On Polarity Frames: Applications to Substructural and Lattice-based Logics

Tomoyuki Suzuki [1]

Institute of Computer Science, Academy of Sciences of the Czech Republic
Pod Vodarenskou vezi 271/2,
182 07, Prague 8, Czech Republic

Abstract

In this paper, on one hand, we address topology on polarities via general polarity frames by analogy of the relationship between topology on sets and general Kripke frames. Based on the topology on polarities, we provide the topological characterisation of descriptive polarity frames. On the other hand, we introduce disjoint unions and amalgamations of polarity frames with additional relations and constants. As applications of these constructions, we establish the Goldblatt-Thomason's theorem for (distributive) substructural logic and the amalgamation property for some lattice-based algebras.

Keywords: Relational semantics, substructural and lattice-based logics, lattice expansions, topological characterisation, Goldblatt-Thomason's theorem and amalgamation property.

1 Introduction

The study of *polarities*, i.e. triples of two sets and a binary relation, is already found in the first edition of [2]. We can find the same structures in the context of formal concept analysis e.g. [7,18]. Interestingly, in non-classical logics, polarities have attracted attentions from the generalisation of the canonicity problem on the setting of modal logics to lattice-based logics such as substructural logic and distributive modal logics. Whilst the method in [21] was extended mainly by [12,11,13], it was not clear whether we had to restrict the Sahlqvist argument for non-Boolean based logics. However, by means of the Ghilardi-Meloni canonicity methodology [14], we found that the same argument works for lattice-based logics [40] and even for poset expansions with minor restriction [42].

[1] *Email:tomoyuki.suzuki@cs.cas.cz*
The author would like to thank Marta Bílková, Rostislav Horčík, Adam Přenosil, Jiří Velebil, and anonymous referees for their valuable comments. The author is supported by grant GAP202/10/1826 of the Czech Science Foundation and RV067985807.

Bi-approximation semantics for substructural logic was introduced in [39] to explicate the Ghilardi-Meloni canonicity methodology [14] from the relational semantic viewpoints. On one hand, as the semantics enjoys the Sahlqvist-type canonicity and elementary results [44], many lattice-based logics are sound and complete with respect to elementary classes. On the other hand, it evaluates not just formulae but also sequents (logical consequences) based on a polarity. Note that the idea of semantics on polarities are also found in e.g. [10,9], although their main targets are duality and algebraic proof theory.

In contrast, the motivation of our research for bi-approximation semantics is to introduce logical properties or well established proof methods on Kripke semantics to substructural and lattice-based logics. The main question of this research line may boil down to the following:

By replacing sets by polarities, can we obtain universal relational semantics?

More precisely, we are interested in whether the relationships between modal logics and Kripke semantics can be universally extended to the ones between lattice-based logics and bi-approximation semantics. Towards the goal, the current author has introduced morphisms [43] for bi-approximation semantics and addressed the interpretations between Kripke semantics for distributive lattice-based logics and distributive polarity frames [38,41]. In the current paper, we will address topology on polarities, definable classes of polarity frames and amalgamations of polarities.

Topology. The Stone representation and topological duality of Boolean algebras and sets was already established in [36,37]. Later, by [32,33], the relationship was generalised to distributive lattices and ordered Stone spaces, see also [19,8]. A further generalisation can be found, e.g. in [47].

However, unlike what happens in the setting over distributive lattices, the notion of topology does not seem clear for lattices, because topology forms algebraically a distributive lattice with respect to $\cap$ and $\cup$, *a priori*. Hence, it is natural to ask what is the appropriate notion of topology over lattice-based algebras. For the question, we give a possible answer by pulling back the connection between general Kripke frames and topological spaces in [34] via general polarity frames.

Goldblatt-Thomason's theorem. The Goldblatt-Thomason's theorem was originally shown in [46,17,15] to account for classes of first-order models (with a single binary relation) which are definable by modal formulae, see also [3,16]. The theorem was also applied to other languages: hybrid language [45], graded modal language [35] or coalgebraic logic [23,1].

Since our motivation is to justify that polarities are sufficiently qualified as a replacement of sets, in the present paper, we introduce the notion of disjoint unions of polarities and, via the Birkhoff's variety theorem [4], we establish the Goldblatt-Thomason's theorem for substructural and lattice-based languages on polarity models.

Amalgamation property. In non-classical logics, the amalgamation property is known as a foundational property to explain logical properties in an algebraic term e.g. [31,28,24]. In substructural logic, the correspondence between logical properties and algebraic properties is presented in [22]. Along the same research direction, we can also find in the correspondence over lattice-based algebras and logics in [26].

Whilst these correspondences between algebras and logics have been provided, it must be interesting if a systematic proof method can universally account for logics and algebras which have the amalgamation property. In fact, [25] found a sufficient condition for this question. However, considering non-distributive lattice-based algebras, the condition seems too strong. In this paper, we establish a basic scheme to explicate the amalgamation property for lattice-based algebras by introducing amalgamations of polarity frames via the dual representation. Note that our results include existing results such as the amalgamation property of lattices [20], distributive lattices [30], see also [18]. However, our main concern for this topic is a "schematic" account. [2]

Outline. In Section 2, we briefly recall necessary definitions, properties and theorems on bi-approximation semantics in the current author's previous publications. In Section 3, we provide topology on polarities and general polarity frames. Based on the setting, we establish the topological characterisation of descriptive polarity frames. In Section 4, we introduce disjoint unions of polarity frames and show their fundamental properties. Accordingly, by means of the Birkhoff's variety theorem, we achieve the Goldblatt-Thomason's theorem on polarity models. In Section 5, we explain our methodology to prove the amalgamation property via the dual representation and amalgamations of polarity frames. In Section 6, as concluding remarks, we list forthcoming work.

2 Preliminaries for polarity frames

Polarity frames. Details are found for example in [2,7]. A *polarity frame* is a triple $\mathbb{F} = \langle X, Y, B \rangle$ with non-empty sets X and Y, and a binary relation $B \subseteq X \times Y$. Note that, X and Y are not necessarily disjoint. However, if the equality is in the frame language, we must guarantee that, if $x = y$ then xBy for $x \in X$ and $y \in Y$. Given a polarity $\mathbb{P}$, a natural pre-order $\leq_B$ on $X \cup Y$ is introduced as follows: for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$,

(i) $x_1 \leq_B x_2 \iff \forall y' \in Y. [x_2 B y' \Longrightarrow x_1 B y']$,

(ii) $y_1 \leq_B y_2 \iff \forall x' \in X. [x' B y_1 \Longrightarrow x' B y_2]$,

(iii) $x \leq_B y \iff xBy$,

(iv) $y \leq_B x \iff \forall x' \in X, y' \in Y. [xBy' \text{ and } x'By \Longrightarrow x'By']$.

It is straightforward to check that the equality, if exists, is subsumed by $\leq_B$-equivalence. We may omit the subscript $_B$ when it is clear from the context. Note that, throughout this paper, we treat non-trivial polarity frames (i.e. $B \neq X \times Y$) only.

[2] In the literature, we can also find the amalgamation property for distributive lattice-based algebras such as *tetravalent modal algebras* [5] or varieties of lattice-ordered commutative groups and many-valued algebras [27].

Galois stable lattice. Given a polarity frame $\mathbb{F} = \langle X, Y, B\rangle$, let $\wp(X)$ and $\wp(Y)^\partial$ be the powerset poset of X ordered by the set-theoretic inclusion $\subseteq$ and the powerset poset of Y ordered by the set-theoretic *reverse* inclusion $\supseteq$. We introduce two functions $\lambda\colon \wp(X) \to \wp(Y)^\partial$ and $\upsilon\colon \wp(Y)^\partial \to \wp(X)$ as follows: for all $\mathfrak{X} \in \wp(X)$ and $\mathfrak{Y} \in \wp(Y)^\partial$, $\lambda(\mathfrak{X}) := \{y \in Y \mid \forall x \in \mathfrak{X}.\ xBy\}$ and $\upsilon(\mathfrak{Y}) := \{x \in X \mid \forall y \in \mathfrak{Y}.\ xBy\}$. It is known that λ and υ form a Galois connection, hence the images $\lambda[\wp(X)]$ and $\upsilon[\wp(Y)^\partial]$ are isomorphic. A pair of $\mathfrak{X} \in \wp(X)$ and $\mathfrak{Y} \in \wp(Y)^\partial$ is called a *Galois stable pair*, a.k.a *Dedekind-cut*, if they satisfy $\lambda(\mathfrak{X}) = \mathfrak{Y}$ and $\upsilon(\mathfrak{Y}) = \mathfrak{X}$.

Definition 2.1 [Galois stable lattice] *The Galois stable lattice* $\mathsf{G}_\mathbb{F}$ *on a polarity frame* $\mathbb{F}$ is the subposet of all Galois stable pairs of the product poset $\wp(X) \times \wp(Y)^\partial$, where the lattice operations are defined as follows: for all $(\mathfrak{X}_1, \mathfrak{Y}_1)$ and $(\mathfrak{X}_2, \mathfrak{Y}_2)$,

(i) $(\mathfrak{X}_1, \mathfrak{Y}_1) \vee (\mathfrak{X}_2, \mathfrak{Y}_2) := (\upsilon(\mathfrak{Y}_1 \cap \mathfrak{Y}_2), \mathfrak{Y}_1 \cap \mathfrak{Y}_2)$,

(ii) $(\mathfrak{X}_1, \mathfrak{Y}_1) \wedge (\mathfrak{X}_2, \mathfrak{Y}_2) := (\mathfrak{X}_1 \cap \mathfrak{X}_2, \lambda(\mathfrak{X}_1 \cap \mathfrak{X}_2))$.

Note that both $\mathsf{G}_\mathbb{F}$ and $\wp(X) \times \wp(Y)^\partial$ are complete lattices, but their lattice operations do not coincide in general. Also, note that the Galois stable lattice of $\mathbb{F}$ is isomorphic to the dual algebra of $\mathbb{F}$, denoted by $\mathbb{F}^+$ (so, throughout this paper, we think of $\mathsf{G}_\mathbb{F}$ as the definition of $\mathbb{F}^+$).

Distributivity. Details are in [38]. Given a polarity frame $\mathbb{F} = \langle X, Y, B\rangle$, an element $x \in X$ (resp. $y \in Y$) is a *splitter*, if there exists y_x (called a splitting counterpart of x) such that xBy_x does not hold and, for each $y \in Y$, if xBy does not hold, $y \leq y_x$ holds. (resp. there exists x_y such that x_yBy does not hold and, for each $x \in X$, if xBy does not hold, $x_y \leq x$ holds). It is known that every splitting counterpart is also a splitter, and splitting counterparts are unique up to $\leq$-equivalence. We call a pair of a splitter and its splitting counterpart *splitting pair.*

Definition 2.2 [Distributive polarity frame] A polarity frame $\mathbb{F} = \langle X, Y, B\rangle$ is *distributive*, if it satisfies

Splitting: for all $x \in X$ and $y \in Y$, if xBy does not hold, there exists a splitting pair (x_s, y_s) such that $x_s \leq x$ and $y \leq y_s$.

Note that the splitting condition is a first-order sentence.

Theorem 2.3 *For a distributive polarity frame* $\mathbb{F}$*, the dual algebra* $\mathbb{F}^+$ *satisfies the distributive law.*

Unary modality. For diamond $\Diamond$ and box $\Box$ (adjoints pair), polarity frames are extended with a binary relation $S \subseteq X \times Y$. On a polarity frame $\mathbb{F} = \langle X, Y, B\rangle$, the binary relation S is extended to a binary relation $S^\Diamond$ on X and a binary relation $S^\Box$ on Y as follows: $S^\Diamond(x, x') \iff \forall y \in Y. [S(x, y) \Longrightarrow x'By]$ and $S^\Box(y', y) \iff \forall x \in X. [S(x, y) \Longrightarrow xBy']$. We assume that the binary relation S on $\mathbb{F}$ satisfies the following conditions:

S-transitivity: $\forall x, x' \in X, y, y' \in Y. [x' \leq x, y \leq y' \,\&\, S(x, y) \Longrightarrow S(x', y')]$,

$\Diamond$-adjoint: $\forall x \in X, \forall y \in Y. [\forall x' \in X. [S^{\Diamond}(x, x') \Longrightarrow x'By] \Longrightarrow S(x, y)]$,

$\Box$-adjoint: $\forall x \in X, y \in Y. [\forall y' \in Y. [S^{\Box}(y', y) \Longrightarrow xBy'] \Longrightarrow S(x, y)]$.

Note that the adjoint conditions [3] are intuitively to obtain the following adjointness condition: $S^{\Diamond}(x, x')By \iff xBS^{\Diamond}(y', y)$, and S is the generator of the adjointness.

Theorem 2.4 *The dual algebra is a lattice with adjoint unary modality of $\Diamond$ and $\Box$.*

Example 2.5 A quadruple $\langle X, Y, B, S\rangle$ forms a polarity frame for lattice-based modal logic. Lattice-based modal formulae are given by $\phi ::= p \mid \phi \vee \phi \mid \phi \wedge \phi \mid \Diamond\phi \mid \Box\phi$.

De Morgan negation. For the de Morgan negation $\neg$, polarity frames are extended with two binary relations $C \subseteq X \times X$ and $D \subseteq Y \times Y$. On a polarity frame $\mathbb{F} = \langle X, Y, B\rangle$, these binary relations C and D are extended to binary relations $\tilde{C}, \tilde{D} \subseteq X \times Y$ as follows: $\tilde{C}(x, y) \iff \forall x' \in X. [C(x, x') \Longrightarrow x'By]$ and $\tilde{D}(x, y) \iff \forall y' \in Y. [D(y', y) \Longrightarrow xBy']$. We assume that these binary relations satisfy the following conditions:

C-symmetry: $\forall x, x' \in X. [C(x, x') \Longrightarrow C(x', x)]$,

D-symmetry: $\forall y, y' \in Y. [D(y', y) \Longrightarrow D(y, y')]$,

C-transitivity: $\forall x_1, x_1', x_2, x_2' \in X. [x_1' \leq x_1, x_2' \leq x_2 \,\&\, C(x_1, x_2) \Rightarrow C(x_1', x_2')]$,

D-transitivity: $\forall y_1, y_1', y_2, y_2' \in Y. [y_1 \leq y_1', y_2 \leq y_2' \,\&\, D(y_1, y_2) \Rightarrow D(y_1', y_2')]$,

C-interdefinability: $\forall x, x' \in X. \left[\exists y \in Y. \left[\tilde{D}(x, y) \,\&\, x'By\right] \Longrightarrow C(x, x')\right]$,

D-interdefinability: $\forall y, y' \in Y. \left[\exists x \in X. \left[\tilde{C}(x, y) \,\&\, xBy'\right] \Longrightarrow D(y', y)\right]$,

C-duality: $\forall x \in X, \exists y \in Y. \left[\tilde{C}(x, y) \,\&\, \tilde{D}(x, y)\right]$,

D-duality: $\forall y \in Y, \exists x \in X. \left[\tilde{D}(x, y) \,\&\, \tilde{C}(x, y)\right]$.

Note that these conditions are dependent on each other. Also, we mention that the binary relations C and D are interdefinable. However, to keep nice symmetry, we choose the above conditions.

Theorem 2.6 *The dual algebra is a lattice with the de Morgan negation $\neg$.*

Example 2.7 A quintuple $\langle X, Y, B, C, D\rangle$ forms a polarity frame for (unbounded) ortholattice logic. Ortholattice formulae are given by $\phi ::= p \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi$.

Residuality. Details are in [39]. For fusion $\circ$ and the residuals $\rightarrow$ and $\leftarrow$, polarity frames are extended by a ternary relation $R \subseteq X \times X \times Y$. In addition, for constants $\mathbf{t}$ and $\mathbf{f}$, polarity frames are extended with two subsets of X and two subsets of Y, i.e. $O_X (\neq \emptyset)$, O_Y, N_X and N_Y. We

[3] Adjoint conditions were called tightness in the current author's papers.

assume that (O_X, O_Y) and (N_X, N_Y) form Galois stable pairs. On a polarity frame $\mathbb{F} = \langle X, Y, B\rangle$, the ternary relation R is extended to three ternary relations $R^\circ \subseteq X \times X \times X$, $R^\rightarrow \subseteq X \times Y \times Y$ and $R^\leftarrow \subseteq Y \times X \times Y$ as follows: $R^\circ(x_1, x_2, x) \iff \forall y \in Y. [R(x_1, x_2, y) \Longrightarrow xBy]$, $R^\rightarrow(x_1, y_2, y) \iff \forall x_2 \in X. [R(x_1, x_2, y) \Longrightarrow x_2 B y_2]$ and $R^\leftarrow(y_1, x_2, y) \iff \forall x_1 \in X. [R(x_1, x_2, y) \Longrightarrow x_1 B y_1]$. We assume that the ternary relation R satisfies the following conditions:

R-order: $\forall x, x' \in X. [x' \leq x \iff \exists o \in O_X. [R^\circ(x, o, x') \text{ or } R^\circ(o, x, x')]]$,

R-identity: $\forall x \in X. [\exists o \in O_X. [R^\circ(x, o, x)] \mathbin{\&} \exists o' \in O_X. [R^\circ(o', x, x)]]$,

R-transitivity: $\forall x_1, x_1', x_2, x_2' \in X, y, y' \in Y.$

$$[x_1' \leq x_1, x_2' \leq x_2, y \leq y' \mathbin{\&} R(x_1, x_2, y) \Longrightarrow R(x_1', x_2', y')]\,,$$

R-associativity: for all $x_1, x_2, x_3, x \in X$,

$$\begin{aligned}&\exists x' \in X. [R^\circ(x_1, x_2, x') \mathbin{\&} R^\circ(x', x_3, x)]\\ &\qquad\iff \exists x'' \in X. [R^\circ(x_1, x'', x) \mathbin{\&} R^\circ(x_2, x_3, x'')]\,,\end{aligned}$$

$\circ$-adjoint: $\forall x_1, x_2 \in X, y \in Y. [\forall x{\in}X. [R^\circ(x_1, x_2, x) \Rightarrow xBy] \Rightarrow R(x_1, x_2, y)]$,

$\rightarrow$-adjoint: $\forall x_1, x_2{\in}X, y{\in}Y. [\forall y_2{\in}Y. [R^\rightarrow(x_1, y_2, y) \Rightarrow x_2 B y_2] \Rightarrow R(x_1, x_2, y)]$,

$\leftarrow$-adjoint: $\forall x_1, x_2{\in}X, y{\in}Y. [\forall y_1{\in}Y. [R^\leftarrow(y_1, x_2, y) \Rightarrow x_1 B y_1] \Rightarrow R(x_1, x_2, y)]$.

Note that these conditions are not independent. The adjoint conditions intuitively tell

$$R^\circ(x_1, x_2, x)By \iff x_2 B R^\rightarrow(x_1, y_2, y) \iff x_1 B R^\leftarrow(y_1, x_2, y)$$

and R is the generator of this residuality.

Example 2.8 An octuple $\langle X, Y, B, R, O_X, O_Y, N_X, N_Y\rangle$ forms a polarity frame for substructural logic. Substructural formulae are given by $\phi ::= p \mid \mathbf{t} \mid \mathbf{f} \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \circ \phi \mid \phi \rightarrow \phi \mid \phi \leftarrow \phi$.

Dedekind-cut preserving morphism. Details are in [43]. Given two polarity frames $\mathbb{F} = \langle X_1, Y_1, B_1\rangle$ and $\mathbb{G} = \langle X_2, Y_2, B_2\rangle$, a pair of functions $\sigma\colon X_1 \to X_2$ and $\tau\colon Y_1 \to Y_2$ forms a *Dedekind-cut preserving morphism* (*d-morphism* for short), denoted by $\langle\sigma|\tau\rangle$, if it satisfies

(i) $\forall x \in X_1, y \in Y_1. [\sigma(x) B_2 \tau(y) \Longrightarrow x B_1 y]$,

(ii) $\forall x \in X_1, y' \in Y_2. [\forall y \in Y_1. [y' \leq_2 \tau(y) \Longrightarrow x B_1 y] \Longrightarrow \sigma(x) B_2 y']$,

(iii) $\forall x' \in X_2, y \in Y_1. [\forall x \in X_1. [\sigma(x) \leq_2 x' \Longrightarrow x B_1 y] \Longrightarrow x' B_2 \tau(y)]$.

A d-morphism $\langle\sigma|\tau\rangle\colon \mathbb{F} \to \mathbb{G}$ is called

***B*-embedding:** if $\forall x \in X_1, y \in Y_1. [x B_1 y \Longrightarrow \sigma(x) B_2 \tau(y)]$,

***B*-separating:** if, for all $x' \in X_2$ and $y' \in Y_2$,

$$\forall x \in X_1, y \in Y_1. [\sigma(x) \leq_2 x' \text{ and } y' \leq_2 \tau(y) \Longrightarrow x B_1 y] \Longrightarrow x' B_2 y'.$$

B-reflecting: if B-embedding and B-separating.

Notice that B-separating and B-reflecting hold the same invariance of validity of logical formulae (sequents) as surjective and isomorphic, hence we may depict the arrows for B-embedding and B-separating with $\rightarrowtail$ and $\twoheadrightarrow$.

For additional relations S, C, D and R, we require the following conditions:

(iv) $\forall x \in X_1, y \in Y_1. [S_2(\sigma(x), \tau(y)) \Longrightarrow S_1(x, y)]$,

(v) $\forall x \in X_1, y' \in Y_2. [\forall y \in Y_1. [y' \leq_2 \tau(y) \Longrightarrow S_1(x, y)] \Longrightarrow S_2(\sigma(x), y')]$,

(vi) $\forall x' \in X_2, y \in Y_1. [\forall x \in X_1. [\sigma(x) \leq_2 y' \Longrightarrow S_1(x, y)] \Longrightarrow S_2(x', \tau(y))]$,

(vii) $\forall x_1, x_2 \in X_1. [C_2(\sigma(x_1), \sigma(x_2)) \Longrightarrow C_1(x_1, x_2)]$,

(viii) $\forall x_1 \in X_1, x_2' \in X_2. [\forall x_2 {\in} X_1. [\sigma(x_2) \leq_2 x_2' \Rightarrow C(x_1, x_2)] \Rightarrow C_2(\sigma(x_1), x_2')]$,

(ix) $\forall y_1, y_2 \in Y_1. [D_2(\tau(y_1), \tau(y_2)) \Longrightarrow D_1(y_1, y_2)]$,

(x) $\forall y_1' \in Y_2, y_2 \in Y_2. [\forall y_1 \in Y_1. [y_1' \leq_2 \tau(y_1) \Rightarrow D_1(y_1, y_2)] \Rightarrow D_2(y_1', \tau(y_2))]$,

(xi) $\forall x_1, x_2 \in X_1, y \in Y_1. [R_2(\sigma(x_1), \sigma(x_2), \tau(y)) \Longrightarrow R_1(x_1, x_2, y)]$,

(xii) $\forall x_1', x_2' \in X_2, y \in Y_1$,

$$\forall x_1, x_2 {\in} X_1. [\sigma(x_1) {\leq_2} x_1' \,\&\, \sigma(x_2) {\leq_2} x_2' \Rightarrow R_1(x_1, x_2, y)] \Rightarrow R_2(x_1', x_2', \tau(y)),$$

(xiii) $\forall x_1' \in X_2, x_2 \in X_1, y' \in Y_2$,

$$\forall x_1 {\in} X_1, y {\in} Y_1. [\sigma(x_1) {\leq_2} x_1' \,\&\, y' {\leq_2} \tau(y) {\Rightarrow} R_1(x_1, x_2, y)] \Rightarrow R_2(x_1', \sigma(x_2), y'),$$

(xiv) $\forall x_1 \in X_1, x_2' \in X_2, y' \in Y_2$,

$$\forall x_2 {\in} X_1, y {\in} Y_1. [\sigma(x_2) {\leq_2} x_2' \,\&\, y' {\leq_2} \tau(y) {\Rightarrow} R_1(x_1, x_2, y)] \Rightarrow R_2(\sigma(x_1), x_2', y').$$

Note that the conditions for C and D can be reduced by means of interdefinability. If we include constants, we also need to preserve their Dedekind-cuts.

Theorem 2.9 *Every d-morphism preserves Dedekind-cuts.*

Dual representation. Details are in [39]. Given a lattice $\mathbb{L} = \langle L, \vee, \wedge \rangle$, we let $\mathcal{F}$ and $\mathcal{I}$ be the set of filters and the set of ideals. Also, between $\mathcal{F}$ and $\mathcal{I}$, we define a binary relation $\sqsubseteq$ as follows: $F \sqsubseteq I \iff F \cap I \neq \emptyset$. Then, $\mathbb{L}_+ = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq \rangle$ forms a polarity frame. We call $\mathbb{L}_+$ the dual frame.

For additional logical operations $\Diamond$, $\Box$, $\neg$, $\circ$, $\rightarrow$ and $\leftarrow$, we define their relations S, C, D and R as follows: for all $F, G \in \mathcal{F}$ and $I, J \in \mathcal{I}$, we let

(i) $S(F, I) \iff \Diamond F \sqsubseteq I \iff F \sqsubseteq \Box I$,

(ii) $C(F, G) \iff F \sqsubseteq \neg G$,

(iii) $D(I, J) \iff \neg I \sqsubseteq J$,

(iv) $R(F, G, I) \iff F \circ G \sqsubseteq I \iff G \sqsubseteq F \rightarrow I \iff F \sqsubseteq I \leftarrow G$,

where $\Diamond F := \{a \in L \mid \exists f \in F.\ \Diamond f \leq a\}$, $\Box I := \{a \in L \mid \exists i \in I.\ a \leq \Box i\}$, $\neg F := \{a \in L \mid \exists f \in F.\ a \leq \neg f\}$, $\neg I := \{a \in L \mid \exists i \in I.\ \neg i \leq a\}$, $F \circ G := \{a \in$

$L \mid \exists f \in F, g \in G.\ f \circ g \leq a\}$, $F \to I := \{a \in L \mid \exists f \in F, i \in I.\ a \leq f \to i\}$ and $I \leftarrow G := \{a \in L \mid \exists g \in G, i \in I.\ a \leq i \leftarrow g\}$.

Remark 2.10 Note that, even if we do not have full adjoint pairs in our language, on the dual frame, we can introduce them: see [40, Lemma 5.8].

Theorem 2.11 *The dual frame satisfies the appropriate relational conditions for each logical connectives.*

For a strict homomorphism $h\colon \mathbb{L} \to \mathbb{M}$, we define two functions $h_+\colon \mathcal{F}_2 \to \mathcal{F}_1$ and $h_-\colon \mathcal{I}_2 \to \mathcal{I}_1$ with $h_+(F) := h^{-1}[F]$ and $h_-(I) := h^{-1}[I]$.

Theorem 2.12 *For a strict homomorphism $h\colon \mathbb{L} \to \mathbb{M}$, the functions h_+ and h_- form a d-morphism, i.e. $\langle h_+ | h_- \rangle\colon \mathbb{M}_+ \to \mathbb{L}_+$. Furthermore, we have*

(i) *if h is injective, $\langle h_+ | h_- \rangle$ is B-separating,*

(ii) *if h is surjective, $\langle h_+ | h_- \rangle$ is B-embedding.*

For a polarity frame $\mathbb{F}$, additional structures S, C, D and R (and constants) yield appropriate lattice operations $\Diamond$, $\Box$, $\neg$, $\circ$, $\to$ and $\leftarrow$ (and constants) on the dual algebra $\mathbb{F}^+$ as follows: recall that $\mathbb{F}^+$ is isomorphic to the Galois stable lattice $\mathsf{G}_{\mathbb{F}}$. For all $(\mathfrak{X}, \mathfrak{Y})$, $(\mathfrak{X}_1, \mathfrak{Y}_1)$ and $(\mathfrak{X}_2, \mathfrak{Y}_2)$, we let

(i) $\Diamond(\mathfrak{X}, \mathfrak{Y}) := (\upsilon(\mathfrak{X}^\Diamond), \mathfrak{X}^\Diamond)$ where $\mathfrak{X}^\Diamond := \{y \in Y \mid \forall x \in \mathfrak{X}.\ S(x, y)\}$,

(ii) $\Box(\mathfrak{X}, \mathfrak{Y}) := (\mathfrak{Y}^\Box, \lambda(\mathfrak{Y}^\Box))$ where $\mathfrak{Y}^\Box := \{x \in X \mid \forall y \in \mathfrak{Y}.\ S(x, y)\}$,

(iii) $\neg(\mathfrak{X}, \mathfrak{Y}) := (\mathfrak{X}^\neg, \mathfrak{Y}^\neg)$
where $\mathfrak{X}^\neg := \{x{\in}X \mid \forall x'{\in}X.\ C(x, x')\}$, $\mathfrak{Y}^\neg := \{y{\in}Y \mid \forall y'{\in}Y.\ D(y, y')\}$,

(iv) $(\mathfrak{X}_1, \mathfrak{Y}_1) \circ (\mathfrak{X}_2, \mathfrak{Y}_2) := (\upsilon(\mathfrak{X}_1 \circ \mathfrak{X}_2), \mathfrak{X}_1 \circ \mathfrak{X}_2)$
where $\mathfrak{X}_1 \circ \mathfrak{X}_2 := \{y \in Y \mid \forall x_1 \in \mathfrak{X}_1, x_2 \in \mathfrak{X}_2.\ R(x_1, x_2, y)\}$,

(v) $(\mathfrak{X}_1, \mathfrak{Y}_1) \to (\mathfrak{X}_2, \mathfrak{Y}_2) := (\mathfrak{X}_1 \to \mathfrak{Y}_2, \lambda(\mathfrak{X}_1 \to \mathfrak{Y}_2))$
where $\mathfrak{X}_1 \to \mathfrak{Y}_2 := \{x_2 \in X \mid \forall x_1 \in \mathfrak{X}_1, y \in \mathfrak{Y}_2.\ R(x_1, x_2, y)\}$,

(vi) $(\mathfrak{X}_2, \mathfrak{Y}_2) \leftarrow (\mathfrak{X}_1, \mathfrak{Y}_1) := (\mathfrak{Y}_2 \leftarrow \mathfrak{X}_1, \lambda(\mathfrak{Y}_2 \leftarrow \mathfrak{X}_1))$
where $\mathfrak{Y}_2 \leftarrow \mathfrak{X}_1 := \{x_1 \in X \mid \forall x_2 \in \mathfrak{X}_1, y \in \mathfrak{Y}_2.\ R(x_1, x_2, y)\}$.

Theorem 2.13 *For a d-morphism $\langle \sigma | \tau \rangle\colon \mathbb{F} \to \mathbb{G}$, the function $\langle \sigma^+ | \tau^- \rangle\colon \mathbb{G}^+ \to \mathbb{F}^+$, defined by $\langle \sigma^+ | \tau^- \rangle(\mathfrak{X}_2, \mathfrak{Y}_2) := (\sigma^{-1}[\mathfrak{X}_2], \tau^{-1}[\mathfrak{Y}_2])$ for $(\mathfrak{X}_2, \mathfrak{Y}_2) \in \mathfrak{G}^+$, is well-defined and homomorphic. Moreover, we have*

(i) *if $\langle \sigma | \tau \rangle$ is B-embedding, $\langle \sigma^+ | \tau^- \rangle$ is surjective,*

(ii) *if $\langle \sigma | \tau \rangle$ is B-separating, $\langle \sigma^+ | \tau^- \rangle$ is injective.*

3 Topology and general polarity frames

In this section, we will extend the notion of topology from sets to polarities, address the topological representation between lattices and general polarity frames, and establish the topological characterisation of descriptive polarity frames: see e.g. [6] for the arguments on Kripke frames.

Definition 3.1 [B-topology] Let $\mathbb{P} = \langle X, Y, B \rangle$ be a polarity. A *B-topology* $\mathcal{O}$ of $\mathbb{P}$ is a collection of Galois stable pairs of $\mathbb{P}$, i.e. $\mathcal{O} \subseteq \mathsf{G}_{\mathbb{P}}$, satisfying

(i) for all $(\mathfrak{X}_1, \mathfrak{Y}_1), \ldots, (\mathfrak{X}_n, \mathfrak{Y}_n) \in \mathcal{O}$. $(\mathfrak{X}_1 \cap \cdots \cap \mathfrak{X}_n, \lambda(\mathfrak{X}_1 \cap \cdots \cap \mathfrak{X}_n)) \in \mathcal{O}$,

(ii) for all $(\mathfrak{X}_1, \mathfrak{Y}_1), \ldots, (\mathfrak{X}_n, \mathfrak{Y}_n) \in \mathcal{O}$. $(\upsilon(\mathfrak{Y}_1 \cap \cdots \cap \mathfrak{Y}_n), \mathfrak{Y}_1 \cap \cdots \cap \mathfrak{Y}_n) \in \mathcal{O}$,

(iii) $\bigcup\{\mathfrak{X} \mid (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{O}\} = X$,

(iv) $\bigcup\{\mathfrak{Y} \mid (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{O}\} = Y$.

We may call the pair $\langle \mathbb{P}, \mathcal{O} \rangle$ a *B-topological space* on $\mathbb{P}$, or simply *B-topology*.

One may feel that Definition 3.1 is far from the topology on sets: for example, there is no condition for arbitrary unions nor for X and $\emptyset$. This is because Galois stable pairs are not closed under unions and we do not know whether X and $\emptyset$ can be natural requirements for topological representations (in particular unbounded cases). Instead, a B-topology employs a possible generalisation of the bound condition, called *covering conditions*: items (iii) and (iv). From the topological viewpoints, the covering conditions tell us that (iii) for each point in X, there exists at least one open set to which the point belongs, and (iv) for each point in Y, there exists at least one open set to which the point belongs. It is obvious that, if $(X, \lambda(X))$ and $(\upsilon(Y), Y)$ are in $\mathcal{O}$, it satisfies the covering conditions. Note that the non-trivial difference disappears over distributive polarity frames (with one-side infinitary extension), hence it could be a natural generalisation of topology on sets.

We introduce the following notions: A B-topology $\mathcal{O}$ on $\mathbb{P}$ is

differentiated: if $\forall x \in X, y \in Y. [xBy \iff \exists(\mathfrak{X}, \mathfrak{Y}) \in \mathcal{O}. [x \in \mathfrak{X} \,\&\, y \in \mathfrak{Y}]]$,

compact: for all subfamilies $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{O}$, if $\pi_1[\mathcal{X}](:= \{\mathfrak{X} \mid (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{X}\})$ and $\pi_2[\mathcal{Y}](:= \{\mathfrak{Y} \mid (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{Y}\})$ have the finite intersection property and $\mathcal{X} \cap \mathcal{Y} = \emptyset$, there are $x \in X$ and $y \in Y$ such that xBy does not hold, $x \in \bigcap \pi_1[\mathcal{X}]$ and $y \in \bigcap \pi_2[\mathcal{Y}]$.

Given two B-topological spaces $\langle \mathbb{P}, \mathcal{O}_P \rangle$ and $\langle \mathbb{Q}, \mathcal{O}_Q \rangle$, a *continuous map* is defined as an extension of a d-morphism $\langle \sigma | \tau \rangle \colon \mathbb{P} \to \mathbb{Q}$ with

(i) $\forall (\mathfrak{X}', \mathfrak{Y}') \in \mathcal{O}_Q.\ \left(\sigma^{-1}[\mathfrak{X}'], \tau^{-1}[\mathfrak{Y}']\right) \in \mathcal{O}_P$,

(ii) $\forall (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{O}_P, \exists (\mathfrak{X}', \mathfrak{Y}'), (\mathfrak{X}'', \mathfrak{Y}'') \in \mathcal{O}_Q. [\sigma[\mathfrak{X}] \subseteq \mathfrak{X}'$ and $\mathfrak{Y}'' \supseteq \tau[\mathfrak{Y}]]$.

Note that item (i) is exactly the same as the continuity of topology on sets and item (ii) is the strictness condition of lattice homomorphisms. Since B-topology does not always include $(X, \lambda(X))$ or $(\upsilon(Y), Y)$, it is necessary to introduce the condition. We also mention the algebraic view of B-topology.

Definition 3.2 [Covering sublattice] Let $\mathbb{L} = \langle L, \vee, \wedge, \bot, \top \rangle$ be a complete lattice. A sublattice $\mathbb{L}' = \langle L', \vee, \wedge \rangle$ of $\mathbb{L}$ is *covering*, if it satisfies $\top = \bigvee_{a \in L'} a$ and $\bot = \bigwedge_{b \in L'} b$.

Therefore, a B-topology on a polarity $\mathbb{P}$ is a covering sublattice of $\mathsf{G}_{\mathbb{P}}$.

Definition 3.3 [General polarity frame & continuous d-morphism] A quadruple $\mathbb{F} = \langle X, Y, B, P \rangle$ is a *general polarity frame* if $\langle X, Y, B \rangle$ is a polarity frame and P is a covering sublattice of $\mathsf{G}_{\mathbb{F}}$. Given two general polarity frames $\mathbb{F} = \langle X_1, Y_1, B_1, P \rangle$ and $\mathbb{G} = \langle X_2, Y_2, B_2, Q \rangle$, a continuous map from $\mathbb{F}$ to $\mathbb{G}$ is a *continuous d-morphism*: the terminology (B-embedding, B-separating and B-reflecting) is inherited in the obvious way. Note that, for the B-embedding, the following condition is also necessary as in the case of general Kripke frames:

$$\forall(\mathfrak{X},\mathfrak{Y}) \in P, \exists(\mathfrak{X}',\mathfrak{Y}') \in Q. \left[\sigma^{-1}[\mathfrak{X}'] = \mathfrak{X} \,\&\, \tau^{-1}[\mathfrak{Y}'] = \mathfrak{Y}\right].$$

The dual representation is also defined in a usual way. Given a lattice $\mathbb{L} = \langle L, \vee, \wedge \rangle$, the dual general frame $\mathbb{L}_*$ is a pair of the dual frame $\mathbb{L}_+$ with $\hat{L} := \{(\lfloor a \rfloor, \lceil a \rceil) \mid a \in L\}$, where $\lfloor a \rfloor := \{F \in \mathcal{F} \mid a \in F\}$ and $\lceil a \rceil := \{I \in \mathcal{I} \mid a \in I\}$: i.e. $\mathbb{L}_* = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq, \hat{L} \rangle$. For a strict homomorphism $h \colon \mathbb{L} \to \mathbb{M}$, the dual morphism $\langle h_\times | h_\div \rangle \colon \mathbb{M}_* \to \mathbb{L}_*$ is obtained as the dual morphism of the underlying frame$\langle h_+ | h_- \rangle$.

Theorem 3.4 *For lattices $\mathbb{L}$ and $\mathbb{M}$, and a strict homomorphism $h \colon \mathbb{L} \to \mathbb{M}$, the dual general frames $\mathbb{L}_*$ and $\mathbb{M}_*$ are general polarity frames and the dual morphism $\langle h_\times | h_\div \rangle$ is a continuous d-morphism from $\mathbb{M}_*$ to $\mathbb{L}_*$.*

For a general polarity frame $\mathbb{F} = \langle X, Y, B, P \rangle$, the dual algebra $\mathbb{F}^*$ is the covering sublattice P of $\mathsf{G}_\mathbb{F}$ itself. For a continuous d-morphism $\langle \sigma | \tau \rangle \colon \mathbb{F} \to \mathbb{G}$, the dual homomorphism $\langle \sigma^\times | \tau^\div \rangle \colon \mathbb{G}^* \to \mathbb{F}^*$ is the same as $\langle \sigma^+ | \tau^- \rangle$.

Theorem 3.5 *For general polarity frames $\mathbb{F}$ and $\mathbb{G}$, and a continuous d-morphism $\langle \sigma | \tau \rangle \colon \mathbb{F} \to \mathbb{G}$, the dual algebras $\mathbb{F}^*$ and $\mathbb{G}^*$ are lattices and the dual homomorphism $\langle \sigma^\times | \tau^\div \rangle$ is a strict homomorphism, i.e. $\langle \sigma^\times | \tau^\div \rangle \colon \mathbb{G}^* \to \mathbb{F}^*$.*

Theorem 3.6 *For a strict homomorphism $h \colon \mathbb{L} \to \mathbb{M}$ and a continuous d-morphism $\langle \sigma | \tau \rangle \colon \mathbb{F} \to \mathbb{G}$, we have*

(i) *if h is injective then $\langle h_\times | h_\div \rangle$ is B-separating,*

(ii) *if h is surjective then $\langle h_\times | h_\div \rangle$ is B-embedding,*

(iii) *if $\langle \sigma | \tau \rangle$ is B-embedding then $\langle \sigma^\times | \tau^\div \rangle$ is surjective,*

(iv) *if $\langle \sigma | \tau \rangle$ is B-separating then $\langle \sigma^\times | \tau^\div \rangle$ is injective.*

Topological characterisation of descriptive general polarity frames is addressed as follows: for a general polarity frame $\mathbb{F} = \langle X, Y, B, P \rangle$, we introduce two functions $\mathfrak{s} \colon X \to \wp(X)$ and $\mathfrak{t} \colon Y \to \wp(Y)$ with $\mathfrak{s}(x) := \{(\mathfrak{X},\mathfrak{Y}) \in P \mid x \in \mathfrak{X}\}$ and $\mathfrak{t}(y) := \{(\mathfrak{X},\mathfrak{Y}) \in P \mid y \in \mathfrak{Y}\}$.

Proposition 3.7 *$\mathfrak{s}(x)$ and $\mathfrak{t}(y)$ are a filter and an ideal over P, hence $\mathfrak{s} \colon X \to \mathcal{F}(P)$ and $\mathfrak{t} \colon Y \to \mathcal{I}(P)$ are well-defined.*

Definition 3.8 [Descriptive polarity frame] A general polarity frame $\mathbb{F} = \langle X, Y, B, P \rangle$ is *descriptive*, if the maps $\mathfrak{s}$ and $\mathfrak{t}$ form a B-reflecting continuous d-morphism from $\mathbb{F}$ to $(\mathbb{F}^*)_*$, i.e. $\langle \mathfrak{s} | \mathfrak{t} \rangle \colon \mathbb{F} \to (\mathbb{F}^*)_*$.

To obtain the topological characterisation of descriptive polarity frames, [4] we show two lemmata.

[4] The result on the similar setting can be found in [29].

Lemma 3.9 *For each lattice* $\mathbb{L} = \langle L, \vee, \wedge \rangle$*, the bi-dual lattice* $(\mathbb{L}_*)^* = \langle \hat{L}, \vee, \wedge \rangle$ *is isomorphic to* $\mathbb{L}$ *via the canonical map* $\mathbb{L} \to (\mathbb{L}_*)^*$ *with* $(\lfloor a \rfloor, \lceil a \rceil) \in \hat{L}$ *for each* $a \in L$.

Lemma 3.10 *A differentiated B-topological space on a polarity* $\langle \mathbb{P}, \mathcal{O} \rangle$ *satisfies*

(i) *for all* $x_1, x_2 \in X$, $x_1 \leq x_2 \iff \forall (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{O}. [x_2 \in \mathfrak{X} \Longrightarrow x_1 \in \mathfrak{X}]$,

(ii) *for all* $y_1, y_2 \in Y$, $y_1 \leq y_2 \iff \forall (\mathfrak{X}, \mathfrak{Y}) \in \mathcal{O}. [y_1 \in \mathfrak{Y} \Longrightarrow y_2 \in \mathfrak{Y}]$.

Theorem 3.11 *A general polarity frame* $\mathbb{F} = \langle X, Y, B, P \rangle$ *is descriptive if and only if it is differentiated and compact as a B-topological space.*

4 Goldblatt-Thomason's theorem

In this section, we introduce disjoint unions of polarity frames. Also, based on the setting, we establish the Goldblatt-Thomason's theorem for substructural logic via the Birkhoff's variety theorem. Hence, throughout this section, we consider polarity frames for substructural logic. However, the results can be naturally applied for the other relational structures and variants of distributive lattice-based logics.

The disjoint union. To discuss disjoint unions of polarity frames, it seems natural to introduce the following notions: for a polarity frame $\mathbb{F} = \langle X, Y, B \rangle$, an element $x \in X$ is a *bottom element*, denoted by $\bot$, if $\forall y \in Y. [xBy]$, and an element $y \in Y$ is a *top element*, denoted by $\top$, if $\forall x \in X. [xBy]$. In general we do not assume the existence of bottom elements and top elements, but, if they exist, they are unique up to $\leq_B$-equivalence.

Let $\mathbb{F}_i$ for $i \in I$ be polarity frames for substructural logic. *The disjoint union* $\biguplus_{i \in I} \mathbb{F}_i$ consists of the set-theoretical disjoint unions $\biguplus_{i \in I} X_i$, $\biguplus_{i \in I} Y_i$, $\biguplus_{i \in I} O_{X\,i}$, $\biguplus_{i \in I} O_{Y\,i}$, $\biguplus_{i \in I} N_{X\,i}$, $\biguplus_{i \in I} N_{Y\,i}$, and the following relations $B_\uplus$ and $R_\uplus$: for all $x, x_1, x_2 \in \biguplus_{i \in I} X_i$ and $y \in \biguplus_{i \in I} Y_i$, we let

$$xB_\uplus y \iff \begin{cases} xB_i y & x \in X_i, y \in Y_i \\ \text{always holds} & x \in X_i, y \in Y_j, i \neq j \end{cases}$$

$$R_\uplus(x_1, x_2, y) \iff \begin{cases} R_i(x_1, x_2, y) & x_1, x_2 \in X_i, y \in Y_i \\ \text{always holds} & x_1 \in X_i, x_2 \in X_j, y \in Y_k, \neq\{i, j, k\} \end{cases}$$

where (and hereafter) $\neq\{i, j, k\}$ means that at least one index is different from the others.

On the disjoint union $\biguplus_{i \in I} \mathbb{F}_i$, the extended relations satisfy the following.

Proposition 4.1 *For* $x, x_1, x_2 \in \biguplus_{i \in I} X_i$ *and* $y, y_1, y_2 \in \biguplus_{i \in I} Y_i$*, we have*

(i) $x_1 \leq_\uplus x_2 \iff \begin{cases} x_1 \leq_i x_2 & x_1, x_2 \in X_i \\ x_1 = \bot_i & x_1 \in X_i, x_2 \in X_j, i \neq j \end{cases}$

(ii) $y_1 \leq_\uplus y_2 \iff \begin{cases} y_1 \leq_i y_2 & y_1, y_2 \in Y_i \\ y_2 = \top_j & y_1 \in Y_i, y_2 \in Y_j, i \neq j \end{cases}$

(iii) $R^{\circ}_{\uplus}(x_1, x_2, x) \iff \begin{cases} R^{\circ}_i(x_1, x_2, x) & x, x_1, x_2 \in X_i \\ x = \bot_k & x_1 \in X_i, x_2 \in X_j, x \in X_k, \neq\{i, j, k\} \end{cases}$

(iv) $R^{\rightarrow}_{\uplus}(x_1, y_2, y) \iff \begin{cases} R^{\rightarrow}_i(x_1, y_2, y) & x_1 \in X_i, y_2, y \in Y_i \\ y_2 = \top_j & x_1 \in X_i, y_2 \in Y_i, y \in Y_k, \neq\{i, j, k\} \end{cases}$

(v) $R^{\leftarrow}_{\uplus}(y_1, x_2, y) \iff \begin{cases} R^{\leftarrow}_i(y_1, x_2, y) & x_2 \in X_i, y_1, y \in Y_i \\ y_1 = \top_i & y_1 \in Y_i, x_2 \in X_j, y \in Y_k, \neq\{i, j, k\} \end{cases}$

where $=$ is a shorthand for $\leq_B$-equivalence.

Theorem 4.2 *For polarity frames for substructural logic $\mathbb{F}_i$ (for $i \in I$), the disjoint union $\biguplus_{i\in I} \mathbb{F}_i$ is also a polarity frame for substructural logic. Hence, the class of polarity frames for substructural logic is closed under disjoint unions.*

We can also have the natural canonical embeddings as well.

Theorem 4.3 *Let $\mathbb{F}_i$ (for $i \in I$) be polarity frames for substructural logic. For each index $i \in I$, the canonical functions $\iota^X_i : X_i \to \biguplus_{i\in I} X_i$ (i.e. $\iota^X_i(x) = x$) and $\iota^Y_i : Y_i \to \biguplus_{i\in I} Y_i$ (i.e. $\iota^Y_i(y) = y$) form a B-embedding d-morphism from $\mathbb{F}_i$ to the disjoint union $\biguplus_{i\in I} \mathbb{F}_i$, i.e. $\langle \iota^X_i | \iota^Y_i \rangle : \mathbb{F}_i \to \biguplus_{i\in I} \mathbb{F}_i$ for each $i \in I$.*

Goldblatt-Thomason's theorem. To state the theorem, we introduce the following: for polarity frames (with additional structures) $\mathbb{F}$ and $\mathbb{G}$, $\mathbb{F}$ is a *subframe* of $\mathbb{G}$, if there exists a B-embedding d-morphism from $\mathbb{F}$ to $\mathbb{G}$, i.e. $\mathbb{F} \rightarrowtail \mathbb{G}$; $\mathbb{G}$ is a *separating image* of $\mathbb{F}$, if there exists a B-separating d-morphism from $\mathbb{F}$ to $\mathbb{G}$, i.e. $\mathbb{F} \twoheadrightarrow \mathbb{G}$, and $(\mathbb{F}^+)_+$ is *the filter-ideal extension* of $\mathbb{F}$.

Lemma 4.4 *Let $\mathbb{F}_i$ for $i \in I$ be polarity frames for substructural logic.*

$$\left(\biguplus_{i\in I} \mathbb{F}_i\right)^+ \cong \prod_{i\in I} (\mathbb{F}^+_i)$$

Theorem 4.5 *The first-order definable class of polarity frames for substructural logic is definable by substructural formulae, if and only if it is closed under subframes, separating images and disjoint unions, and reflects filter-ideal extensions.*

The above statements hold for distributive polarity frames, the results also apply for distributive substructural and lattice-based logics as well.

5 Amalgamation property

In this section, we will discuss the amalgamation property based on the dual representation for canonical lattice-based logics. As we shall see below, the argument goes from the basic structure, i.e. lattices. But, surprisingly, the amalgamation property of the other variants of lattice-based algebras are also schematically proved on the base result as well.

Definition 5.1 [Amalgamation property] A class $\mathcal{C}$ of lattice-based algebras has *the amalgamation property*, if for all $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathcal{C}$ with injections $f : \mathbb{A} \to \mathbb{B}$

and $g\colon \mathbb{A} \to \mathbb{C}$, there are an algebra $\mathbb{D} \in \mathcal{C}$ and two injections $i\colon \mathbb{B} \to \mathbb{D}$ and $j\colon \mathbb{C} \to \mathbb{D}$ such that $i \circ f = j \circ g$.

The recipe is as follows:

(i) Given algebras $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and two injections $f\colon \mathbb{A} \to \mathbb{B}$ and $g\colon \mathbb{A} \to \mathbb{C}$, dualise the algebras and injections, i.e. $\langle f_+|f_-\rangle\colon \mathbb{B}_+ \to \mathbb{A}_+$ and $\langle g_+|g_-\rangle\colon \mathbb{C}_+ \to \mathbb{A}_+$ (they are B-separating: see Theorem 2.13). Note that if given algebras are unbounded or injections are not strict, we embed them into bounded algebras and strict injections in the standard way.

(ii) Construct, by amalgamating polarity frames $\mathbb{B}_+$ and $\mathbb{C}_+$, a polarity frame $\mathbb{F}_D$ endowed with two B-separating d-morphisms $\langle \sigma_B|\tau_B\rangle$ and $\langle \sigma_C|\tau_C\rangle$ to $\mathbb{B}_+$ and $\mathbb{C}_+$.

(iii) Check the commutativity $\langle \sigma_B|\tau_B\rangle \circ \langle f_+|f_-\rangle = \langle \sigma_C|\tau_C\rangle \circ \langle g_+|g_-\rangle$.

(iv) Dualise the commutative diagram to the dual algebras.

(v) Connect the original algebras to the bi-dual algebras with the canonical embeddings, i.e. $\mathfrak{c}_A\colon \mathbb{A} \to (\mathbb{A}_+)^+$. Note that the canonical embedding is injective, and concatenations of injections are injective, hence $\langle \sigma_B^+|\tau_B^-\rangle \circ \mathfrak{c}_B \circ f = \langle \sigma_C^+|\tau_C^-\rangle \circ \mathfrak{c}_C \circ g$.

Theorem 5.2 *Lattices admit the amalgamation property. Also, lattices extended with the distributivity, adjoint unary modality ($\Diamond \dashv \Box$), de Morgan negation $\neg$ admit the amalgamation property.*

Instead of the proof of this theorem, which requires a lot of space, we show the construction of the amalgamation $\mathbb{F}_D$ and B-separating d-morphisms.

Let $\mathbb{A}_+$, $\mathbb{B}_+$, $\mathbb{C}_+$ be the dual polarity frames and $\langle f_+|f_-\rangle\colon \mathbb{B}_+ \to \mathbb{A}_+$, $\langle g_+|g_-\rangle\colon \mathbb{C}_+ \to \mathbb{A}_+$ the dual morphisms of injections $f\colon \mathbb{A} \to \mathbb{B}$ and $g\colon \mathbb{A} \to \mathbb{C}$. The amalgamation $\mathbb{F}_D$ is constructed as the disjoint union of $\mathbb{B}_+$ and $\mathbb{C}_+$ with additional requirements for $\mathbb{A}_+$. That is, $\mathbb{F}_D = \langle \mathcal{F}_B \uplus \mathcal{F}_C, \mathcal{I}_B \uplus \mathcal{I}_C, \sqsubseteq_D\rangle$ with

$$F \sqsubseteq_D I \iff \begin{cases} F \sqsubseteq_B I & \text{if } F \in \mathcal{F}_B, I \in \mathcal{I}_B \\ F \sqsubseteq_C I & \text{if } F \in \mathcal{F}_C, I \in \mathcal{I}_C \\ f_+(F) \sqsubseteq_A g_-(I) & \text{if } F \in \mathcal{F}_B, I \in \mathcal{I}_C \\ g_+(F) \sqsubseteq_A f_-(I) & \text{if } F \in \mathcal{F}_C, I \in \mathcal{I}_B \end{cases}$$

For adjoint unary modality $\Diamond \dashv \Box$, the relation S_D on $\mathbb{F}_D$ is defined as follows:

$$S_D(F, I) \iff \begin{cases} S_B(F, I) & \text{if } F \in \mathcal{F}_B, I \in \mathcal{I}_B \\ S_C(F, I) & \text{if } F \in \mathcal{F}_C, I \in \mathcal{I}_C \\ S_A(f_+(F), g_-(I)) & \text{if } F \in \mathcal{F}_B, I \in \mathcal{I}_C \\ S_A(g_+(F), f_-(I)) & \text{if } F \in \mathcal{F}_C, I \in \mathcal{I}_B \end{cases}$$

For the de Morgan negation $\neg$, the two relations C_D and D_D on $\mathbb{F}_D$ are defined as follows: $C_D(F,G)$ and $D_D(I,J)$ are

$$\begin{cases} C_B(F,G) & F,G \in \mathcal{F}_B \\ C_C(F,G) & F,G \in \mathcal{F}_C \\ C_A(f_+(F),g_+(G)) & F \in \mathcal{F}_B, G \in \mathcal{F}_C \\ C_A(g_+(F),f_+(G)) & F \in \mathcal{F}_C, G \in \mathcal{F}_B \end{cases} \quad \begin{cases} D_B(I,J) & I,J \in \mathcal{I}_B \\ D_C(I,J) & I,J \in \mathcal{I}_C \\ D_A(f_-(I),g_-(J)) & I \in \mathcal{I}_B, J \in \mathcal{I}_C \\ D_A(g_-(I),f_-(J)) & I \in \mathcal{I}_C, J \in \mathcal{I}_B \end{cases}$$

The d-morphisms $\langle \sigma_B | \tau_B \rangle \colon \mathbb{F}_D \to \mathbb{B}_+$ and $\langle \sigma_C | \tau_C \rangle \colon \mathbb{F}_D \to \mathbb{C}_+$ are defined as follows:

$$\sigma_B(F) := \begin{cases} F & \text{if } F \in \mathcal{F}_B \\ \uparrow f[g_+(F)] & \text{if } F \in \mathcal{F}_C \end{cases} \qquad \tau_B(I) := \begin{cases} I & \text{if } I \in \mathcal{I}_B \\ \downarrow f[g_-(I)] & \text{if } I \in \mathcal{I}_C \end{cases}$$

where $\uparrow f[g_+(F)]$ and $\downarrow f[g_-(I)]$ are the generated filter of the image of f of $g_+(F)$ and the generated ideal of the image of f of $g_-(I)$. The definition of $\langle \sigma_C | \tau_C \rangle$ is analogous.

The (distributive) polarity frame $\mathbb{F}_D$ (endowed with the above relations) and the d-morphisms $\langle \sigma_B | \tau_B \rangle$ and $\langle \sigma_C | \tau_C \rangle$ satisfy our requirements. Therefore Theorem 5.2 holds.

6 Conclusion

In this paper, we have introduced the notion of topology on polarities, named B-topology, and general polarity frames, the disjoint union of polarity frames and amalgamation of polarity frames. Based on these notions and constructions, we have provided the topological characterisation of descriptive polarity frames as in the case of modal logics, established the Goldblatt-Thomason's theorem on polarity frames, and shown the amalgamation property for some lattice-based algebras.

As concluding remarks, we shortly list the current author's forthcoming work (with collaborators). For the topological representation, we will study the persistence properties for substructural and lattice-based formulae. For the Goldblatt-Thomason's theorem, we will also provide the model-theoretic proof and model theory on polarities. For the amalgamation property, we are investigating sufficient conditions for the amalgamation property for lattice-base algebras. Note that, unfortunately, our approach for the amalgamation property seems containing strong requirements to amalgamate polarity frames with (more than) ternary relations, hence for substructural logics, we can universally discuss the amalgamation property only above intuitionistic logic.

Appendix

A Proof of Theorem 3.11

Proof. ($\Rightarrow$). Assume that $\langle \mathfrak{s} | \mathfrak{t} \rangle$ forms a B-reflecting d-morphism. For all $x \in X$ and $y \in Y$ satisfying that xBy does not hold and $(\mathfrak{X}, \mathfrak{Y}) \in P$ does, if

$x \in \mathfrak{X}$ then $y \notin \mathfrak{Y}$, otherwise, as $\mathfrak{X} = \upsilon(\mathfrak{Y})$, we obtain xBy which contradicts to the assumption that xBy does not hold. Conversely, suppose xBy for $x \in X$ and $y \in Y$. Since $\langle \mathfrak{s}|\mathfrak{t} \rangle$ is d-embedding, we have $\mathfrak{s}(x) \sqsubseteq \mathfrak{t}(y)$. So there exists $(\mathfrak{X}, \mathfrak{Y}) \in P$ such that $(\mathfrak{X}, \mathfrak{Y}) \in \mathfrak{s}(x) \cap \mathfrak{t}(y)$. By definition, $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$, hence $\mathbb{F}$ is differentiated. For all subfamilies $\mathcal{X}, \mathcal{Y} \subseteq P$ satisfying $\pi_1[\mathcal{X}]$ and $\pi_2[\mathcal{Y}]$ have finite intersection property, and $\mathcal{X} \cap \mathcal{Y} = \emptyset$. Let $F_\mathcal{X}$ and $I_\mathcal{Y}$ be the generated filter by $\mathcal{X}$ and the generated ideal by $\mathcal{Y}$ over P, i.e. $F_\mathcal{X} := \uparrow\mathcal{X}$ and $I_\mathcal{Y} := \downarrow\mathcal{Y}$. Suppose $F_\mathcal{X} \sqsubseteq I_\mathcal{Y}$. It contradicts to $\mathcal{X} \cap \mathcal{Y} = \emptyset$, so $F_\mathcal{X} \not\sqsubseteq I_\mathcal{I}$. As $\langle \mathfrak{s}|\mathfrak{t} \rangle$ is d-separating, there exist $x \in X$ and $y \in Y$ such that $\mathfrak{s}(x) \sqsubseteq F_\mathcal{X}$ and $I_\mathcal{Y} \sqsubseteq \mathfrak{t}(y)$ hold, but xBy does not. For arbitrary $(\mathfrak{X}_f, \mathfrak{Y}_f) \in F_\mathcal{X}$ and $(\mathfrak{X}_i, \mathfrak{Y}_i) \in I_\mathcal{Y}$, since $F_\mathcal{X} \subseteq \mathfrak{s}(x)$ and $I_\mathcal{Y} \subseteq \mathfrak{t}(y)$, we have $(\mathfrak{X}_f, \mathfrak{Y}_f) \in \mathfrak{s}(x)$ and $(\mathfrak{X}_i, \mathfrak{Y}_i) \in \mathfrak{t}(y)$. By definition, $x \in \mathfrak{X}_f$ and $y \in \mathfrak{Y}_i$. So $x \in \bigcap \pi_1[\mathcal{X}]$ and $y \in \bigcap \pi_2[\mathcal{Y}]$. Therefore $\mathbb{F}$ is compact.

($\Leftarrow$). Because of Proposition 3.7 and Lemma 3.9, it suffices to show that $\langle \mathfrak{s}|\mathfrak{t} \rangle$ is a d-morphism, i.e. items (i) - (iii), B-embedding and B-separating for differentiated and compact B-topological spaces. **Item (i)**: for all $x \in X$ and $y \in Y$, suppose that $\mathfrak{s}(x) \sqsubseteq \mathfrak{t}(y)$. Then there exists $(\mathfrak{X}, \mathfrak{Y}) \in P$ such that $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$. Since $(\mathfrak{X}, \mathfrak{Y})$ is a Galois stable pair, we have $X = \upsilon(\mathfrak{Y})$ and $\lambda(\mathfrak{X}) = \mathfrak{Y}$, which concludes xBy. **Item (ii)**: we prove the contraposition. Suppose $\mathfrak{s}(x) \not\sqsubseteq I$ for arbitrary $x \in X$ and $I \in \mathcal{I}_P$. Since $\mathfrak{s}(x)$ is a filter over P and I is an ideal over P, they have intersection property. Because $\mathbb{F}$ is compact, there exist $x_c \in X$ and $y_c \in Y$ such that $x_c \in \bigcap \pi_1[\mathfrak{s}(x)]$ and $y_c \in \bigcap \pi_2[I]$ hold, but $x_c B y_c$ does not. By definition, for each $\mathfrak{X} \in \pi_1[P]$, if $\mathfrak{X} \in \pi_1[\mathfrak{s}(x)]$ then $\mathfrak{X} \in \pi_1[\mathfrak{s}(x_c)]$. Moreover, for each $\mathfrak{Y} \in \pi_2[P]$, if $\mathfrak{Y} \in \pi_2[I]$ then $\mathfrak{Y} \in \pi_2[\mathfrak{t}(y_c)]$. Hence $\mathfrak{s}(x_c) \sqsubseteq \mathfrak{s}(x)$ and $I \sqsubseteq \mathfrak{t}(y_c)$. As $\mathbb{F}$ is differentiated, by Lemma 3.10, we have $x_c \leq x$. Further, since $x_c B y_c$ does not hold, neither does xBy_c. **Item (iii)**: this is analogous to item (ii). ***B*-embedding**: for arbitrary $x \in X$ and $y \in Y$, if xBy, since it is differentiated, there exists $(\mathfrak{X}, \mathfrak{Y}) \in P$ such that $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$. Because of the definitions $\mathfrak{s}(x)$ and $\mathfrak{t}(y)$, we have $(\mathfrak{X}, \mathfrak{Y}) \in \mathfrak{s}(x) \cap \mathfrak{t}(y)$, hence $\mathfrak{s}(x) \sqsubseteq \mathfrak{t}(y)$. ***B*-separating**: we prove the contraposition. for arbitrary $F \in \mathcal{F}_P$ and $I \in \mathcal{I}_P$, assume $F \not\sqsubseteq I$. Since F and I are subfamilies of P, they have finite intersection property. In addition, by our assumption, $F \cap I = \emptyset$. Because of the compactness, there exist $x \in X$ and $y \in Y$ such that $x \in \bigcap \pi_1[F]$ and $y \in \bigcap \pi_2[I]$ hold, but xBy does not. As $x \in \bigcap \pi_1[F]$, each $\mathfrak{X} \in \pi_1[F]$ is in $\mathfrak{s}(x)$, that is, $\mathfrak{s}(x) \sqsubseteq F$. Also, as $y \in \bigcap \pi_2[I]$, each $\mathfrak{Y} \in \pi_2[I]$ is in $\mathfrak{t}(y)$, that is, $I \sqsubseteq \mathfrak{t}(y)$. □

B Hints for Section 4

Theorem 4.2 follows from Propositions 4.1, B.1 and B.2.

Proposition B.1 *For all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, we have*

(i) *if y is a top element then $R(x_1, x_2, y)$,*

(ii) *if x_2 is a bottom element then $R(x_1, x_2, y)$,*

(iii) *if x_1 is a bottom element then $R(x_1, x_2, y)$,*

(iv) *if x is a bottom element then $R^{\circ}(x_1, x_2, x)$,*

(v) *if y_2 is a top element then $R^{\rightarrow}(x_1, y_2, y)$,*

(vi) *if y_1 is a top element then $R^{\leftarrow}(y_1, x_2, y)$.*

Proposition B.2 *For all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$, we have*

(i) *if $x \neq \bot$ and $R^{\circ}(x_1, x_2, x)$ then $x_1 \neq \bot$ and $x_2 \neq \bot$,*

(ii) *if $y_2 \neq \top$ and $R^{\rightarrow}(x_1, y_2, y)$ then $x_1 \neq \bot$ and $y \neq \top$,*

(iii) *if $y_1 \neq \top$ and $R^{\leftarrow}(y_1, x_2, y)$ then $x_2 \neq \bot$ and $y \neq \top$.*

For Lemma 4.4, it suffices to show that the following function η is a well defined isomorphism. We define $\eta\colon \left(\biguplus_{i\in I} \mathbb{F}_i\right)^+ \to \prod_{i\in I}(\mathbb{F}_i^+)$ as follows: for each Galois stable pair $(\mathfrak{X}, \mathfrak{Y}) \in \mathsf{G}_{\uplus \mathbb{F}_i}$,

$$\eta(\mathfrak{X}, \mathfrak{Y}) := ((\mathfrak{X} \cap X_1, \mathfrak{Y} \cap Y_1), \ldots, (\mathfrak{X} \cap X_i, \mathfrak{Y} \cap Y_i), \ldots).$$

A proof of Theorem 4.5 is sketched as follows: the ($\Rightarrow$)-direction follows from the invariance of validity of sequents via B-embedding d-morphisms and B-separating d-morphisms. The ($\Leftarrow$)-direction is as follows: let $\mathcal{P}$ be a class of polarity frames satisfying the condition. For any polarity frame $\mathbb{F}$ validating the substructural formulae of $\mathcal{P}$, the dual algebra $\mathbb{F}^+$ is a model of the equational theory of the class of dual algebras of $\mathcal{P}$.[5] Due to the Birkhoff's variety theorem, $\mathbb{F}^+$ is in the variety, which means $\mathbb{F}^+$ is a homomorphic image of a subalgebra of a product of dual algebras of polarity frames in $\mathcal{P}$. Because of Lemma 4.4, products of dual algebras of polarity frames are isomorphic to dual algebras of disjoint unions of the same polarity frames. By dualising the HSP conditions, the filter-ideal extension $(\mathbb{F}^+)_+$ is a subframe of a separating image of the filter-ideal extension of the disjoint union of polarity frames in $\mathcal{P}$. Hence, $\mathbb{F}$ is in $\mathcal{P}$.

Note that, for variants of distributive lattice-based logics, it is easy to check the construction of disjoint unions straightforwardly admits the splitting condition. For the connection between distributive polarity frames and Kripke frames can be found in [38].

C Hints for Section 5

Theorem C.1 *Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be lattices, and $f\colon \mathbb{A} \to \mathbb{B}$ and $g\colon \mathbb{A} \to \mathbb{C}$ be injective. There exists a polarity frame $\mathbb{F}_D$ with two B-separating d-morphisms $\langle \sigma_B | \tau_B \rangle\colon \mathbb{F}_D \to \mathbb{B}_+$ and $\langle \sigma_C | \tau_C \rangle\colon \mathbb{F}_D \to \mathbb{C}_+$ satisfying $\langle f_+ | f_- \rangle \circ \langle \sigma_B | \tau_B \rangle = \langle g_+ | g_- \rangle \circ \langle \sigma_C | \tau_C \rangle$.*

Lemma C.2 *Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be lattices, and $f\colon \mathbb{A} \to \mathbb{B}$ and $g\colon \mathbb{A} \to \mathbb{C}$ injective homomorphisms. On the dual frames, we have*

(i) *for arbitrary $F \in \mathcal{F}_B$ and $I \in \mathcal{I}_C$,*

$$F \sqsubseteq_B {\downarrow} f[g_-(I)] \iff f_+(F) \sqsubseteq_A g_-(I) \iff {\uparrow} g[f_+(F)] \sqsubseteq_C I,$$

[5] Each sequent corresponds to an inequality, but on lattices, it is naturally translated to an equality.

(ii) *for arbitrary* $F \in \mathcal{F}_C$ *and* $I \in \mathcal{I}_B$,

$$\uparrow f[g_+(F)] \sqsubseteq_B I \iff g_+(F) \sqsubseteq_A f_-(I) \iff F \sqsubseteq_C \downarrow g[f_-(I)],$$

(iii) *for each* $F \in \mathcal{F}_A$, $F = f_+(\uparrow f[F])$ *and* $F = g_+(\uparrow g[F])$,

(iv) *for each* $I \in \mathcal{I}_A$, $I = f_-(\downarrow f[I])$ *and* $I = g_-(\downarrow g[I])$,

(v) *for each* $F \in \mathcal{F}_B$, $F \sqsubseteq_B \uparrow f[f_+(F)]$,

(vi) *for each* $F \in \mathcal{F}_C$, $F \sqsubseteq_C \uparrow g[g_+(F)]$,

(vii) *for each* $I \in \mathcal{I}_B$, $\downarrow f[f_-(I)] \sqsubseteq_B I$,

(viii) *for each* $I \in \mathcal{I}_C$, $\downarrow g[g_-(I)] \sqsubseteq_C I$.

Proposition C.3 *For all filters* $F, G \in X_D$ *and ideals* $I, J \in Y_D$, *we have*

$$F \sqsubseteq_D G \iff \begin{cases} F \sqsubseteq_B G & F, G \in \mathcal{F}_B \\ F \sqsubseteq_C G & F, G \in \mathcal{F}_C \\ F \sqsubseteq_B \uparrow f[g_+(G)] \ \&\ \uparrow g[f_+(F)] \sqsubseteq_C G & F \in \mathcal{F}_B, G \in \mathcal{F}_C \\ \uparrow f[g_+(F)] \sqsubseteq_B G \ \&\ F \sqsubseteq_C \uparrow g[f_+(G)] & F \in \mathcal{F}_C, G \in \mathcal{F}_B \end{cases}$$

$$I \sqsubseteq_D J \iff \begin{cases} I \sqsubseteq_B J & I, J \in \mathcal{I}_B \\ I \sqsubseteq_C J & I, J \in \mathcal{I}_C \\ I \sqsubseteq_B \downarrow f[g_-(J)] \ \&\ \downarrow g[f_-(I)] \sqsubseteq_C J & I \in \mathcal{I}_B, J \in \mathcal{I}_C \\ \downarrow f[g_-(I)] \sqsubseteq_B J \ \&\ I \sqsubseteq_C \downarrow g[f_-(J)] & I \in \mathcal{I}_C, J \in \mathcal{I}_B \end{cases}$$

Proposition C.4 *The dualised commutative diagram also commutes.*

Distributive lattices. We extend the previous result to distributive lattice along completely the same construction and the method.

Theorem C.5 *For distributive lattices* $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$ *and injective homomorphisms* $f\colon \mathbb{A} \to \mathbb{B}$ *and* $g\colon \mathbb{A} \to \mathbb{C}$, *there exists a distributive polarity frame* $\mathbb{F}_D$ *equipped with two d-separating morphisms* $\langle \sigma_B | \tau_B \rangle\colon \mathbb{F}_D \to \mathbb{B}_+$ *and* $\langle \sigma_C | \tau_C \rangle\colon \mathbb{F}_D \to \mathbb{C}_+$ *such that* $\langle f_+ | f_- \rangle \circ \langle \sigma_B | \tau_B \rangle = \langle g_+ | g_- \rangle \circ \langle \sigma_C | \tau_C \rangle$.

Theorem C.6 *Distributive lattices admit the amalgamation property.*

Modal operators. Now we consider adjoint unary modal operators $\Diamond \dashv \Box$.

Theorem C.7 *For lattices with adjoint unary modality* ($\Diamond \dashv \Box$) $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$, *and two injective homomorphisms* $f\colon \mathbb{A} \to \mathbb{B}$ *and* $g\colon \mathbb{A} \to \mathbb{C}$, *there exists a polarity frame with adjoint unary modality* $\mathbb{F}_D$ *endowed with two d-separating morphisms* $\langle \sigma_B | \tau_B \rangle\colon \mathbb{F}_D \to \mathbb{B}_+$ *and* $\langle \sigma_C | \tau_C \rangle\colon \mathbb{F}_D \to \mathbb{C}_+$ *such that* $\langle f_+ | f_- \rangle \circ \langle \sigma_B | \tau_B \rangle = \langle g_+ | g_- \rangle \circ \langle \sigma_C | \tau_C \rangle$.

To prove the main statement, Lemma C.8 and Proposition C.9 are useful.

Lemma C.8 *Let* $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$ *be lattices with adjoint unary modality* $\Diamond \dashv \Box$, *and* $f\colon \mathbb{A} \to \mathbb{B}$ *and* $g\colon \mathbb{A} \to \mathbb{C}$ *injective homomorphisms. On the dual frames,*

(i) *for arbitrary* $F \in \mathcal{F}_B$ *and* $I \in \mathcal{I}_C$,

$$S_B(F, \downarrow f[g_-(I)]) \iff S_A(f_+(F), g_-(I)) \iff S_C(\uparrow g[f_+(F)], I),$$

(ii) *for arbitrary* $F \in \mathcal{F}_C$ *and* $I \in \mathcal{I}_B$,

$$S_B(\uparrow f[g_+(F)], I) \iff S_A(g_+(F), f_-(I)) \iff S_C(F, \downarrow g[f_-(I)]),$$

(iii) *for arbitrary* $F \in \mathcal{F}_B$ *and* $I \in \mathcal{I}_B$,

$$S_B(F, I) \Longleftarrow S_A(f_+(F), f_-(I)) \iff S_C(\uparrow g[f_+(F)], \downarrow g[f_-(I)]),$$

(iv) *for arbitrary* $F \in \mathcal{F}_C$ *and* $I \in \mathcal{I}_C$,

$$S_B(\uparrow f[g_+(F)], \downarrow f[g_-(I)]) \iff S_A(g_+(F), g_-(I)) \Longrightarrow S_C(F, I).$$

Proposition C.9 *For all* $F, G \in X_D$ *and* $I, J \in Y_D$, *we have*

$$S_D^{\diamond}(F, G) \iff \begin{cases} S_B^{\diamond}(F, G) & F, G \in \mathcal{F}_B \\ S_C^{\diamond}(F, G) & F, G \in \mathcal{F}_C \\ S_B^{\diamond}(F, \uparrow f[g_+(G)]) \ \&\ S_C^{\diamond}(\uparrow g[f_+(F)], G) & F \in \mathcal{F}_B, G \in \mathcal{F}_C \\ S_B^{\diamond}(\uparrow f[g_+(F)], G) \ \&\ S_C^{\diamond}(F, \uparrow g[f_+(G)]) & F \in \mathcal{F}_C, G \in \mathcal{F}_B \end{cases}$$

$$S_D^{\square}(J, I) \iff \begin{cases} S_B^{\square}(J, I) & I, J \in \mathcal{I}_B \\ S_C^{\square}(J, I) & I, J \in \mathcal{I}_C \\ S_B^{\square}(\downarrow f[g_-(J)], I) \ \&\ S_C^{\square}(J, \downarrow g[f_-(I)]) & I \in \mathcal{I}_B, J \in \mathcal{I}_C \\ S_B^{\square}(J, \downarrow f[g_-(I)]) \ \&\ S_C^{\square}(\downarrow g[f_-(J)], I) & I \in \mathcal{I}_C, J \in \mathcal{I}_B \end{cases}$$

Theorem C.10 *Lattices with adjoint unary modality* ($\diamond \dashv \square$) *admit the amalgamation property.*

Corollary C.11 *Distributive lattices with adjoint unary modality admit the amalgamation property.*

De Morgan negation. Now we consider the de Morgan negation $\neg$.

Theorem C.12 *For lattices with the de Morgan negation* ($\neg$) $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$, *and two injective homomorphisms* $f\colon \mathbb{A} \to \mathbb{B}$ *and* $g\colon \mathbb{A} \to \mathbb{C}$, *there exists a polarity frame with the de Morgan negation* $\mathbb{F}_D$ *endowed with two d-separating morphisms* $\langle \sigma_B | \tau_B \rangle\colon \mathbb{F}_D \to \mathbb{B}_+$ *and* $\langle \sigma_C | \tau_C \rangle\colon \mathbb{F}_D \to \mathbb{C}_+$ *such that* $\langle f_+ | f_- \rangle \circ \langle \sigma_B | \tau_B \rangle = \langle g_+ | g_- \rangle \circ \langle \sigma_C | \tau_C \rangle$.

Lemma C.13 *Let* $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$ *be lattices with the de Morgan negation, and* $f\colon \mathbb{A} \to \mathbb{B}$ *and* $g\colon \mathbb{A} \to \mathbb{C}$ *injective homomorphisms. On the dual frame,*

(i) *for arbitrary* $F \in \mathcal{F}_B$, $G \in \mathcal{F}_C$,

$$C_B(F, \uparrow f[g_+(G)]) \iff C_A(f_+(F), g_+(G)) \iff C_C(\uparrow g[f_+(F)], G),$$

(ii) *for arbitrary* $I \in \mathcal{I}_B$, $J \in \mathcal{I}_C$,

$$D_B(I, \downarrow f[g_-(I)]) \iff D_A(f_-(I), g_-(J)) \iff D_C(\downarrow g[f_-(I)], J),$$

(iii) *for arbitrary* $F, G \in \mathcal{F}_B$,

$$C_B(F, G) \Longleftarrow C_A(f_+(F), f_+(G)) \iff C_C(\uparrow g[f_+(F)], \uparrow g[f_+(G)]),$$

(iv) *for arbitrary* $F, G \in \mathcal{F}_C$,

$$C_B(\uparrow f[g_+(F)], \uparrow f[g_+(G)]) \iff C_A(g_+(F), g_+(G)) \Longrightarrow C_C(F, G),$$

(v) *for arbitrary* $I, J \in \mathcal{I}_B$,

$$D_B(I, J) \Longleftarrow D_A(f_-(I), f_-(J)) \iff D_C(\downarrow g[f_-(I)], \downarrow g[f_-(J)]),$$

(vi) *for arbitrary* $I, J \in \mathcal{I}_C$,

$$D_B(\downarrow g[f_-(I)], \downarrow g[f_-(J)]) \iff D_A(f_-(I), f_-(J)) \Longrightarrow D_C(I, J).$$

Proposition C.14 *For all* $F \in X_D$, $I \in Y_D$, *we have*

$$\tilde{C}_D(F, I) \iff \begin{cases} \tilde{C}_B(F, I) & F \in \mathcal{F}_B, I \in \mathcal{I}_B \\ \tilde{C}_C(F, I) & F \in \mathcal{F}_C, I \in \mathcal{F}_C \\ \tilde{C}_B(F, \downarrow f[g_-(I)]) \ \& \ \tilde{C}_C(\uparrow g[f_+(F)], I) & F \in \mathcal{F}_B, I \in \mathcal{I}_C \\ \tilde{C}_B(\uparrow f[g_+(F)], I) \ \& \ \tilde{C}_C(F, \downarrow g[f_-(I)]) & F \in \mathcal{F}_C, I \in \mathcal{I}_B \end{cases}$$

$$\tilde{D}_D(F, I) \iff \begin{cases} \tilde{D}_B(F, I) & F \in \mathcal{F}_B, I \in \mathcal{I}_B \\ \tilde{D}_C(F, I) & F \in \mathcal{F}_C, I \in \mathcal{I}_C \\ \tilde{D}_B(F, \downarrow f[g_-(I)]) \ \& \ \tilde{D}_C(\uparrow g[f_+], I) & F \in \mathcal{F}_B, I \in \mathcal{I}_C \\ \tilde{D}_B(\uparrow f[g_+(F)], I) \ \& \ \tilde{D}_C(F, \downarrow g[f_-(I)]) & F \in \mathcal{F}_C, I \in \mathcal{I}_B \end{cases}$$

Theorem C.15 *Lattices with the de Morgan negation admit the amalgamation property.*

Corollary C.16 *Distributive lattices with the de Morgan negation, lattices with the de Morgan negation and adjoint unary modality and distributive lattices with the de Morgan negation and adjoint unary modality admit the amalgamation property.*

References

[1] Bílková, M., R. Horčík and J. Velebil, *Distributive substructural logics as coalgebraic logics over posets.*, in: *AiML*, 2012.

[2] Birkhoff, G., "Lattice Theory," AMS Colloquium Publications, 1973.

[3] Blackburn, P., M. de Rijke and Y. Venema, "Modal logic," Cambridge UP, 2002.

[4] Burris, S. and H. Sankappanavar, "A course in universal algebra," Springer-Verlag, 1981.

[5] Celani, S. A., *Classical modal de morgan algebras.*, Studia Logica **98** (2011).

[6] Chagrov, A. and M. Zakharyaschev, "Modal logic," Oxford Science Publications, 1997.

[7] Davey, B. and H. Priestley, "Introduction to Lattices and Order," Cambridge UP, 2002.

[8] Esakia, L., *Topological Kripke models*, Soviet Math. Dokl. **15** (1974).

[9] Galatos, N. and P. Jipsen, *Residuated frames with applications to decidability*, Trans. AMS **365** (2013).

[10] Gehrke, M., *Generalized Kripke frames*, Studia Logica **84** (2006).

[11] Gehrke, M. and J. Harding, *Bounded lattice expansions*, J. Alg. **239** (2001).

[12] Gehrke, M. and B. Jónsson, *Bounded distributive lattice expansions*, Math. Scand. **94** (2004).

[13] Gehrke, M., H. Nagahashi and Y. Venema, *A Sahlqvist theorem for distributive modal logic*, Ann. Pure Appl. Log. **131** (2005).
[14] Ghilardi, S. and G. Meloni, *Constructive canonicity in non-classical logics*, Ann. Pure Appl. Log. **86** (1997).
[15] Goldblatt, R., "Mathematics of modality," CSLI Publications, 1993.
[16] Goldblatt, R., *Fine's theorem on first-order complete modal logics*, in: *Metaphysics, Meaning, and Modality. Themes From Kit Fine*, Oxford UP, to appear .
[17] Goldblatt, R. and S. Thomason, *Axiomatic classes in propositional modal logic*, in: *Algebra and Logic*, Springer Berlin Heidelberg, 1975 .
[18] Grätzer, G., "General Lattice Theory," Birkhäuser Verlag, 1996.
[19] Johnstone, P. T., "Stone spaces," Cambridge UP, 1982.
[20] Jónsson, B., *Universal relational systems*, Math. Scand. **4** (1956).
[21] Jónsson, B., *On the canonicity of Sahlqvist identities*, Studia Logica **53** (1994).
[22] Kihara, H. and H. Ono, *Interpolation properties, Beth definability properties and amalgamation properties for substructural logics*, J. Log. Comp. **20** (2010).
[23] Kurz, A. and J. Rosicky, *The Goldblatt-Thomason theorem for coalgebras*, in: *Proc. CALCO'07* (2007).
[24] Maksimova, L., *Definability and interpolation in non-classical logics*, Studia Logica **82** (2006).
[25] Maksimova, L., *Amalgamation, interpolation, and implicit definability in varieties of algebras*, Proc. Steklov Institute Math. **278** (2012).
[26] Maksimova, L. and E. Orlowska, *The Beth property and interpolation in lattice-based algebras and logics*, Algebra and Logic **47** (2008).
[27] Metcalfe, G., F. Montagna and C. Tsinakis, *Amalgamation and interpolation in ordered algebras*, J. Alg. **402** (2014).
[28] Ono, H., *Interpolation and the Robinson property for logics not closed under the boolean operations*, Alg. Univ. **23** (1986).
[29] Paoli, F., "Substructural Logics: A Primer," Kluwer Academic Publishers, 2002.
[30] Pierce, R., "Introduction to the Theory of Abstract Algebras," Holt, Rinehart and Winston, 1968.
[31] Pigozzi, D., *Amalgamation, congruence-extension, and interpolation properties in algebras*, Alg. Univ. **1** (1971).
[32] Priestley, H. A., *Representation of distributive lattices by means of ordered stone spaces*, Bull. London Math. Soc. **2** (1970).
[33] Priestley, H. A., *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **24** (1972).
[34] Sambin, G. and V. Vaccaro, *Topology and duality in modal logic*, Ann. Pure Appl. Log. **37** (1988).
[35] Sano, K. and M. Ma, *Goldblatt-Thomason-style theorems for graded modal language.*, in: *AiML*, 2010.
[36] Stone, M. H., *The theory of representations for Boolean algebras*, Trans. AMS **40** (1936).
[37] Stone, M. H., *Topological representations of distributive lattices and Brouwerian logics*, Casopis Pest. Mat. Fys. **67** (1937).
[38] Suzuki, T., *The distributivity on bi-approximation semantics*, NDJFL Accepted.
[39] Suzuki, T., *Bi-approximation semantics for substructural logic at work.*, in: *AiML*, 2010.
[40] Suzuki, T., *Canonicity results of substructural and lattice-based logics*, Rev. Symb. Log. **4** (2011).
[41] Suzuki, T., *First-order definability for distributive modal substructural logics: an indirect method* (2011), draft: available on *http://www2.cs.cas.cz/˜suzuki/papers.html.*
[42] Suzuki, T., *On canonicity of poset expansions*, Alg. Univ. **66** (2011).
[43] Suzuki, T., *Morphisms on bi-approximation semantics.*, in: *AiML*, 2012.
[44] Suzuki, T., *A Sahlqvist theorem for substructural logic*, Rev. Symb. Log. **6** (2013).
[45] ten Cate, B., "Model Theory for Extended Modal Languages," Ph.D. thesis, University of Amsterdam (2005).
[46] Thomason, S., *Reduction of second-order logic to modal logic*, Math. Log. Quar. **21** (1975).
[47] Urquhart, A., *A topological representation theory for lattices*, Alg. Univ. **8** (1978).