A_1, A_0, \Box_0)$. In Section 4, we give the definition of such a functor F in the case where V is the class of GL-algebras, and apply it to the particular case of the finitely generated free algebras $\mathbb{F}_{\mathsf{GL}}(k)$. Definition 4.1 is an instance of the general definition in [9, Section 2] of a free image-total functor associated to a set of quasi-equations of degree ≤ 1. It is important to note that the functor F depends on the particular axiomatization of the class V, also see the conclusion.
The free image-total functor F is often most conveniently described using duality. Dual to any partial modal algebra is a *partial frame* (called "q-frame" in [9]), which consists of a set equipped with an equivalence relation $\sim$ and a Kripke accessibility relation R which respects $\sim$, cf. Definition 3.1 below for the precise definition. Under this duality, the construction F on partial modal algebras corresponds to a dual construction G on partial frames. Since the construction that underlies F is a pushout in the category of Boolean algebras ([5, Prop. 5]), i.e., a quotient of a coproduct, the construction that underlies G is a pullback in the category of sets, i.e., a subset of a product. As a result, the action on objects of the functor G is usually easier to identify than that of the functor F. Indeed, in Definition 4.4 we will give a direct combinatorial description of the functor G.

Outline of the paper. Section 2 contains the necessary algebraic definitions, notably, the definition of partial GL-algebra. In Section 3 we recall duality for partial modal algebras, and specialize it to the case of partial GL-algebras. In Section 4 we define the free image-total functor for GL and characterize its dual. In Section 5 we apply the functor F to obtain a construction of the free finitely generated GL-algebra, and in Section 6 we use G to obtain a canonical graded model for GL.

2 Partial GL-algebras

In this paper, we make extensive use of a generalization of modal algebras, namely *partial* modal algebras.

Definition 2.1 A *partial modal algebra* is a tuple $(A, B, \Box)$, where A is a Boolean algebra, B is a Boolean subalgebra of A, and $\Box : B \to A$ is a function which preserves $\wedge$ and 1. The algebra A is called the *underlying Boolean algebra* of $(A, B, \Box)$. A *homomorphism* from a partial modal algebra $(A, B, \Box)$ to a partial modal algebra $(A', B', \Box')$ is a map $h : A \to A'$ such that $h(B) \subseteq B'$ and $h(\Box b) = \Box' h(b)$ for all $b \in B$. A homomorphism h is an *isomorphism* if h is bijective and $h(B) = B'$. A *congruence* on a partial modal algebra $(A, B, \Box)$ is a Boolean algebra congruence θ on A such that moreover, for all $b, b' \in B$, if $b\theta b'$, then $\Box b \theta \Box b'$. We denote by $(A/\theta, B/\theta, \Box/\theta)$ the quotient of $(A, B, \Box)$ by θ. Note that, for any congruence θ, $(A/\theta, B/\theta, \Box/\theta)$ is again a partial modal algebra. A partial modal algebra $(A, B, \Box)$ is called *total* if $B = A$.

Remark 2.2 Note that the category of partial modal algebras with homomorphisms is equivalent to the category whose objects are diagrams of shape $B \underset{i}{\overset{\Box}{\rightrightarrows}} A$ where A, B are Boolean algebras, i is an injective Boolean homomorphism, and $\Box$ is a meet-semilattice morphism, and whose morphisms are given by the obvious commuting diagrams. Therefore, the category of partial modal algebras is equivalent to a full subcategory of the category of one-step modal algebras introduced in [5].

Example 2.3 Let $(A, \Box)$ be a finitely generated modal algebra, with generators $a_1, \dots, a_k$. Let A_0 be the Boolean algebra generated by $a_1, \dots, a_k$, and let A_1 be the Boolean algebra generated by $a_1, \dots, a_k, \Box a_1, \dots, \Box a_k$. The operation $\Box$ on C restricts to an operation $\Box : A_0 \to A_1$, and it follows that $(A_0, A_1, \Box)$ is a partial modal algebra. Repeating this process, by taking all elements of A_1 as set of generators in the next step, one obtains, after n repetitions, the partial modal algebra $(A_{n+1}, A_n, \Box_n)$ defined in equation (2) in the introduction.
In particular, if $\mathbf{L}$ is a normal modal logic, we may take for A the k-variable Lindenbaum algebra for $\mathbf{L}$, i.e., the modal algebra of $\mathbf{L}$-equivalence classes of modal formulas in variables $c_1, \dots, c_k$. In this case, the partial modal algebras $(A_{n+1}, A_n, \Box)$ defined in the previous paragraph are the partial modal algebras of $\mathbf{L}$-equivalence classes of k-variable modal formulas of degree at most $n + 1$.

We now identify a subclass $\mathbf{V}$ of the class of partial modal algebras such that the total algebras in $\mathbf{V}$ are exactly the GL-algebras. There is a choice to be made here: several such varieties $\mathbf{V}$ exist, and not all of them are suitable for our purposes (also cf. the conclusion).

Definition 2.4 A partial modal algebra $(A, B, \Box)$ is a *partial* GL-*algebra* if, for all $a, b \in B$:

$$\text{if } b \leq \Box a \to a \text{ then } \Box b \leq \Box a. \tag{3}$$

The proof of the following fact is a direct application of a procedure which is known in the literature as *Ackermann's lemma*, or *flattening*.

Lemma 2.5 *Let $(A, \Box)$ be a modal algebra. The partial modal algebra $(A, A, \Box)$ is a partial GL-algebra if, and only if, $(A, \Box)$ is a GL-algebra.*

Proof. If (3) holds in $(A, A, \Box)$, then, for any $a \in A$, we can instantiate (3) with $b := \Box a \to a$ to obtain $\Box(\Box a \to a) \leq \Box a$. Conversely, if $(A, \Box)$ is a GL-algebra and $a, b \in A$ are such that $b \leq \Box a \to a$, then we get

$$\Box b \leq \Box(\Box a \to a) \leq \Box a,$$

where we have used that $\Box : A \to A$ is order-preserving and that $(A, \Box)$ is a GL-algebra. □

Remark 2.6 Definition 2.4 was inspired by [3, Section 5], where a rule-based formulation for GL due to Avron [2] was used. Avron's rule, when translated to a quasi-equation, says that, for all $a, b \in B$,

$$\text{if } b \wedge \Box b \leq \Box a \to a \text{ then } \Box b \leq \Box a. \tag{4}$$

Avron [2] used the rule (4) to obtain a sequent calculus for GL which has cut-elimination, and used it to prove Kripke completeness of GL. It is possible to prove syntactically that the quasi-equations (3) and (4) are equivalent for any partial modal algebra. The reason why we used (3) rather than (4) in Definition 2.4 is that (3) is simply a flattening of the usual GL-axiom, while showing the equivalence of (4) with the usual GL-axiom requires some ingenuity.

3 Duality for partial GL-algebras

We now recall the facts about duality for finite partial modal algebras that we need in this paper, and we recall how this duality specializes to finite partial GL-algebras. For more details about duality for finite partial modal algebras, cf., e.g., [9, Sec. 4] or [5, Sec. 3.2].

Definition 3.1 A *partial frame*[5] is a tuple $(X, \sim, R)$, where X is a set, $\sim$ is an equivalence relation on X, and $R \subseteq X \times X$ is a relation such that $xRy \sim y'$ implies xRy'. A *bounded morphism* from a partial frame $(X, \sim_X, R)$ to a partial frame $(Y, \sim_Y, S)$ is a function f from X to Y such that $x \sim_X x'$ implies $f(x) \sim_Y f(x')$, and $f(x)Sy$ if, and only if, there exists $x' \in X$ such that xRx' and $f(x') \sim_Y y$. A *partial generated subframe* of a partial frame $(X, \sim, R)$ is a partial frame $(Y, \approx, Q)$ such that $Y \subseteq X$, the relations $\approx$ and Q are the restrictions to Y of the relations $\sim$ and R, respectively, and, for all $y \in Y$ and $x \in X$, if yRx, then there exists $x' \in Y$ such that $x' \sim x$. A partial frame is called *total* if the equivalence relation $\sim$ is the diagonal $\Delta = \{(x, x) \mid x \in X\}$.

[5] These were called q-frames in [9], and a generalization of these were called one-step frames in [5].

Notation. In a partial frame $(X, \sim, R)$, for $x \in X$, we use the notation $R(x)$ for the set $\{y \in X \mid xRy\}$ and $[x]_\sim$ for the $\sim$-equivalence class of y.

For any partial frame $(X, \sim, R)$, we define its *dual partial modal algebra* $(A, B, \Box)$ by

$$\begin{aligned} A &:= \mathcal{P}(X), \\ B &:= \mathcal{P}_\sim(X) = \{b \in \mathcal{P}(X) \mid \text{ if } x \in b \text{ and } x \sim x' \text{ then } x' \in b\}, \\ \text{for } b \in B,\ \Box b &:= \{x \in X \mid \text{ if } xRy \text{ then } y \in b\}. \end{aligned}$$

Note that indeed $(A, B, \Box)$ is a partial modal algebra. For a bounded morphism $f : (X, \sim_X, R) \to (Y, \sim_Y, S)$, if $(A, B, \Box)$, $(A', B', \Box')$ are the partial modal algebras dual to X and Y respectively, we define a homomorphism $h : A' \to A$ by $h(c) := f^{-1}(c)$. Note that h is indeed a homomorphism of partial modal algebras. Moreover, these assignments define a contravariant functor from the category of partial frames to the category of partial modal algebras. When restricted to *finite* partial frames, this functor becomes part of a dual equivalence, as is obvious by combining Remark 2.2 with the well-known fact that the category of finite Kripke frames is dually equivalent to the category of finite modal algebras.

Theorem 3.2 (Duality for finite partial modal algebras) *The category of finite partial frames is dually equivalent to the category of finite partial modal algebras.*

Proof. (Sketch) The functor from finite partial frames to finite partial modal algebras takes a finite partial frame to its (finite) dual partial modal algebra. The functor in the other direction takes a finite partial modal algebra to its set of atoms, equipped with the appropriate structure of a partial frame. See Theorem A.1 in the appendix for more details. $\Box$

Theorem 3.2 in particular implies that epimorphisms in the category of finite partial modal algebras dually correspond to monomorphisms of finite partial frames, and vice versa. We formulate these facts in some detail in the following two corollaries.

Corollary 3.3 *Let $(A, B, \Box)$ be a finite partial modal algebra with dual finite partial frame $(X, \sim, R)$. There is a Galois connection between $\mathcal{P}(A \times A)$ and $\mathcal{P}(X)$, given by the assignments:*

$$\begin{aligned} E \in \mathcal{P}(A \times A) &\mapsto Y_E := \{x \in X \mid \forall (a, b) \in E : x \le a \iff x \le b\} \\ Y \in \mathcal{P}(X) &\mapsto \theta_Y := \{(a, b) \in A \times A \mid \forall y \in Y \ : \ y \le a \iff y \le b\}. \end{aligned}$$

This Galois connection restricts to an order-reversing isomorphism between the poset of congruences on A and the poset of partial generated subframes of X, both ordered by inclusion. Moreover, for any $E \in \mathcal{P}(A \times A)$, the congruence θ_{Y_E} is the smallest congruence containing E.

Corollary 3.4 *Let $f : (X, \sim_X, R) \to (Y, \sim_Y, S)$ be a bounded morphism between partial frames and let $h : (A', B', \Box') \to (A, B, \Box)$ be the homomorphism* *of partial modal algebras dual to it. Then f is surjective if, and only if, h is injective.*

We now recall some constructions and facts about partial modal algebras that will be used later in this paper.[6] We first define, given a finite Boolean algebra A, a partial modal algebra $(A+V(A), A, \Box)$, and then describe its dual. Recall that the *coproduct* of any two Boolean algebras A and B exists, and can be characterized as the (up to isomorphism unique) Boolean algebra $A+B$ which contains A and B as subalgebras and such that any pair of homomorphisms $(A \to C,\ B \to C)$ factors uniquely through $A+B$.
In the rest of this section, let A and B be arbitrary finite Boolean algebras.

Fact 3.5 *The dual of $A+B$ is the Cartesian product $\mathrm{At}(A) \times \mathrm{At}(B)$ and the Boolean algebra $A+B$ is isomorphic to $\mathcal{P}(\mathrm{At}(A) \times \mathrm{At}(B))$.*

The *Vietoris algebra* $V(A)$ over a finite Boolean algebra A is defined to be the algebra of Boolean combinations of formal elements $\Box a$, for $a \in A$, quotiented by the equalities $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$.

Fact 3.6 *The dual of $V(A)$ is $\mathcal{P}(\mathrm{At}(A))$ and the Boolean algebra $V(A)$ is isomorphic to $\mathcal{P}(\mathcal{P}(\mathrm{At}(A)))$.*

Combining the above two constructions, for any Boolean algebra A we can form a partial modal algebra $(A+V(A), A, \Box)$, where $\Box$ sends an element $a \in A$ to $\Box a \in V(A)$.

Fact 3.7 *The dual of $A+V(A)$ is $X \times \mathcal{P}(X)$, where $X := \mathrm{At}(A)$ and $A+V(A)$ is isomorphic to $\mathcal{P}(X \times \mathcal{P}(X))$. The subalgebra A of $A+V(A)$ is dual to the equivalence relation $\approx$ on $X \times \mathcal{P}(X)$ defined by $(x,T) \approx (y,S)$ if, and only if, $x = y$. The dual of the operation $\Box : A \to A+V(A)$ is the relation Q on $X \times \mathcal{P}(X)$ defined by $(x,T)Q(y,S)$ if, and only if, $y \in T$.*

We end this section by specializing the duality in Theorem 3.2 to finite partial GL-algebras. The following definition identifies the objects in the category of finite partial frames that are dual to finite partial GL-algebras.

Definition 3.8 A *partial* GL*-frame* is a partial frame $(X, \sim, R)$ such that, for any $x, y \in X$, if xRy, then there exists $y' \sim y$ such that $R(y') \subsetneq R(x)$.

It is easy (but not entirely trivial) to show that the finite partial GL-frames of the form $(X, =, R)$ are exactly the frames for which R is irreflexive and transitive. This also follows from Lemma 2.5 and Proposition 3.9 below.
The condition in Definition 3.8 was first derived in [4, Section 9.2] using general correspondence theory for one-step algebras. We give a direct proof of the correspondence between finite partial GL-algebras and finite partial GL-frames.

Proposition 3.9 *Let $(X, \sim, R)$ be a finite partial frame with dual partial modal algebra $(A, B, \Box)$. The following are equivalent:*

[6] The proofs of the well-known Facts 3.5–3.7, which amount to an algebraic explanation of Fine's normal forms (1), are given in the appendix (Facts A.2–A.4).

(i) $(A, B, \Box)$ *is a partial* GL*-algebra;*

(ii) $(X, \sim, R)$ *is a partial* GL*-frame.*

Proof. First suppose that $(A, B, \Box)$ is a partial GL-algebra, and let $x, y \in X$ with xRy. Define $b := R(x)$ and $a := R(x) \setminus [y]_\sim$. Note that $x \in \Box b$, but $x \notin \Box a$, since xRy and $y \notin a$. Thus, $\Box b \nleq \Box a$, and since $(A, B, \Box)$ is a partial GL-algebra we get that $b \nleq \Box a \to a$. Pick $y' \in b$ such that $y' \in \Box a$ and $y' \notin a$. We then have $y' \in b \setminus a = [y]_\sim$, and, since $y' \in \Box a$, we get $R(y') \subseteq a \subsetneq R(x)$.
Conversely, assume that $(X, \sim, R)$ is a partial GL-frame, and let $a, b \in B$ be such that $b \leq \Box a \to a$. Reasoning towards a contradiction, suppose that $\Box b \nleq \Box a$.
Pick $x_0 \in \Box b \setminus \Box a$. Since $x_0 \notin \Box a$, pick $x_1' \in X$ with $x_0 R x_1'$ and $x_1' \notin a$. Since $(X, \sim, R)$ is a partial GL-frame, pick $x_1 \sim x_1'$ such that $R(x_1) \subsetneq R(x_0)$. Since $x_0 R x_1' \sim x_1$, we have $x_0 R x_1$. Therefore, since $x_0 \in \Box b$, we have $x_1 \in b$. Using the assumption $b \leq \Box a \to a$, we get that $x_1 \in \Box a \to a$. Since $a \in B = \mathcal{P}_\sim(X)$, from $x_1' \notin a$ and $x_1' \sim x_1$ we get $x_1 \notin a$. Thus, since $x_1 \in \Box a \to a$, we must have $x_1 \notin \Box a$. Also, since $R(x_1) \subseteq R(x_0)$ and $x_0 \in \Box b$, we have $x_1 \in \Box b$.
In the preceding paragraph, starting from a point $x_0 \in \Box b \setminus \Box a$, we have constructed a point $x_1 \in \Box b \setminus \Box a$ such that $R(x_1) \subsetneq R(x_0)$. Repeating this argument at most $n = |R(x_0)|$ times, we will obtain a point x_n with $x_n \notin \Box a$, but $R(x_n) = \emptyset$, which is the desired contradiction. $\Box$

Combining Proposition 3.9 with Theorem 3.2, we see that the category of finite partial GL-algebras is dually equivalent to the category of finite partial GL-frames.

4 The free image-total GL-algebra and its dual

In this section, we will apply the construction of the *free image-total algebra* for a variety of partial modal algebras [9, Section 3] to the particular case of partial GL-algebras. The idea is to construct, given a partial GL-algebra $(A, B, \Box)$, in which $\Box a$ is only defined for $a \in B$, a larger partial GL-algebra in which the value of $\Box a$ is defined *for all* $a \in A$. To this end, we first build a partial modal algebra which consists of all Boolean combinations of elements from A with formal elements '$\Box a$' for all $a \in A$. Next, we take the largest possible quotient of this partial modal algebra which is a partial GL-algebra. A crucial fact, to be proved in Theorem 4.7 below, is that this quotient does not identify any elements from the original partial GL-algebra $(A, B, \Box)$.

Definition 4.1 Let $(A, B, \Box)$ be a partial modal algebra. Let θ be the smallest congruence on the partial modal algebra $(A+V(A), A, \boxast)$ [7] such that θ contains $E := \{(\Box b, \boxast b) \mid b \in B\}$ and the quotient by θ is a partial GL-algebra. Let $F(A, B, \Box)$ be the partial GL-algebra $((A + V(A))/\theta, A/\theta, \boxast/\theta)$, and let i be the natural homomorphism $(A, B, \Box) \to F(A, B, \Box)$ which sends $a \in A$ to $[a]_\theta \in (A + V(A))/\theta$.

[7] We use the notation $\boxast$ to distinguish the formal box operation $A \to A + V(A)$ from the already existing box operation $B \to A$.

The following proposition says that $F(A, B, \Box)$ defined in the above definition is the *free image-total* GL*-algebra* over $(A, B, \Box)$, cf. [9, Definition 2.10].

Proposition 4.2 *Let $(A, B, \Box)$ be a partial modal algebra. For any homomorphism $h : (A, B, \Box) \to (C, D, \Box)$ such that $(C, D, \Box)$ is a partial GL-algebra and $h(A) \subseteq D$, there exists a unique homomorphism $\bar{h} : F(A, B, \Box) \to (C, D, \Box)$ such that $\bar{h} \circ i = h$.*

Proof. This is a special case of [9, Lemma 3.12]. □

Our next aim is to directly describe the dual of the construction F in Definition 4.1, by giving a construction which associates to any finite partial frame $(X, \sim, R)$ a partial GL-frame $G(X, \sim, R)$ and a bounded morphism $p : G(X, \sim, R) \to (X, \sim, R)$. Recall from Fact 3.7 that the dual of the construction $(A, B, \Box) \mapsto (A + V(A), A, \boxast)$ is $(X, \sim, R) \mapsto (X \times \mathcal{P}(X), \approx, Q)$. Since $F(A, B, \Box)$ is a certain quotient of $(A + V(A), A, \boxast)$, we have that $G(X, \sim, R)$ is a certain partial generated subframe of $(X \times \mathcal{P}(X), \approx, Q)$, by Corollary 3.3. We now give a definition which will be seen to characterize exactly which points are in $G(X, \sim, R)$.

Definition 4.3 Let $(X, \sim, R)$ be a finite partial frame. An element $(x, T) \in X \times \mathcal{P}(X)$ will be called GL-*suitable*[8] if $R(x) = [T]_\sim$ and, for any $y \in T$, there exists $S \subsetneq T$ such that (y, S) is GL-suitable.

Note that Definition 4.3 is recursive: in order to determine whether (x, T) is GL-suitable, one first needs to know whether (y, S) is GL-suitable for $y \in T$ and $S \subsetneq T$. The recursion terminates: for any $x \in X$, the element $(x, \emptyset)$ is GL-suitable if, and only if, $R(x) = \emptyset$.

Definition 4.4 Let $(X, \sim, R)$ be a finite partial frame. Let $G(X, \sim, R)$ be the partial frame $(Y, \approx, Q)$ defined by:

$$\begin{aligned} Y &:= \{(x, T) \in X \times \mathcal{P}(X) \mid (x, T) \text{ is } \mathsf{GL}\text{-suitable}\}, \\ (x, T) &\approx (y, S) \iff x = y, \\ (x, T) Q (y, S) &\iff y \in T. \end{aligned}$$

Proposition 4.5 *Let $(X, \sim, R)$ be a finite partial frame with dual partial modal algebra $(A, B, \Box)$. Then the partial frame $G(X, \sim, R)$ is dual to the partial GL-algebra $F(A, B, \Box)$.*

Proof. By Fact 3.7, $(A + V(A), A, \boxast)$ is dual to $(X \times \mathcal{P}(X), \approx, Q)$. Let θ be the congruence from Definition 4.1; by definition, $F(A, B, \Box)$ is the quotient of $(A + V(A), A, \boxast)$ by θ. Combining Corollary 3.3 and Proposition 3.9, this quotient is dual to the largest partial generated sub-GL-frame of $(X \times \mathcal{P}(X), \approx, Q)$ that is contained in the subset $Y_E \subseteq X$ corresponding to $E = \{(\Box b, \boxast b) \mid b \in B\}$. Therefore, we need to prove the following two properties for $G(X, \sim, R) = (Y, \approx, Q)$ defined in Definition 4.4:

[8] Our terminology is inspired by terms such as "$\mathsf{K4}$-suitable", as used in [10].

(i) The partial frame $(Y, \approx, Q)$ is a partial generated subframe of the partial frame $(X \times \mathcal{P}(X), \approx, Q)$, is a partial GL-frame, and is contained in Y_E.

(ii) If $Z \subseteq X \times \mathcal{P}(X)$ is such that $(Z, \approx, Q)$ is a partial generated subframe of the partial frame $(X \times \mathcal{P}(X), \approx, Q)$, is a partial GL-frame, and is contained in Y_E, then $Z \subseteq Y$.

For (i), first note that $(Y, \approx, Q)$ is a partial generated subframe of $X \times \mathcal{P}(X)$: if $(x, T) \in Y$ and $y \in T$, then there exists S with $(y, S) \in Y$, by GL-suitability of (x, T). To see that $(Y, \approx, Q)$ is a partial GL-frame (Definition 3.8), let $(x, T), (y, S) \in Y$ with $(x, T)Q(y, S)$, i.e., $y \in T$. Since (x, T) is GL-suitable, there exists $S' \subsetneq T$ such that (y, S') is GL-suitable. Now $(y, S) \approx (y, S')$, and $Q((y, S')) \subsetneq Q((x, T))$.
We now prove that $(Y, \approx, Q)$ satisfies all equalities in E. Note first that, for $b \in B$ and $(x, T) \in X \times \mathcal{P}(X)$, we have $(x, T) \in \boxast b$ if, and only if, $T \subseteq b$, and $(x, T) \in \Box b$ if, and only if, $R(x) \subseteq b$. Hence, a pair $(x, T) \in X \times \mathcal{P}(X)$ satisfies an equality $(\Box b, \boxast b) \in E$ iff $T \subseteq b \iff R(x) \subseteq b$. Now, since the elements of B are the $\sim$-saturated subsets, it easily follows that a pair (x, T) satisfies *all* equalities in E iff $[T]_\sim = R(x)$ (cf. [9, Lemma 5.11]).
To prove (ii), let $Z \subseteq X \times \mathcal{P}(X)$ be as in the assumptions of (ii). We need to show that all elements in Z are GL-suitable. First of all, by the argument in the previous paragraph, since Z satisfies all equalities in E, any pair $(x, T) \in Z$ satisfies $[T]_\sim = R(x)$. Now, since X is finite, it suffices to prove the following statement for all n:

$$\text{For any } (x, T) \in X \times \mathcal{P}(X) \text{ such that } |T| \leq n, \text{ if } (x, T) \in Z \text{ then } (x, T) \in Y. \tag{H_n}$$

To prove (H_0), note that if $(x, \emptyset) \in Z$ then $R(x) = \emptyset$, so $(x, \emptyset)$ is GL-suitable. Now assume (H_n) holds. Let $(x, T) \in Z$ with $|T| = n + 1$. To prove that (x, T) is GL-suitable, we have already noted that $[T]_\sim = R(x)$. Let $y \in T$ be arbitrary. Since Z is a partial generated subframe of $X \times \mathcal{P}(X)$, there exists S such that $(y, S) \in Z$. Since Z is a partial GL-frame and $(x, T)Q(y, S)$, there exists $(y, S') \in Z$ such that $S' \subsetneq T$. By the induction hypothesis (H_n) applied to (y, S'), we see that (y, S') is GL-suitable. Thus, (x, T) is GL-suitable, since y was arbitrary. This concludes the proof of (H_{n+1}). $\Box$

Note that, by Proposition 4.5, the homomorphism $i : (A, B, \Box) \to F(A, B, \Box)$ is dual to the function $p : G(X, \sim, R) \to (X, \sim, R)$ which sends (x, T) to x.

Proposition 4.6 *Let $(X, \sim, R)$ be a finite partial GL-frame. Then the bounded morphism $p : G(X, \sim, R) \to (X, \sim, R)$ is surjective.*

Proof. We will prove that, for any $x \in X$, the pair (x, T_x) is in $G(X, \sim, R)$, where

$$T_x := \{y \mid xRy \text{ and } R(y) \subsetneq R(x)\}.$$

To this end, we use induction on the number of elements in $R(x)$. Specifically, we will show that, for each n,

$$\text{For any } x \in X \text{ such that } |R(x)| \leq n, (x, T_x) \text{ is } \mathsf{GL}\text{-suitable.} \tag{I_n}$$

To prove (I_0), note that if $R(x) = \emptyset$ then $T_x = \emptyset$ and $(x, \emptyset)$ is GL-suitable. Now assume (I_n) holds. Let $x \in X$ such that $|R(x)| = n + 1$. Towards proving that (x, T_x) is GL-suitable, we first show that $R(x) = [T_x]_\sim$. If $y \in R(x)$, then since $(X, \sim, R)$ is a partial GL-frame, there exists $y' \sim y$ such that $R(y') \subsetneq R(x)$. We then also have xRy', since xRy, and therefore $y' \in T_x$, so $y \in [T_x]_\sim$. Conversely, since $R(x)$ is $\sim$-saturated and contains T_x, it also contains $[T_x]_\sim$. Now let $y \in T_x$. We need to show that there exists $S \subsetneq T_x$ such that (y, S) is GL-suitable; we will show that T_y is an instance of such an S. Since $y \in T_x$, we have $R(y) \subsetneq R(x)$, so $|R(y)| \leq n$, and by the induction hypothesis (I_n) applied to y, we see that (y, T_y) is GL-suitable. We now prove that $T_y \subsetneq T_x$. If $z \in T_y$, then yRz, and since $y \in T_x$ we have $R(y) \subsetneq R(x)$, so xRz. Also, $R(z) \subsetneq R(y) \subsetneq R(x)$, proving that $z \in T_x$. Finally, we have that $y \in T_x$ but clearly $y \notin T_y$, so T_y is strictly contained in T_x. This concludes the proof of (I_{n+1}). We now easily deduce that p is surjective: for any $x \in X$, we have $p((x, T_x)) = x$, and (x, T_x) is GL-suitable by $I_{|R(x)|}$. $\square$

The following is now an easy but important consequence of the results in this section.

Theorem 4.7 *Let $(A, B, \Box)$ be a partial GL-algebra. Then the homomorphism $i : (A, B, \Box) \to F(A, B, \Box)$ is injective.*

Proof. Since $p : G(X, \sim, R) \to (X, \sim, R)$ is surjective by Proposition 4.6, the result follows by Corollary 3.3. $\square$

In the terminology of [3], Theorem 4.7 shows that $F(A, B, \Box)$ is an *injective one-step extension* of $(A, B, \Box)$, and (hence) that the class of partial GL-algebras has the *extension property* [3, Def. 9]. A proof of a closely related fact is given by different methods in [3, Sec. 9.2, Thm. 4]. The latter theorem, however, shows that the dual map p is surjective without identifying exactly which points are in the frame $G(X, \sim, R)$, but just showing that there are enough points. As such, the arguments given in the proofs of Propositions 4.5 and 4.6 are the main technical contributions of this paper. In the following sections, we will apply these results to construct free algebras and graded models for GL.

5 Application: free algebras for GL

Throughout the rest of the paper, fix a finite set $P = \{p_1, \ldots, p_k\}$ of propositional variables.

Definition 5.1 Let $\mathbb{F}_{\mathsf{K}}(P)$ denote the free modal algebra over P and let $(A_{n+1}, A_n, \Box_n)$ be the increasing chain of sub-partial modal algebras of $\mathbb{F}_{\mathsf{K}}(P)$ where A_n is the Boolean algebra of K-equivalence classes of modal formulas of degree $\leq n$, as in Example 2.3.

Let $(B_1, B_0, \Box_0) \overset{i_0}{\hookrightarrow} (B_2, B_1, \Box_1) \overset{i_1}{\hookrightarrow} \cdots$ be a countable chain of embeddings of partial modal algebras with $i_n(B_{n+1}) \subseteq B_{n+1}$. For any $v : P \to B_0$, there exists, for each $n \geq 0$, a natural *interpretation function* v_n from A_n to B_n, defined as follows:

- v_0 is the unique Boolean algebra homomorphism extending v;

- v_{n+1} is the unique Boolean algebra homomorphism such that, for all ϕ of modal degree $\leq n$, $v_{n+1}(\Box\phi) = \Box_n v_n(\phi)$ and $v_{n+1}(\phi) = i_n(v_n(\phi))$.

Note that each interpretation function v_n is a well-defined homomorphism of partial modal algebras $(B_{n+1}, B_n, \Box_n) \to (A_{n+1}, A_n, \Box_n)$.

We now apply the results in the previous section to give an incremental construction of the free k-generated GL-algebra, for any $k \geq 0$. Let B_0 be the free Boolean algebra over P. Let $B_1 := B_0 + V(B_0)$. Then B_0 is a subalgebra of B_1, and we define $\Box_0 : B_0 \to B_1$ to be the map which sends $a \in B_0$ to the formal element $\Box a \in V(B_0)$. We thus obtain a finite partial modal algebra $(B_1, B_0, \Box_0)$. The finite partial frame $(X_1, \sim_1, R_1)$ that is dual to $(B_1, B_0, \Box_0)$ for the 1-generated case is depicted in the figure below. Note that, for any k, the partial frame $(X_1, \sim_1, R_1)$ is a partial GL-frame, simply because each of the 2^k equivalence classes contains a blind point. Therefore, $(B_1, B_0, \Box_0)$ is a partial GL-algebra by Proposition 3.9.

Now, for $n \geq 1$, we inductively define $(B_{n+1}, B_n, \Box_n) := F(B_n, B_{n-1}, \Box_{n-1})$, and we let i_n be the natural map $B_n \to B_{n+1}$. Then, for each n, $(B_{n+1}, B_n, \Box_n)$ is a partial GL-algebra, and, hence, by Theorem 4.7, each i_n is injective.

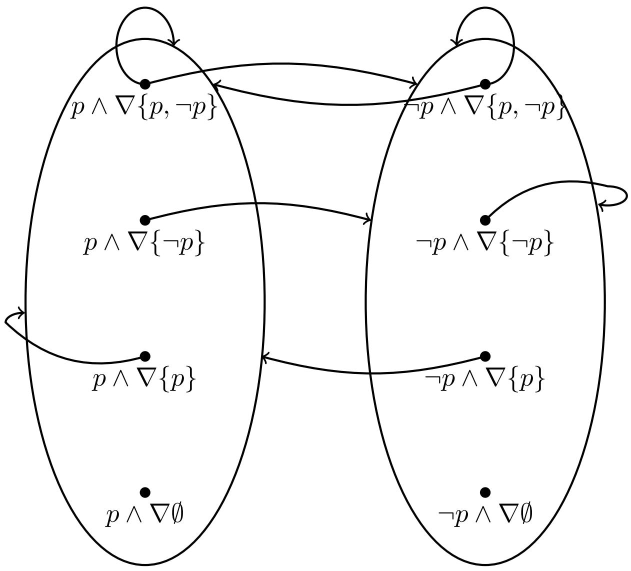

Fig. 1. The partial frame $(X_1, \sim_1, R_1)$ in the case $P = \{p\}$. In this diagram, the points are labelled by the atoms of the Boolean algebra A_1 that they represent. The classes of the equivalence relation $\sim_1$ are depicted by the two ellipses. The relation R_1 is depicted by arrows from points to classes; for instance, the arrow from the point $x := p \wedge \nabla\{p\}$ to the class $[x]_{\sim_1}$ indicates that xR_1y for any point y in $[x]_{\sim_1}$.

Theorem 5.2 *For each $n \geq 0$, the partial GL-algebra $(B_{n+1}, B_n, \Box_n)$ is isomorphic to the partial modal subalgebra $(A'_{n+1}, A'_n, \Box_n)$ of the free GL-algebra which consists of the GL-equivalence classes of modal formulas of degree $\leq n$.*

Theorem 5.2 is a straightforward consequence of the results in Section 3 of [9], in particular cf. Thm. 3.15. The isomorphism $A'_n \to B_n$ is given by factoring the natural interpretation function $A_n \to B_n$ (Definition 5.1) through the quotient $A_n \twoheadrightarrow A'_n$ of A_n. The reader is referred to [9, Section 2–3] for a more detailed proof.

Corollary 5.3 *The free k-generated* GL*-algebra is isomorphic to $(B_\omega, \Box)$, where B_ω is the Boolean algebra colimit of the chain $(i_n : B_n \hookrightarrow B_{n+1})_{n\geq 0}$ of embeddings of Boolean algebras, and $\Box$ is the unique operation $B_\omega \to B_\omega$ such that, for all $n \geq 0$, $\Box \circ j_n = j_{n+1} \circ \Box_n$ (here, $j_n : B_n \hookrightarrow B_\omega$ denotes the colimit embedding).*

6 Application: Graded models for GL

In this section, we give a semantic interpretation of the above results on finitely generated free GL-algebras.

We first translate the algebraic definition of the interpretation functions (Definition 5.1) into a dual definition of a notion of *graded model* and a satisfaction relation for it.

Definition 6.1 A *graded frame* X is a chain $\{(X_n, \sim_n, R_n)\}_{n\geq 1}$ of partial frames such that $X_{n+1}/{\sim_{n+1}} \cong X_n$ for all $n \geq 1$. A *graded model* is a pair (X, f) where X is a graded model and $f : X_1 \to \mathcal{P}(P)$ is a function such that, for each $p \in P$, $f^{-1}(p)$ is $\sim_1$-saturated.

We inductively define, for any formula ϕ of modal degree $\leq n$, and any $x \in X_n$, the relation "$X, x \models \phi$" as follows:

- For the case $n = 1$: let $x \in X_1$.
  - · For $\phi = p_i$, we define $X, x \models p_i$ if, and only if, $p_i \in f(x)$, and we extend this definition to all $\Box$-free formulas in the usual way;
  - · For $\phi = \Box\psi$, where ψ is $\Box$-free, we define $X, x \models \Box\phi$ if, and only if, for all $y \in X_1$ such that xR_1y, we have $X, y \models \phi$.
- For the induction step, assume that the relation "$X, x \models \phi$" has been defined for all formulas ϕ of modal degree $\leq n$ and all $x \in X_n$. Let $x \in X_{n+1}$.
  - · For ϕ of modal degree $\leq n$, we define $X, x \models \phi$ if, and only if, $X, [x]_{\sim_{n+1}} \models \phi$;
  - · For ϕ of modal degree $\leq n$, we define $X, x \models \Box\phi$ if, and only if, for all $y \in X_{n+1}$ such that $xR_{n+1}y$, we have $X, [y]_{\sim_{n+1}} \models \phi$;
  - · Now, for an arbitrary ϕ of modal degree $\leq n+1$, we write ϕ as a Boolean combination of formulas $\phi_1 \ldots, \phi_r$, and $\Box\phi_{r+1}, \ldots, \Box\phi_{r+s}$, where each ϕ_i is of modal degree $\leq n$. We now define $X, x \models \phi$ in the usual way.

Proposition 6.2 *Let (X, f) be a graded model. For $n \geq 1$, let $(A_n, A_{n-1}, \Box_n)$ be the partial modal algebra dual to the partial frame $(X_n, \sim_n, R_n)$. For any ϕ of modal degree $\leq n$ and $x \in X_n$, we have*

$$x \in v_n(\phi) \iff X, x \models \phi.$$

Proof. Clear from the definitions of the functions v_n, the satisfaction relation, and the dual partial modal algebra of a partial frame. □

Definition 6.3 Fix $P = \{p_1, \ldots, p_k\}$. Let $X_0 := \mathcal{P}(P)$, $X_1 := X_0 \times \mathcal{P}(X_0)$. For $n \geq 1$, $(X_{n+1}, \sim_{n+1}, R_{n+1}) := G(X_n, \sim_n, R_n)$, where G is defined as in Definition 4.4. Let (X, f) be the graded model given by these partial frames and the valuation $f : X_1 \to \mathcal{P}(P)$ given by projection onto the first coordinate. We call (X, f) the *canonical graded* GL*-model* (on k generators).
For each $n \geq 0$, we now associate to each $x \in X_n$ a formula $\phi(x)$ of degree $\leq n$. For $x \in X_0$, let

$$\phi(x) := \bigwedge_{p_i \in x} p_i \wedge \bigwedge_{p_i \notin x} \neg p_i.$$

For $n \geq 0$, assume that the formulas $\phi(x)$ have been defined for all $x \in X_n$, and let $y \in X_{n+1}$. Since $y \in X_n \times \mathcal{P}(X_n)$, we may write $y = (x, T)$, and we define

$$\phi(y) := \phi(x) \wedge \nabla\{\phi(t) \mid t \in T\}.$$

The following corollary is a consequence of combining Proposition 4.5, Theorem 5.2 and Proposition 6.2.

Corollary 6.4 *Any modal formula ψ of degree $\leq n$ is* GL*-equivalent to the finite (possibly empty) disjunction of the set*

$$N_\psi := \{\phi(x) \mid x \in X_n \text{ such that } X, x \models \psi\}.$$

7 Conclusion

In this paper we applied the general theory of incremental constructions of free modal algebras using partial modal algebras ([12], [9], [5]) to the particular case of GL. We obtained a bottom-up construction of the free GL-algebra and normal forms for GL. Normal forms play a role in the proofs of interpolation and definability theorems for GL, cf. e.g. [13] and [18]. It is now perhaps also a natural and feasible question whether constructions like the one given in this paper could be applied to provability and interpretability logics [20] other than GL. We leave it as an open question for further research whether the results in this paper could yield new insights about such proofs. We have also not been able to discuss more proof-theoretical aspects of the incremental approach; more about this can be found [4] of Bezhanishvili and Ghilardi.
The reader will have noted that the normal forms that we defined for GL in Definition 6.3 are of a slightly different form than Fine's normal forms as in (1) above. In particular, while Fine's normal forms are the conjunction of a ∇-formula with a modal formula of degree 0, here we only obtain a conjunction of a ∇-formula with a modal formula of lesser degree. We expect that an equivalence with Fine-type normal forms for GL can be obtained by either a syntactic or semantic argument, but we leave the precise formulation of such a result to further research.
A phenomenon that seems to emerge from the approach taken in this paper is that either a *two-sorted* or a *bi-modal* variant of a logic can be better behaved

and easier to study than the one-sorted or uni-modal logic that one intended to study in the first place. We regard it as an important question to obtain a more structural understanding from this phenomenon, as it could also be interesting in the study of other modal logics than GL.

On a related topic, it remains rather mysterious why, of the possible quasi-equational flattenings of the Gödel-Löb axiom, one flattening (namely (3)) is "better" than another, namely

$$\text{if } b \leq \Box a \text{ then } \Box(b \to a) \leq \Box a, \tag{5}$$

which does not admit a construction as the one in this paper, cf. [4, Sec. 9.2, Ex. 3]. In order to understand this phenomenon, it would probably help to take the idea seriously that multi-sortedness/polymodality plays an important role here. To make the question a bit more concrete: is there is an apparent structural, maybe Sahlqvist-like, explanation for the fact that the quasi-equation (3) is better behaved than the quasi-equation (5)?

The results in this paper are close in spirit to [14], where a construction of a "canonical exact model" for GL is given, in an analogous way to the universal model for intuitionistic propositional logic. However, note that the approach used in this paper uses a different "slicing into finite parts" than the approach of exact/universal models. Whereas the latter organizes normal/canonical formulas according to the height of the corresponding point in the canonical frame, our method organizes formulas according to their modal degree. It is another interesting direction for further research to investigate how these two different methods are precisely related, both in the context of GL and in the context of intuitionistic propositional logic. We refer to [11, Section 8] for a more detailed study of the relationship between the universal model and the approximating chain in the context of intuitionistic propositional logic.

Moss [17] gives a one-sorted account of canonical formulas and compares his approach to filtrations and Fine's original approach. It would be equally interesting to compare the recent results of the "incremental construction" kind to the more common modal logic method of filtrations.

The nabla connective, ∇, is important in coalgebraic modal logic [16] and we think it is an important question how the methods discussed in this paper, or more generally in the algebraic theory of normal forms, relate to coalgebraic modal logic. As a first step in this direction, a coalgebraic account of Ghilardi's method for the incremental construction of free algebras, but only in the case of equations of rank exactly 1, was given in [6]. It would be an interesting research project to give a similar coalgebraic account of the incremental construction of free algebras for varieties axiomatized by quasi-equations of rank at most 1. A first pointer in this direction may be [9, Section 2], where one can find a category-theoretic formulation of the results about incremental constructions that were used in this paper.

Appendix

A Proofs of basic facts in Section 2

Theorem A.1 (Duality for finite partial modal algebras) *The category of finite partial frames is dually equivalent to the category of finite partial modal algebras.*

Proof. We have already defined a contravariant functor from partial frames to partial algebras, and this functor clearly sends *finite* partial frames to *finite* partial modal algebras. Conversely, given a finite partial modal algebra $(A, B, \Box)$, its dual finite partial frame $(X, \sim, R)$ is defined by:

$$X := \mathrm{At}(A) = \{x \in A \mid x \text{ is an atom of } A\},$$
$$\text{for } x, x' \in X,\ x \sim x' \overset{\text{def}}{\iff} \text{for all } b \in B, x \leq b \text{ if and only if } x' \leq b,$$
$$xRx' \overset{\text{def}}{\iff} \text{for all } b \in B, x \leq \Box b \text{ implies } x' \leq b.$$

Given a homomorphism of partial modal algebras $h : (A, B, \Box) \to (A', B', \Box')$, its dual bounded morphism $f : (Y, \sim_Y, S) \to (X, \sim_X, R)$ is defined by $f(y) := \bigwedge\{a \in A \mid y \leq h(a)\}$. It is standard to show that these assignments define a functor from finite partial modal algebras to finite partial frames which is an inverse, up to natural isomorphisms, of the functor from finite partial frames to finite partial modal algebras. (Cf., e.g., [9, Thm. 4.3].) □

Fact A.2 *The dual of $A + B$ is the Cartesian product $\mathrm{At}(A) \times \mathrm{At}(B)$ and the Boolean algebra $A + B$ is isomorphic to $\mathcal{P}(\mathrm{At}(A) \times \mathrm{At}(B))$.*

Proof. Immediate from finite duality. The isomorphism in the second part of the statement sends $u \in \mathcal{P}(\mathrm{At}(A) \times \mathrm{At}(B))$ to the element $\bigvee_{(x,y)\in u}(x \wedge y)$ in $A + B$. □

Fact A.3 *The dual of $V(A)$ is $\mathcal{P}(\mathrm{At}(A))$ and the Boolean algebra $V(A)$ is isomorphic to $\mathcal{P}(\mathcal{P}(\mathrm{At}(A)))$.*

Proof. Note that atoms of $V(A)$ correspond to homomorphisms $V(A) \to \mathbf{2}$, which correspond to meet-preserving functions $A \to \mathbf{2}$, which correspond to subsets of $\mathrm{At}(A)$ by finite duality. The isomorphism in the second part of the statement sends an element $u \in \mathcal{P}(\mathcal{P}(\mathrm{At}(A)))$ to the element $\bigvee_{T\in u} \nabla T$ of $V(A)$, where we use the 'nabla'-notation $\nabla T := \Box\left(\bigvee_{t\in T} t\right) \wedge \left(\bigwedge_{t\in T} \Diamond t\right)$. □

Fact A.4 *The dual of $A+V(A)$ is $X \times \mathcal{P}(X)$, where $X := \mathrm{At}(A)$ and $A+V(A)$ is isomorphic to $\mathcal{P}(X \times \mathcal{P}(X))$. The subalgebra A of $A + V(A)$ is dual to the equivalence relation $\approx$ on $X \times \mathcal{P}(X)$ defined by $(x, T) \approx (y, S)$ if, and only if, $x = y$. The dual of the operation $\Box : A \to A + V(A)$ is the relation Q on $X \times \mathcal{P}(X)$ defined by $(x, T)Q(y, S)$ if, and only if, $y \in T$.*

Proof. Combining the previous two facts, the isomorphism between $A+V(A)$ and $\mathcal{P}(X \times \mathcal{P}(X))$ is given by sending $u \in \mathcal{P}(X \times \mathcal{P}(X))$ to the element $\bigvee_{(x,T)\in u}(x \wedge \nabla T)$. For the other statements, cf., e.g., [9, Lemma 5.10]). □

References

[1] Abramsky, S., *A Cook's Tour of the Finitary Non-Well-Founded Sets*, in: S. Artemov, H. Barringer, A. d'Avila Garcez, L. C. Lamb and J. Woods, editors, *We Will Show Them*, Essays in honour of Dov Gabbay **1** (2005), pp. 1–18, based on unpublished lecture notes (1988).

[2] Avron, A., *On modal systems having arithmetical interpretations*, J. Symbolic Logic **49** (1984), pp. 935–942.

[3] Bezhanishvili, N. and S. Ghilardi, *Bounded proofs and step frames*, in: D. Galmiche and D. Larchey-Wendling, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Computer Science **8123** (2013), pp. 44–58.

[4] Bezhanishvili, N. and S. Ghilardi, *Bounded proofs and step frames* (2013), preprint available at `http://users.mat.unimi.it/users/ghilardi/allegati/TR_NickSilvio.pdf`.

[5] Bezhanishvili, N., S. Ghilardi and M. Jibladze, *Free modal algebras revisited: the step-by-step method*, in: *Leo Esakia on duality in modal and intuitionistic logics*, Trends in Logic: Outstanding Contributions, Springer, 2014, to appear. Preprint available at `http://dspace.library.uu.nl/handle/1874/257814`.

[6] Bezhanishvili, N. and A. Kurz, *Free Modal Algebras: A Coalgebraic Perspective*, in: T. Mossakowski, U. Montanari and M. Haveraaen, editors, *Algebra and Coalgebra in Computer Science*, Lecture Notes in Computer Science **4624**, Springer, 2007 .

[7] Boolos, G., *On deciding the truth of certain statements involving the notion of consistency*, J. Symbolic Logic **41** (1976), pp. 779–781.

[8] Boolos, G., *On the nonexistence of certain normal forms in the logic of provability*, J. Symbolic Logic **47** (1982), pp. 638–640.

[9] Coumans, D. C. S. and S. J. van Gool, *On generalizing free algebras for a functor*, J. Logic Computation **23** (2013), pp. 645–672.

[10] Fine, K., *Normal forms in modal logic*, Notre Dame J. of Formal Logic **XVI** (1975).

[11] Gehrke, M., *Canonical extensions, Esakia spaces, and universal models*, in: *Leo Esakia on duality in modal and intuitionistic logics*, Trends in Logic: Outstanding Contributions, Springer, 2014, to appear. Preprint available at `http://www.liafa.univ-paris-diderot.fr/~mgehrke/Ge12.pdf`.

[12] Ghilardi, S., *An algebraic theory of normal forms*, Ann. Pure Appl. Logic. **71** (1995), pp. 189–245.

[13] Gleit, Z. and W. Goldfarb, *Characters and Fixed Points in Provability Logic*, Notre Dame J. of Formal Logic **31** (1990), pp. 26–36.

[14] Hendriks, L. and D. De Jongh, *Finitely generated magari algebras and arithmetic*, in: A. Ursini and P. Aglianò, editors, *Logic and Algebra*, Lecture Notes in Pure and Applied Mathematics **180**, Marcel Dekker, Inc., 1996 .

[15] Magari, R., *The diagonalizable algebras (the algebraization of the theories which express theor. ii)*, Boll. Un. Mat. Ital. **12** (1975), pp. 117–125.

[16] Moss, L. S., *Coalgebraic Logic*, Ann. Pure Appl. Logic. **96** (1999), pp. 277–317.

[17] Moss, L. S., *Finite models constructed from canonical formulas*, J. Philosophical Logic **36** (2007), pp. 605–640.

[18] Shavrukov, V. Y., *Subalgebras of diagonalizable algebras of theories containing arithmetic*, Dissertationes Mathematicae **323** (1993).

[19] Solovay, R. M., *Provability interpretations of modal logic*, Israel J. of Math. **25** (1976), pp. 287–304.

[20] Visser, A., *An overview of interpretability logic*, in: M. Kracht, M. de Rijke, H. Wansing and M. Zakharyaschev, editors, *Advances in modal logic, Volume 1*, CSLI Lecture Notes **87** (1998), pp. 307–359.