The modal logic
of almost sure frame validities in the finite

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Abstract

A modal formula is almost surely valid in the finite if the probability that it is valid in a randomly chosen finite frame with n states is asymptotically 1 as n grows unboundedly. This paper studies the normal modal logic $\mathbf{ML}^{\mathsf{as}}$ of all modal formulae that are almost surely valid in the finite. Because of the failure of the zero-one law for frame validity in modal logic, the logic $\mathbf{ML}^{\mathsf{as}}$ extends properly the modal logic of the countable random frame $\mathbf{ML}^{\mathsf{r}}$, which was completely axiomatized in a 2003 paper by Goranko and Kapron. The present work studies the logic $\mathbf{ML}^{\mathsf{as}}$, provides a model-theoretic characterisation of its additional validities beyond those in $\mathbf{ML}^{\mathsf{r}}$, and raises some open problems and conjectures regarding the missing additional axioms over $\mathbf{ML}^{\mathsf{r}}$ and the explicit description of the complete axiomatisation of $\mathbf{ML}^{\mathsf{as}}$ which may turn out to hinge on difficult combinatorial-probabilistic arguments and calculations.

Keywords: modal logic, asymptotic probabilities, almost sure frame validities, 0-1 laws, countable random frame, bounded morphisms, axiomatisation

1 Introduction: asymptotic probabilities of logical formulae, 0-1 laws, and almost sure validities

[1] What is the probability that a given modally defined property of Kripke frames holds of a randomly chosen finite Kripke frame? What does it mean for such a property to be 'almost surely valid' in finite Kripke frames? These questions have a good intuitive sense and some potential practical importance (to be discussed briefly further) but, as currently stated, they are imprecise and cannot be answered in general. To make these questions precise, one has to (at least) specify a probability distribution over the class of all finite frames. There is no unique natural such distribution, for at least two reasons:

(i) Finite frames may be considered as *labelled structures* over a concrete finite domain, e.g. a finite set of natural numbers, or as abstract, *unlabelled*

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[1] This is a long, but hopefully useful for modal logicians, introduction to the topic.

structures, defined up to isomorphism. These two notions of a structure define two different sample spaces (but, see further).

(ii) In either case, the result is a countably infinite space of structures, which admits uncountably many probability distributions, and none of them can be uniform over all finite structures, because of the countable additivity of the probability measure.

To address the second point we will make some standard assumptions, viz that we first relativise the question above to all finite frames (in either sense) of a fixed size (number of possible worlds) n, all of which we assume to have the same domain $U_n = \{1, ..., n\}$. Then we consider a uniform distribution over all frames with that domain. The latter is equivalent to assuming that a random frame of size n is constructed by assigning with probability $1/2$ an arrow (transition) to each ordered pair of possible worlds in the domain U_n. The difference between the cases of labelled and unlabelled frames is that two randomly constructed frames over U_n that turn out isomorphic are considered the same as unlabelled frames, but not as labelled frames, unless they are identically labelled. Thus, one can define *labelled* and *unlabelled* probability [2] of a given frame property P to hold in a randomly chosen/constructed frame of size n. Then we consider the asymptotic behaviour of these probabilities and their limits as n increases without bound. If these exist, they define the *labelled (resp. unlabelled) (asymptotic) probability in the finite of the property P*. In particular, we define the respective probabilities $\mathrm{Pr}_l(\phi)$ and $\mathrm{Pr}_u(\phi)$ for the frame validity of any modal formula ϕ. It turns out, as shown in [10] (cf. also [18]), that *these probabilities coincide*. The reason for this is that

i) the property of a frame to be *rigid*, i.e. not to have non-trivial automorphisms, has asymptotic probability 1, and

ii) every rigid n-element frame has the same number, viz. $n!$, of non-isomorphic labellings, whence the equality of the asymptotic probabilities.

When $\mathrm{Pr}(P) = 1$ we say that P is *almost surely true in the finite*, while if $\mathrm{Pr}(P) = 0$ we say that P is *almost surely false in the finite*. These apply respectively to first-order (FO) sentences, in terms of the frame properties they define. Since in modal logic we traditionally talk about *validity* and *non-validity* of a modal formula in a given frame, rather than truth and falsity, we say that ϕ is *almost surely valid in the finite* when $\mathrm{Pr}(\phi) = 1$, while ϕ is *almost surely invalid in the finite* if $\mathrm{Pr}(\phi) = 0$. See the precise technical details in Section 2.

It turns out that many natural properties of frames are either almost surely true or almost surely false in the finite. In particular, this is the case for *all first-order definable properties*, which is the celebrated *Zero-one Law for first-order logic (FOL)*, proved first in [13] and independently (and quite differently) in [10]. In the latter, Fagin gave an insightful proof of the 0-1 law for the FO logic of arbitrary relational languages of finite signature, with the case of graphs (i.e. a single binary relation) being representative. Fagin related the almost

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[2] Note that computing the labelled probabilities is easy, whereas computing the unlabelled ones is difficult, because it only counts numbers of structures up to isomorphism.

sure truth of FO sentences on finite graphs to the FO theory of the co-called *countable random graph* $\mathfrak{R}$ (aka *Radó graph*) for which Gaifman had proved in [12] that $Th(\mathfrak{R})$ is ω-categorical and axiomatized by an infinite set EXT of *extension axioms*: sentences claiming that every n-tuple of elements in the structure can be extended to an $(n+1)$-tuple in all possible (consistent) ways. The probabilistic aspect of this result is rather surprising: assuming uniform distribution, any randomly constructed countable relational structure is, with probability 1, isomorphic to $\mathfrak{R}$ – thus justifying the term '*the countable random structure*'. That notion and Gaifman's results extend [3] to any finite relational language L. Fagin applied these results, by showing that every extension axiom is almost surely true in the finite. Thus, he provided two purely logical descriptions of the FO sentences σ of any relational language L that are almost surely true in the finite, viz. he showed that the following are equivalent for any FO sentence:

- σ is almost surely true in finite L-structures.
- σ follows from (finitely many) extension axioms in L.
- σ is true in the countable random structure for L.

Consequently, for every FO sentence σ, either it is in $Th(\mathfrak{R})$, hence almost surely true, or its negation is in $Th(\mathfrak{R})$, hence σ is almost surely false; whence the 0-1 law. Fagin's result, which, in particular, states that almost sure truth in the finite is equivalent to logical truth in the respective countable random structure, is often referred to as a *transfer theorem*. That result sparked much interest in the area of finite model theory and further extensive research on 0-1 laws. Such results were proved for several extensions of first-order logic, incl: the extension FOL+LFP of FOL with fixed point operators, in [6]; later subsumed by the 0-1 law for the infinitary logic over bounded number of variables $L^\omega_{\infty,\omega}$, in [23]; for some prefix-defined fragments of monadic second-order logic [22], where also strong relations were established between decidability and 0-1 laws of such fragments, etc. Most of these results were proved, like Fagin's result, by a means of suitable versions of the transfer theorem. For a popular and very readable exposition of 0-1 laws in FOL and some extensions see [18], and for such results in fragments of Σ^1_1 see [24].

On the other hand, the 0-1 law easily fails in the presence of a single constant in the language (consider a sentence saying that a given unary predicate is true at the element interpreting that constant) [4]. In second-order logic the 0-1 law

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[3] A model-theoretic aside: the random graph $\mathfrak{R}$ is a particular example of a countably infinite homogeneous structure that can be constructed as a Fraïssé limit of a family of sets of finite structures satisfying certain natural closure properties. There are deep model-theoretic connections between homogeneous (more generally, homogenizable) structures, extension properties, asymptotic probabilities and almost sure theories, that generalise Gaifman's and Fagin's results and enable further relativisations and refinements of 0-1 laws (and more generally, limit laws), which go beyond the scope of this paper, so I refer the reader e.g. to [25], [2], and further references therein.

[4] Still, it was proved in [30] that a first-order language with only unary functions does have a *limit law*, in sense that every sentence in that language has an asymptotic probability,

fails badly: think, for instance, of the property of having an odd number of elements. Even its monadic existential fragment $M\Sigma_1^1$ contains sentences with no asymptotic probability, as first proved by Kaufmann (see [27] for a very accessible account of Kaufmann's counterexample, and [28], [29] for stronger such results). While for prefix-defined fragments of $M\Sigma_1^1$ the boundary of 0-1 laws seems to be essentially delineated (see [24] for a survey), little is known in general on that for the full (monadic) second-order logic (M)SOL.

Now, what about modal logic? There are at least two basic relevant notions of modal validity: in Kripke models and in Kripke frames. In the former case the 0-1 law follows immediately from 0-1 law in FOL, since validity of a modal formula in a Kripke model is a FO property [19]. However, for the case of frame validity, which is an essentially universal monadic second-order ($M\Pi_1^1$) property, the 0-1 law cannot be claimed as a consequence of Fagin's theorem. Actually, that 0-1 law was claimed to be proved (by complex combinatorial-probabilistic calculations) in [19]. However, later it was proved in [14] that the respective transfer theorem fails for modal frame validity in the finite, which then cast a doubt on the 0-1 law, too. Indeed, that claim turned out to be wrong, as proved by Le Bars in [29], who provided there a very non-trivial counterexample. Soon thereafter, an erratum [20] was published, pointing out the mistake in [19].

A relatively independent from the 0-1 laws concept, which is in the focus of the present work, is the *almost sure theory* Th_L^{as} of a given logical language L, with respect to the notion of truth or validity under consideration – that is, the set consisting precisely of those sentences of L that are almost surely true (resp. valid) in the finite. Clearly, Th_L^{as} is a well-defined logical theory in a very traditional sense: it contains all valid sentences of the logic and is closed under all finitary rules of inference (as the semantic consequence preserves truth and the asymptotic probability measure is finitely additive). What can one say about the theory Th_L^{as}, in terms of axiomatization and deduction in it, decidability, model-theoretic properties, etc?

The cases of classes of FO structures where 0-1 laws holds by way of transfer theorem are generally easy to analyse thoroughly, because in these cases Th_L^{as} is precisely the (ω-categorical and complete, hence decidable) theory of the countable random structure (resp. universal homogeneous structure. cf. [25], [2], [1]). Curiously, as shown in [19], the respective modal logic of almost sure Kripke model validity, turned out to be already known, viz. *Carnap's modal logic* ([7]), the axioms of which are all modal formulae $\Diamond\phi$ where ϕ is a satisfiable propositional formula. (NB: this is not a normal modal logic, as it is not closed under substitutions.)

However, in cases where 0-1 law fails, or when it holds but not by a suitable transfer from a countable random structure, the question of logical character-isation and, in particular, axiomatization of the respective almost sure theory seems generally quite difficult, and very few such results are known. This ques-

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though in general not just 0 or 1.

tion arises, in particular, for the modal logic of almost sure frame validity in the finite, hereafter denoted by **ML**^{as}, and it is the topic of the present study.

What do we know about **ML**^{as} so far? Both much and little. We know that it is a normal modal logic, extending the normal modal logic **ML**^{r} of the *countable random frame* F^{r} (the frame analogue of the countable random graph $\mathfrak{R}$). The logic **ML**^{r} was studied and axiomatized in [14] where it was also proved to be strictly included in **ML**^{as}. (See details in Section 3.) What is not known yet is how the additional axioms, needed to extend the axiomatization of **ML**^{r} to a complete axiomatization of **ML**^{as}, look, and even whether **ML**^{as} is recursively axiomatizable. Such questions are inherently difficult, because **ML**^{as} lacks a priori explicit logical semantics in terms of truth and validity in a specific class of models or frames, but rather involves the class of all finite frames as a whole, so it is an *essentially global* concept. In this paper I study the logic **ML**^{as}, provide partial answers to these questions, and raise conjectures for their solutions.

Lastly, why should one be interested in **ML**^{as}, or in any almost sure theory? Besides being driven by a sheer intellectual curiosity, one can argue that knowing – or being able to identify – the almost sure truths in a given logic may have some practical advantages, e.g. for reducing the *average case* complexity of checking whether a given formula of that logic is valid, by first checking (or, guessing) whether it is almost surely valid. While such argument would probably not make good computational sense for the basic normal modal logic, it may do so for some extensions that are of higher computational complexity, have no finite model property, or are even undecidable. The idea certainly sounds quite reasonable in the case of FOL, which is not only undecidable, but satisfiability of FOL sentences in the finite is not even recursively enumerable (by Trachtenbrot's theorem), whereas the almost sure theory of FOL, being the same as $Th(\mathfrak{R})$, is decidable, and in fact only PSPACE-complete, as proved in [17]. Similar argument might work for other extensions of FOL and (M)SOL, too. As for **ML**^{as}, it seems still early to judge whether and what its practical importance may be. One immediate goal of this paper is to at least attract the attention of the modal logic community to this logic.

The paper is organized as follows. After this long introduction and brief technical preliminaries in Section 2, I introduce and compare the logics **ML**^{r} and **ML**^{as} in Section 3. In Section 4, I explore the question of axiomatization of **ML**^{as} and raise some open problems and conjectures. I conclude briefly in Section 5.

2 Preliminaries on modal logic, asymptotic probabilities and almost sure frame validity

Here I provide some technical details on the basic concepts in this paper introduced informally in the introduction. Besides, I assume that the reader has the necessary background in modal logic, including the notions of Kripke model, Kripke frame, truth and validity of modal formulae in these. Familiarity with some basics of the model theory of modal logic, incl. bounded-morphism (aka

p-morphism) and characteristic (aka Jankov-Fine) formulae would be helpful, but for the reader's convenience I have included the definitions here.

2.1 Bounded morphisms and characteristic formulae

Definition 2.1 Let $F_1 = \langle W_1, R_1 \rangle$ and $F_2 = \langle W_2, R_2 \rangle$ be frames. A mapping $h : W_1 \to W_2$ is a **bounded morphism from** F_1 **to** F_2 if the following hold:

(i) For all $x, y \in W_1$, if xR_1y then $h(x)R_2h(y)$.

(ii) For all $x \in W_1, t \in W_2$, if $h(x)R_2t$ then xR_1y for some $y \in W_1$ such that $h(y) = t$.

If h is onto, F_2 is called a **bounded-morphic image of** F_1.

Following a commonly used notation, I will often denote by $F_1 \twoheadrightarrow F_2$ the claim that F_2 is bounded-morphic image of F_1. An important fact: frame validity of modal formulae is preserved in bounded-morphic images, i.e., if $F_1 \models \phi$ and $F_1 \twoheadrightarrow F_2$, then $F_2 \models \phi$ (cf. [35], [5], and [15], which are also recommended general references on all other modal logic concepts used here).

The **universal modality** (interpreted by the full Cartesian square of the domain, cf. [16]) will be denoted by $[U]$, its dual, **existential modality** – by $\langle U \rangle$, and the basic normal logic **K** extended with $[U]$ – by $\mathbf{K_U}$. The language of **K** will be denoted by ML and the extended one with $[U]$ – by $\mathrm{ML_U}$.

Definition 2.2 ([14]) Let $F = \langle W, R \rangle$ be any finite frame with $W = \{w_1, \ldots, w_n\}$ and let $\{p_1, \ldots, p_n\}$ be fixed different propositional variables. The **characteristic formula**[5] **of** F **over** $\langle p_1, \ldots, p_n \rangle$ is the formula $\chi_F(p_1, \ldots, p_n) := \neg[U]\delta_F(p_1, \ldots, p_n)$, where δ_F is the 'modal diagram' of F:

$$\delta_F(p_1, \ldots, p_n) := \bigwedge_{i=1}^{n} \langle U \rangle p_i \wedge \bigvee_{i=1}^{n} p_i \wedge \bigwedge_{1 \leq i \neq j \leq n} (p_i \to \neg p_j) \wedge$$

$$\bigwedge_{1 \leq i,j \leq n} \{p_i \to \Diamond p_j | w_i R w_j\} \wedge \bigwedge_{1 \leq i,j \leq n} \{p_i \to \neg \Diamond p_j | \neg w_i R w_j\}.$$

When $\{p_1, \ldots, p_n\}$ are fixed or known from the context, I will write simply χ_F.

The following is a variation of a folklore fact (see Remark 2.5). I nevertheless sketch a proof, for the sake of the reader hitherto unfamiliar with it.

Lemma 2.3 ([8], [14]) *For every frame* G *and finite frame* F: $G \twoheadrightarrow F$ *iff* $G \not\models \chi_F$.

Proof. (Sketch) Suppose $G, V \not\models \chi_F$ for some valuation V. Then every point $y \in G$ satisfies exactly one variable $p_{i(y)}$ from $\{p_1, \ldots, p_n\}$. Furthermore, the mapping $f : G \longrightarrow F$ defined by $f(y) = w_{i(y)}$ is a surjective bounded morphism. Vice versa, if $f : G \longrightarrow F$ is a surjective bounded morphism, then the valuation V on G defined by $V(p_i) = f^{-1}(w_i)$ satisfies $\neg\chi_F$. $\qquad\square$

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[5] See Remark 2.5.

When both F and G are finite, the lemma above can be strengthened to the claims of the forthcoming Lemma 2.4, where $\mathbf{ML}(\mathsf{F})$ is the normal modal logic of the validities in the frame F and $\mathbf{K}_\mathsf{U} + \phi$ is the axiomatic extension of the modal logic $\mathbf{K}_\mathsf{U}$ with the axiom scheme ϕ.

Lemma 2.4 ([37]) *For any finite frames F, G the following are equivalent:*

(i) $\mathsf{G} \twoheadrightarrow \mathsf{F}$.

(ii) $\mathsf{G} \not\models \chi_\mathsf{F}$.

(iii) $\mathbf{ML}(\mathsf{G}) \subseteq \mathbf{ML}(\mathsf{F})$.

(iv) $\mathbf{K}_\mathsf{U} + \chi_\mathsf{F} \vdash \chi_\mathsf{G}$.

(v) *For every modal formula ϕ, if $\mathbf{K}_\mathsf{U} + \chi_\mathsf{G} \vdash \phi$ then $\mathbf{K}_\mathsf{U} + \chi_\mathsf{F} \vdash \phi$.*

Proof. Most of these equivalences are straightforward variations (involving $[\mathsf{U}]$) of widely known and frequently re-discovered facts, that can be found scattered elsewhere (cf. [8] for most of them). One implication is not completely trivial, viz. the implication from (i), (ii), or (iii) to (iv). As noted in [37], the implication from (ii) to (iv) holds in a more general form, viz. for any formula ϕ instead of χ_F, with essentially the same proof as for this special case which suffices for our purpose. As [37] has pointed out, the same claim was proved for intuitionistic logic in [33], itself referring to earlier works by Jankov. For further references and more, see the forthcoming Remark 2.5. Nevertheless, I provide here a proof sketch, to make the presentation relatively self-contained for readers not familiar with the more general theory, and also to help them see *why* this result holds, because it is of importance for the logic $\mathbf{ML}^{\mathsf{as}}$, as discussed in Section 4.

(i) $\Rightarrow$ (iv): Suppose $\mathsf{G} \twoheadrightarrow \mathsf{F}$ and fix a bounded morphism $h : \mathsf{G} \to \mathsf{F}$. Let $\mathsf{F} = \langle W_\mathsf{F}, R_\mathsf{F} \rangle$ with $W_\mathsf{F} = \{w_1, ..., w_n\}$ and $\mathsf{G} = \langle W_\mathsf{G}, R_\mathsf{G} \rangle$ with $W_\mathsf{G} = \{u_1, ..., u_m\}$. Suppose $\chi_\mathsf{F} = \chi_\mathsf{F}(p_1, \ldots, p_n) = \neg[\mathsf{U}]\delta_\mathsf{F}(p_1, \ldots, p_n)$ and $\chi_\mathsf{G} = \chi_\mathsf{G}(q_1, \ldots, q_m) = \neg[\mathsf{U}]\delta_\mathsf{G}(q_1, \ldots, q_m)$. Let us define a substitution σ_h on the propositional variables $p_1, ..., p_n$ as follows: for each $i = 1, ..., n$, $\sigma_h(p_i) := \bigvee_{\{j | h(u_j) = w_i\}} q_j$. Intuitively, if we regard $p_1, ..., p_n$ as nominals for $w_1, ..., w_n$ and $q_1, ..., q_m$ respectively as nominals for $u_1, ..., u_m$ then σ_h substitutes each p_i with the syntactic description of the inverse image of w_i in G under h. Now, let us apply σ_h to $\chi_\mathsf{F}(p_1, \ldots, p_n)$ and denote the resulting formula by $\xi_{\mathsf{G} \to \mathsf{F}}(q_1, ..., q_m)$. After simple equivalent transformations in $\mathbf{K}_\mathsf{U}$, for which there is no space here (but, see them illustrated on an example in the Appendix) $\xi_{\mathsf{G} \to \mathsf{F}}(q_1, ..., q_m)$ is transformed to a formula $\xi'_{\mathsf{G} \to \mathsf{F}}(q_1, ..., q_m)$ of the type $\neg[\mathsf{U}]\delta'_\mathsf{F}$, where δ'_F is a (long) conjunction with the following property, which can be seen by direct inspection: every conjunct in δ'_F is either identical, or follows propositionally (essentially by only applying $A \to B \models A \to (B \vee C)$) from a conjunct in $\delta_\mathsf{G}(q_1, \ldots, q_m)$. Thus, $\models \delta_\mathsf{G} \to \delta'_\mathsf{F}$, hence $\models [\mathsf{U}]\delta_\mathsf{G} \to [\mathsf{U}]\delta'_\mathsf{F}$, Thus, $\models \neg\chi_\mathsf{G} \to \neg\xi'_{\mathsf{G} \to \mathsf{F}}$, hence $\models \xi'_{\mathsf{G} \to \mathsf{F}} \to \chi_\mathsf{G}$. Equivalently, $\models \sigma_h(\chi_\mathsf{F}) \to \chi_\mathsf{G}$. Therefore, $\mathbf{K}_\mathsf{U} \vdash \sigma_h(\chi_\mathsf{F}) \to \chi_\mathsf{G}$, by completeness of $\mathbf{K}_\mathsf{U}$, hence $\mathbf{K}_\mathsf{U} + \chi_\mathsf{F} \vdash \chi_\mathsf{G}$. $\square$

Remark 2.5 The characteristic formulae defined here [6] are simplified (due to the availability of the universal modality) variations of the widely called Jankov-Fine formulae, cf. [8, Ch. 9.4]. Such formulae were introduced independently by V.A. Jankov in 1963 [21] (for the intuitionistic logic and finite Heyting algebras) and by D. de Jongh in his 1968 doctoral thesis (for the intuitionistic logic and finite intuitionistic Kripke frames). Their modal logic analogues were invented later, by K. Fine in 1974 [11] for modal logics extending S4 and finite modal algebras, and by W. Rautenberg [31] for modal logics extending K4 and finite Kripke frames. These formulae are at the core of the so called *splitting* techniques and results, initially developed by Jankov (for Heyting algebras), McKenzie (for splitting lattices), Blok, Rautenberg, Kracht, Wolter and others (for splitting lattices of modal logics); see [4] for references. In particular, such formulae were later used by Rautenberg [31] to axiomatize modal logics of finite frames, and generalised and applied further by Kracht [26] and by Zakharyaschev to what he called 'canonical formulae' in [8], used to axiomatize any normal extension of K4. For an algebraic treatment of canonical formulae, see [3].

Thus, the results listed in lemmas 2.3 and 2.4 are essentially not new and apply in a much more general setting [7].

2.2   Asymptotic probabilities and almost sure frame validity in the finite of modal formulae

The class of all finite frames will be denoted by $\mathcal{F}^{\mathsf{fin}}$. Given a modal formula ϕ, the $M\Pi_1^1$-formula expressing the frame condition defined by ϕ (or, any FO sentence equivalent to it, if that frame condition is first-order definable) will be denoted by $FC(\phi)$, and for any class of finite frames $\mathcal{F}$, the subclass of frames in $\mathcal{F}$ where ϕ is valid – by $\mathcal{F}(\phi)$. The set of positive integers is denoted by $\mathbb{N}$.

Given $n \in \mathbb{N}$, let $U_n := \{1, \ldots, n\}$. A **random (labelled) frame of size n** is a frame $\mathsf{F} = (U_n, R)$ obtained by random and independent assignments of truth/falsity of the binary relation R on every pair (x, y) from the set U_n, with probability for truth $p(n)$. The probability space on all n-element frames constructed as above will be denoted by $\mathcal{S}(n, p)$. In this paper I assume $p(n)$ to be the constant 0.5, so the random frame can be obtained by a random assignment of a binary relation on the domain, using uniform distribution. However, the results used and those obtained here hold likewise for any constant probability $p \in (0, 1)$ (cf. [10], [18]).

For any property of frames P, by $\mu_{n,p}(P)$ we denote the classical probability of P in $\mathcal{S}(n, p)$, i.e. the probability that P holds for a randomly constructed n-element frame. In particular, if ϕ is a first-order sentence or a modal formula, $\mu_{n,p}(\phi)$ will denote the probability for ϕ to be true (resp. valid) in a frame from $\mathcal{S}(n, p)$. Note that these are discrete probabilities since $\mathcal{S}(n, p)$ is finite.

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[6] I prefer to work with these formulae, rather than with their negations, as defined in [5], for reasons that will become clear in Section 4.

[7] Thanks to Evgeny Zolin [37] for pointing out these links and references.

Now, let us define $\mu_p(\phi) := \lim_{n\to\infty} \mu_{n,p}(\phi)$. If that limit exists, it will be called **the asymptotic probability** [8] **(in the finite)** of ϕ. As the probability is fixed here to $p = 0.5$ we will omit the subscript. We define likewise $\mu_p(P)$ and $\mu(P)$ for any frame property P. A property P is said to be *almost surely true in the finite* if $\mu(P) = 1$ and, respectively, *almost surely false* if $\mu(P) = 0$.

Definition 2.6 A modal formula ϕ is said to be **almost surely valid (in the finite)** if $\mu(\phi) = 1$; respectively, **almost surely invalid** if $\mu(\phi) = 0$.

Note that, by the 0-1 law for FOL, every first-order definable modal formula is either almost surely valid or almost surely invalid in the finite. Moreover [9], this also holds for all modal formulae that define FO property on *finite frames*. For instance, Gödel-Löb formula $\Box(\Box p \to p) \to \Box p$ defines in the finite the class of irreflexive and transitive *finite* frames (cf. [8]); thus it also satisfies the 0-1 law (being almost surely invalid).

Hereafter, for technical convenience we will assume w.l.o.g., that every finite frame of size n that we consider is defined over the set $U_n = \{1, \ldots, n\}$. Thus, the collection of all finite frames $\mathcal{F}^{\mathsf{fin}}$ can be regarded as a proper set. Now, given any set of finite frames $\mathcal{F}$ which contains at least one frame of every (sufficiently large) size n, the probabilities and concepts defined above readily relativise to $\mathcal{F}$, incl. a modal formula being **almost surely valid (resp. invalid) in** $\mathcal{F}$. Further, we say that a set of finite frames $\mathcal{F}$ has an **asymptotic measure 1 (resp. 0)** if the membership to that set has asymptotic probability 1 (resp. 0). An important observation is that for every set with asymptotic measure 1 the absolute and relativised probabilities are equal, hence the absolute and relativised notions of almost sure validity/invalidity coincide.

2.3   The countable random frame F^{r}

The construction of random frames by means of a random pairwise assignment of a binary relation with a given probability p for truth of the relation can be performed on *infinite* sets, too. The outcome of such a random construction on the set $\mathbb{N}$ of natural numbers is called a **countable random frame**. Using combinatorial-probabilistic argument, it was proved in [10] that any countable random relational structure satisfies with probability 1 an infinite sequence EXT of schemes of first-order sentences, called **extension axioms**. For every $n \in \mathbb{N}$, the extension axiom $(\mathsf{EXT})_n$ for frames (directed graphs with loops) is the conjunction of finitely many sentences, each involving a tuple of n distinct variables $\overline{x} = x_1, ..., x_n$ plus another variable y and parameterised by two subsets $I, J \subseteq U_n$, as follows:

$$(\mathsf{EXT})_n = \forall \overline{x} \exists y \left(\bigwedge_{i \neq j} x_i \neq x_j \to \left(\bigwedge_{i \in U_n} x_i \neq y \wedge T(y,y) \wedge \right.\right.$$

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[8] Note that this probability measure is not countably additive: $\mu(|\mathsf{F}| = n) = 0$ for every fixed n, while $\mu(\exists n (|\mathsf{F}| = n)) = 1$.

[9] Thanks to Evgeny Zolin [37] for this added remark.

$$\bigwedge_{i \in I} Rx_i y \ \wedge \bigwedge_{i \in U_n \setminus I} \neg Rx_i y \ \wedge \bigwedge_{i \in J} Ryx_j \wedge \bigwedge_{j \in U_n \setminus J} \neg Ryx_j \Bigg)\Bigg),$$

where $T(y,y)$ is either Ryy or $\neg Ryy$,

The extension axiom $(\mathsf{EXT})_n$, intuitively says that for every n different points in the frame there is a point which is related to and from each of those, and with itself, in any explicitly prescribed way. Note that if $m < n$ then $(\mathsf{EXT})_n$ implies $(\mathsf{EXT})_m$ on all frames of size at least n. Consequently, every finite set of extension axioms follows almost surely in the finite from a single extension axiom $(\mathsf{EXT})_n$ for a large enough n.

By a result of Gaifman [12] the theory EXT is consistent and ω-categorical, hence complete. The unique countable model F^r of EXT is called **the countable random frame**. Using Gaifman's results, Fagin proved (for graphs) in [10] the following **transfer theorem** that for any sentence ψ of FOL:

(i) If $\mathsf{F}^r \models \psi$ then $\mu(\psi) = 1$.

(ii) If $\mathsf{F}^r \not\models \psi$ then $\mu(\psi) = 0$.

This theorem immediately implies the 0-1 law for FOL for frames: every FO sentence is either almost surely true or almost surely false in the finite. Then, by compactness, every almost surely true FO sentence follows from finitely many extension axioms, hence from some instance of $(\mathsf{EXT})_n$. These claims apply likewise to all FO definable (in terms of frame validity) modal formulae.

3 The modal logics of the countable random frame and of almost sure validity

Here we will explore the two normal modal logics in the focus of this study. Most of the content of this section comes from [14], but is included here for the reader's convenience and self-containment of the paper.

Definition 3.1 $\mathbf{ML}^r$ is the modal logic of all formulae valid in F^r. $\mathbf{ML}^{as}$ is the modal logic of all formulae which are almost surely valid in the finite.

Proposition 3.2 ([14])

(i) $\mathbf{ML}^r$ *and* $\mathbf{ML}^{as}$ *are normal modal logics.*

(ii) *A modal formula* ϕ *is in* $\mathbf{ML}^r$ *iff* $FC(\phi)$ *follows from some extension axiom, hence every such formula is in* $\mathbf{ML}^{as}$. *Consequently,* $\mathbf{ML}^r \subseteq \mathbf{ML}^{as}$.

3.1 Complete axiomatization of $\mathbf{ML}^r$

First, we need some basic facts about the countable random frame F^r, which easily follow from the extension axioms (cf. [14]):

- F^r has a diameter 2: every point can be reached from any point (incl. itself) in 2 R-steps. Indeed, by an instance of the extension axiom scheme $(\mathsf{EXT})_3$: $\mathsf{F}^r \models \forall x \forall y \exists z (Rxz \wedge Rzy)$.
- Every point in F^r has infinitely many R-predecessors and infinitely many R-successors and every finite frame is embeddable as a subframe in F^r.

For some useful validities and non-obvious non-validities in $\mathbf{ML^r}$, see [14].

Since the extension axiom $(\mathsf{EXT})_3$ is almost surely true in the finite, the subset $\mathcal{F}^{\mathsf{d2}}$ of all finite frames of diameter 2 has asymptotic measure 1. This fact will be of crucial importance further, because it enables us to restrict attention from $\mathcal{F}^{\mathsf{fin}}$ to almost sure validity in $\mathcal{F}^{\mathsf{d2}}$ without extensional change of that notion: every property of finite frames is almost surely true in $\mathcal{F}^{\mathsf{fin}}$ iff it is almost surely true in $\mathcal{F}^{\mathsf{d2}}$. Here is the first important consequence. Note that the universal modality $[\mathsf{U}]$ and the existential modality $\langle\mathsf{U}\rangle$ are simply definable in every frame in $\mathcal{F}^{\mathsf{d2}}$:

$$[\mathsf{U}]p \equiv \Box\Box p, \quad \text{respectively} \quad \langle\mathsf{U}\rangle p \equiv \Diamond\Diamond p.$$

Therefore, these equivalences hold in almost every finite frame, and also in $\mathsf{F^r}$. Hereafter, to distinguish the primitives from the definable versions wherever necessary, I use $[\mathsf{U}]$ and $\langle\mathsf{U}\rangle$ as the standard universal/existential modalities, taken as primitives (extending ad hoc the basic modal language) and $[\mathbf{U}]$ and $\langle\mathbf{U}\rangle$ when referring to the operators in the basic modal language defined by the equivalences above. Respectively, $\mathbf{ML^r_U}$ and $\mathbf{ML^{as}_U}$ will denote the extensions of $\mathbf{ML^r}$ and $\mathbf{ML^{as}}$ to the language $\mathrm{ML_U}$. Note, that every formula ϕ of $\mathrm{ML_U}$ is trivially translated into a formula ϕ^d of the basic language ML by replacing all occurrences of $[\mathsf{U}]$ and $\langle\mathsf{U}\rangle$ respectively with $[\mathbf{U}]$ and $\langle\mathbf{U}\rangle$. The important property of that translation is that ϕ and ϕ^d are equivalent, hence equally valid, in every frame from $\mathcal{F}^{\mathsf{d2}}$, hence in almost every finite frame, which will suffice for our purposes. In particular, every axiom in $\mathbf{ML^{as}_U}$ generates its translated axiom in $\mathbf{ML^{as}}$, which will enable me to state most of the claims and results about $\mathbf{ML^{as}}$ in the language $\mathrm{ML_U}$ and for $\mathbf{ML^{as}_U}$, with the understanding that they apply accordingly to the logic $\mathbf{ML^{as}}$ of my primary interest.

Theorem 3.3 ([14]) *The following axiomatic system $\boldsymbol{Ax(}\mathbf{ML^r}\boldsymbol{)}$ is sound and complete for $\mathbf{ML^r}$ (recall that $[\mathbf{U}]$ and $\langle\mathbf{U}\rangle$ are the defined operators):*

($\mathbf{ML^r_1}$) K: $\Box(p \to q) \to (\Box p \to \Box q)$.

($\mathbf{ML^r_2}$) $[\mathbf{U}]p \to p$.

($\mathbf{ML^r_3}$) $[\mathbf{U}]p \to [\mathbf{U}]\Box p$.

($\mathbf{ML^r_4}$) $p \to [\mathbf{U}]\langle\mathbf{U}\rangle p$.

($\mathbf{ML^r_5}$) *Scheme MODEXT, consisting of the following axioms for each $n \in \mathbb{N}$:*

$$\mathsf{MODEXT}_n = \bigwedge_{k=1}^{n} \langle\mathbf{U}\rangle(p_k \wedge \Box q_k) \to \langle\mathbf{U}\rangle \bigwedge_{k=1}^{n}(\Diamond p_k \wedge q_k).$$

The first 4 axiom schemes above come from the axiomatization of $\mathbf{K_U}$ ([16]). It is easy to see that the axiom MODEXT_n is valid in a frame $\mathsf{F} \in \mathcal{F}^{\mathsf{d2}}$ iff for every n points $w_1, \ldots, w_n$ in F there is a point u that is R-reachable from each $w_1, \ldots, w_n$, and each of them is R-reachable from u. This holds for every finite frame, with $[\mathbf{U}]$ and $\langle\mathbf{U}\rangle$ replaced by the primitives $[\mathsf{U}]$, $\langle\mathsf{U}\rangle$. Thus, MODEXT is the modally definable approximation of the extension axioms EXT for FOL.

Proposition 3.4 ([14]) $\mathbf{ML^r}$ *has the finite model property and is decidable, but it is not finitely axiomatizable.*

Thus, $\mathbf{Ax(ML^r)}$ derives the 'well-behaved' formulae in $\mathbf{ML^{as}}$, viz. those that follow from the extension axioms of FOL. As we will see in Prop.4.1, these include all first-order definable formulae in $\mathbf{ML^{as}}$. What about the rest? Maybe, that is all and the logics $\mathbf{ML^r}$and $\mathbf{ML^{as}}$coincide? It turns out, the answer, rather surprisingly on the background of Fagin's Transfer theorem, is 'No', as shown further. To see that, we need to learn more about $\mathsf{F^r}$ and $\mathbf{ML^r}$.

3.2   Kernels in finite frames and in $\mathsf{F^r}$.

Every bounded morphic image F of a given frame G determines a kernel partition $\mathcal{P}_\mathsf{F}$ in G, defined as follows. Given a bounded morphism $h : \mathsf{G} \to \mathsf{F}$, where $\mathsf{F} = \langle W_\mathsf{F}, R_\mathsf{F} \rangle$ and $\mathsf{G} = \langle W_\mathsf{G}, R_\mathsf{G} \rangle$, the **kernel partition** $\mathcal{P}_\mathsf{F}(\mathsf{G})$ in G consists of the family of clusters $\{h^{-1}(w) \mid w \in W_\mathsf{F}\}$. Thus, $\mathcal{P}_\mathsf{F}(\mathsf{G})$ is generated by the equivalence relation $\sim_h$ in W_G, where $u \sim_h v$ holds iff $h(u) = h(v)$. It satisfies the following properties, determined by F and the definition of bounded morphism. For any two clusters $X = h^{-1}(x)$ and $Y = h^{-1}(y)$ in $\mathcal{P}_\mathsf{F}(\mathsf{G})$, either

  (i) for each $u \in X$ there is $v \in Y$ such that $uR_\mathsf{G}v$, (when $xR_\mathsf{F}y$ holds),

or

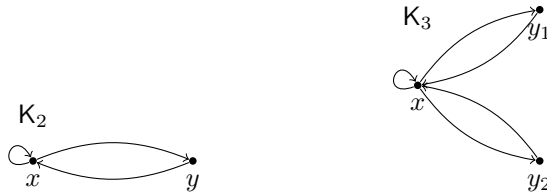 (ii) for no $u \in X$ there is $v \in Y$ such that $uR_\mathsf{G}v$ (when $xR_\mathsf{F}y$ does not hold).

Conversely, for every kernel partition in G generated by mapping $h : \mathsf{G} \to \mathsf{F}$ and satisfying the conditions (i) and (ii) above, the mapping h is a bounded morphism from G onto F.

Thus, kernel partitions are an equivalent, and often more visually intuitive way of describing bounded morphisms. Note that existence of a kernel partition with specific FO-definable properties in a frame, like those above, is a $M\Sigma_1^1$-property and, as stated in lemma 2.3, the existence of the kernel partition determined by F in a (randomly selected) frame G is characterised by the non-validity of the respective χ_F in that frame. Thus, using existence or non-existence of kernel partitions one can show the non-validity or validity in a given frame of various formulae that are not first-order definable. Here I will give two very simple examples, that will suffice to distinguish $\mathbf{ML^r}$ from $\mathbf{ML^{as}}$. Consider the following two frames:

  $\mathsf{K}_2 = \langle \{x, y\}, \{(x, x), (x, y), (y, x)\} \rangle$ and

  $\mathsf{K}_3 = \langle \{x, y_1, y_2\}, \{(x, x), (x, y_1), (x, y_2), (y_1, x), (y_2, x)\} \rangle$.

(Note that K_2 is a bounded morphic image of K_3.)

It turns out that the kernel partition $\mathcal{P}_{K_2}$ that K_2 generates in any frame

G for which K_2 is a bounded morphic image corresponds to the well-known notion of a **kernel** in digraphs (cf.[9] but, taking into account that frames are digraphs with loops), whereas the kernel partition that K_3 generates is called **double kernel** in [14] (see details there). Thus, the ML-translated characteristic formula $\chi^d_{K_2}$ (resp. $\chi^d_{K_3}$) is valid precisely in those frames in $\mathcal{F}^{\mathsf{d2}}$ which *do not* have kernels (resp. double kernels). Here are slightly simpler equivalent formulae, where the falsifying valuation of p, (resp. p and q), in any frame with a kernel (resp. double kernel) is that kernel (resp. each of the two sub-kernels).

NO-KER $= \langle \mathbf{U} \rangle (p \leftrightarrow \Diamond p)$,

NO-DKER $= \langle \mathbf{U} \rangle ((p \vee q) \wedge \Diamond (p \vee q)) \vee \langle \mathbf{U} \rangle (\neg (p \vee q) \wedge (\Box \neg p \vee \Box \neg q))$.

3.3 The finite frames of $\mathbf{ML}^{\mathsf{r}}$.

Even though $\mathbf{ML}^{\mathsf{r}}$ is defined as the logic of the single infinite frame F^{r}, it does have finite frames, as evident from Prop. 3.4. It turns out that they are very simple to describe, as precisely those finite frames that have a 'central point' – a point which is R-related to and from every point (incl. itself). Formally, given a frame $\mathsf{F} = \langle W, R \rangle$, a point $x \in W$ is a **central point** in F if Rxy and Ryx hold for every $y \in W$. The existence of a central point is forced by the axiom scheme MODEXT, and every frame with central point is easily seen to validate MODEXT. Note that both K_2 and K_3 above have central points.

Proposition 3.5 ([14], Lemma 2.4) *For every finite frame F the following are equivalent.*

(i) $\mathsf{F} \models \mathbf{ML}^{\mathsf{r}}$

(ii) F *has a central point.*

(iii) F *is a bounded-morphic image of F^{r}.*

(iv) $\mathsf{F}^{\mathsf{r}} \not\models \chi_{\mathsf{F}}$.

(v) F *can be obtained from F^{r} by filtration.*

In particular, K_2 and K_3 are bounded morphic images of F^{r}, hence both χ_{K_2} and χ_{K_3} fail in F^{r}, i.e. F^{r} has a kernel and a double kernel.

Corollary 3.6 *For every finite frame G without central point: $\mathsf{F}^{\mathsf{r}} \models \chi_{\mathsf{G}}$, and hence existence of kernel partition $\mathcal{P}_{\mathsf{G}}$ is almost surely false in the finite.*

Thus, Corollary 3.6 provides plenty of (generally) non-first-order definable modal formulae in $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}}$, respectively in $\mathbf{ML}^{\mathsf{r}}$.

Here is the main technical result in [14], proved by a non-trivial combinatorial-analytic estimation of the expected number of double kernels in a random finite frame from $\mathcal{F}^{\mathsf{d2}}$.

Theorem 3.7 ([14]) *Existence of a double kernel is almost surely false in finite frames. Consequently, χ_{K_3} is almost surely valid, hence it is in $\mathbf{ML}^{\mathsf{as}}$.*

Thus, $\chi_{\mathsf{K}_3} \in \mathbf{ML}^{\mathsf{as}}$ but $\chi_{\mathsf{K}_3} \notin \mathbf{ML}^{\mathsf{r}}$, hence the inclusion $\mathbf{ML}^{\mathsf{r}} \subset \mathbf{ML}^{\mathsf{as}}$ is proper. Also, Fagin's transfer theorem fails for frame validity in modal logic.

The technique used in the proof of Theorem 3.7 did not help the authors of [14] to prove the same results for single kernels and χ_{K_2}, and these were left as open questions there. They were proved a little later by Le Bars in [29]. He also proved there that the 0-1 law fails for frame validity in modal logic, by showing that a modified kernel property, defined by the formula

$$\mathsf{MODAL\text{-}KERNEL} : \quad \neg p \wedge q \wedge \Box\Box((p \vee q) \to \neg\Diamond(p \vee q)) \to \Diamond\Box\neg p$$

has no asymptotic probability in the finite.

4   On the axiomatization of the modal logic $\mathbf{ML}^{\mathsf{as}}$

What axioms are needed to add to $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}})$ in order to axiomatize completely $\mathbf{ML}^{\mathsf{as}}$? We explore this question here, starting with some useful observations. For technical convenience, most of the results will be stated for $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$, instead of $\mathbf{ML}^{\mathsf{as}}$, but they are readily translated to $\mathbf{ML}^{\mathsf{as}}$. I will denote by $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}})$ the axiomatic system $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}})$ where $[\mathsf{U}]$ and $\langle\mathsf{U}\rangle$ are replaced by the primitives $[\mathsf{U}]$ and $\langle\mathsf{U}\rangle$, with the relevant axioms added (cf. [16]).

4.1   Towards understanding the logic $\mathbf{ML}^{\mathsf{as}}$

Proposition 4.1

  (i) *Every first-order definable modal formula which is in $\mathbf{ML}^{\mathsf{as}}$ is also in $\mathbf{ML}^{\mathsf{r}}$.*

 (ii) *([14]) Every modal formula ϕ in $\mathbf{ML}^{\mathsf{as}}$ that defines a purely universal frame condition $FC(\phi)$ is valid.*

Proof. (i) If $\phi \in \mathbf{ML}^{\mathsf{as}}$ and ϕ is first order definable, then $FC(\phi)$ is almost surely true in the finite, hence it follows from an extension axiom. Therefore, $\phi \in \mathbf{ML}^{\mathsf{r}}$, by Proposition 3.2[ii].

(ii) Suppose ϕ is not valid. Then $\neg\phi$ is satisfiable in a finite frame F. The satisfiability of $\neg\phi$ is an existential property, hence preserved in extensions. As F is embeddable in F^{r}, $\neg\phi$ is satisfiable there, too, which contradicts (i).   □

So, the missing axioms are neither first-order definable, nor purely universal.

More notation: Given a (possibly infinite) set of frames $\mathcal{F}$, a set of formulae Γ, and a formula ϕ, we denote by $\Gamma \models^{\mathsf{fr}}_{\mathcal{F}} \phi$ the claim that ϕ is valid in every frame from $\mathcal{F}$ in which all formulae of Γ are valid [10]. When $\mathcal{F}$ is the class of all frames I will write simply $\Gamma \models^{\mathsf{fr}} \phi$ and when $\mathcal{F} = \mathcal{F}^{\mathsf{fin}}$ I will write $\Gamma \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$. When $\Gamma = \{\psi\}$, I will write just $\psi \models^{\mathsf{fr}}_{\mathcal{F}} \phi$, respectively $\psi \models^{\mathsf{fr}} \phi$ and $\psi \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$.

I denote by $\mathrm{BM}^{-1}(\mathsf{F})$ the set of finite frames G (over $\mathbb{N}$) such that $\mathsf{G} \twoheadrightarrow \mathsf{F}$.

Note that $\mathbf{ML}^{\mathsf{as}}$ (resp. $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$) is closed under $\models^{\mathsf{fr}}_{\mathsf{fin}}$. Now, what are the finite frames for $\mathbf{ML}^{\mathsf{as}}$ like? (Note that they are the same as those for $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$.) A partial answer follows, that essentially employs for our purpose more general facts listed in Lemma 2.4.

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[10] This consequence relation is generally not arithmetically definable, hence not recursively axiomatizable, as first shown in [34] by reduction from logical consequence in second-order logic, cf. also discussion in [35]. However, we are only interested here in very special cases of that consequence relation, so no general results can be assumed a priori to hold.

Proposition 4.2 *For every finite frame* F *and a modal formula* $\phi \in \mathrm{ML}_\mathsf{U}$:

(i) $\mathsf{F} \not\models \phi$ *iff* $\phi \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{F}$ *iff* $\phi \models^{\mathsf{fr}} \chi_\mathsf{F}$.

(ii) $\mathsf{F} \models \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$ *iff* $\chi_\mathsf{F} \notin \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$.

Proof. (i) Let $\mathsf{F} \not\models \phi$. Then $\mathsf{G} \not\models \phi$ for every frame G such that $\mathsf{G} \twoheadrightarrow \mathsf{F}$. Therefore, for every G such that $\mathsf{G} \models \phi$, it follows that $\mathsf{G} \not\twoheadrightarrow \mathsf{F}$, hence $\mathsf{G} \models \chi_\mathsf{F}$. Thus, $\mathsf{F} \not\models \phi$ implies $\phi \models^{\mathsf{fr}} \chi_\mathsf{F}$. Further, $\phi \models^{\mathsf{fr}} \chi_\mathsf{F}$ obviously implies $\phi \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{F}$. Lastly, if $\phi \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{F}$ then $\mathsf{F} \not\models \phi$ because $\mathsf{F} \not\models \chi_\mathsf{F}$.

(ii) By contraposition, if $\chi_\mathsf{F} \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$ then $\mathsf{F} \not\models \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$ because $\mathsf{F} \not\models \chi_\mathsf{F}$. Conversely, take $\phi \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$. If $\mathsf{F} \not\models \phi$ then $\phi \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{F}$ by (i), so $\chi_\mathsf{F} \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$. $\quad\square$

From Proposition 4.2 we immediately obtain the following useful fact.

Corollary 4.3 *For any finite frame* F *and* $\phi \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$, *if* $\mathsf{F} \not\models \phi$ *then* $\chi_\mathsf{F} \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$.

Proposition 4.4 *For any finite frames* F, G:

(i) $\mathsf{G} \twoheadrightarrow \mathsf{F}$ *iff* $\chi_\mathsf{F} \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{G}$.

(ii) *Moreover, if* $\chi_\mathsf{F} \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{G}$ *then* $\mathbf{K}_\mathsf{U} + \chi_\mathsf{F} \vdash \chi_\mathsf{G}$.

Proof.

(i) Directly from Lemma 2.3 and Proposition 4.2.(i)

(ii) By (i) and Lemma 2.4. Also, χ^d_G is derived in the same way in the respectively axiomatized version $\mathbf{K}_{[\mathsf{U}]} + \chi^d_\mathsf{F}$ in ML as sketched in Lemma 2.4.$\square$

As noted in the proof of Lemma 2.4, Proposition 4.4(ii) holds likewise for any formula ϕ instead of χ_F, but the greater generality seems to be of no use in our case, as all conjectured axioms of $\mathbf{ML}^{\mathsf{as}}_\mathsf{U}$ over $\mathbf{ML}^{\mathsf{r}}_\mathsf{U}$ are of the type χ_F, so the respective conjectured axioms of $\mathbf{ML}^{\mathsf{as}}$ over $\mathbf{ML}^{\mathsf{r}}$ are of the type χ^d_F.

4.2 Towards axiomatizing the logics $\mathbf{ML}^{\mathsf{as}}_\mathsf{U}$ and $\mathbf{ML}^{\mathsf{as}}$

From the observations made so far we see that natural candidates for additional axioms of $\mathbf{ML}^{\mathsf{as}}_\mathsf{U}$ over $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}}_\mathsf{U})$ are the almost surely valid formulae of the type χ_F for frames F with central point (recall Corollary 3.6). So, let $\mathcal{C}$ be the set of all finite frames with a central point. Note that $\mathcal{C} \subseteq \mathcal{F}^{\mathsf{d2}}$. Let

$$\Xi^{\mathsf{as}}_\mathsf{U} := \{\chi_\mathsf{F} \mid \mathsf{F} \in \mathcal{C} \text{ and } \chi_\mathsf{F} \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}\}.$$

Then, let Ξ^{as} be the set of translated axioms in ML.

The following conjecture, stated in two equivalent versions, seems natural.

Conjecture 4.5 $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}}_\mathsf{U}) \cup \Xi^{\mathsf{as}}_\mathsf{U}$ *axiomatizes* $\mathbf{ML}^{\mathsf{as}}_\mathsf{U}$.
$\quad$ *Respectively,* $\mathbf{Ax}(\mathbf{ML}^{\mathsf{r}}) \cup \Xi^{\mathsf{as}}$ *axiomatizes* $\mathbf{ML}^{\mathsf{as}}$.

Let us first make an encouraging observation in support of that conjecture. I state the version for $\mathbf{ML}^{\mathsf{as}}_\mathsf{U}$; the one for $\mathbf{ML}^{\mathsf{as}}$ is completely analogous.

Proposition 4.6 *For any* $\phi \in \mathbf{ML}^{\mathsf{as}}_\mathsf{U}$:

(i) $\Xi^{\mathsf{as}}_\mathsf{U}(\phi) \models^{\mathsf{fr}}_\mathcal{C} \phi$, *where* $\Xi^{\mathsf{as}}_\mathsf{U}(\phi) = \Xi^{\mathsf{as}}_\mathsf{U} \cap \{\chi_\mathsf{F} \mid \phi \models^{\mathsf{fr}}_{\mathsf{fin}} \chi_\mathsf{F}\}$.

(ii) $\mathbf{ML}^{\mathsf{r}}_\mathsf{U} \cup \Xi^{\mathsf{as}}_\mathsf{U} \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$.

Proof. Take $\phi \in \mathbf{ML}_U^{as}$ and any finite frame F such that $\mathsf{F} \nVdash \phi$.

Then, by Corollary 4.3, $\chi_\mathsf{F} \in \mathbf{ML}_U^{as}$. Now:

(i) If $\mathsf{F} \in \mathcal{C}$ then $\mathsf{F} \nVdash \Xi_U^{as}(\phi)$ because $\mathsf{F} \nVdash \chi_\mathsf{F}$ and $\chi_\mathsf{F} \in \Xi_U^{as}(\phi)$.

(ii) Consider two cases:

– If $\mathsf{F} \in \mathcal{C}$ then $\mathsf{F} \nVdash \mathbf{ML}_U^r \cup \Xi_U^{as}$ by (i).

– if $\mathsf{F} \notin \mathcal{C}$ then $\mathsf{F} \nVdash \mathbf{ML}_U^r \cup \Xi_U^{as}$ because $\mathsf{F} \nVdash \mathbf{ML}_U^r$.

Thus, in either case, $\mathsf{F} \nVdash \mathbf{ML}_U^r \cup \Xi_U^{as}$.

By contraposition, if $\mathsf{F} \vDash \mathbf{ML}_U^r \cup \Xi_U^{as}$ then $\mathsf{F} \vDash \phi$.

Hence, $\mathbf{ML}_U^r \cup \Xi_U^{as} \vDash_{fin}^{fr} \phi$. □

The proposition above provides a model-theoretic characterisation of the additional validities of $\mathbf{ML}_U^{as}$ (resp. $\mathbf{ML}^{as}$), beyond those in $\mathbf{ML}_U^r$ (resp. $\mathbf{ML}^r$).

Still, there are two major issues with proving Conjecture 4.5, if true at all:

(i) How to identify the axioms in Ξ_U^{as}?

(ii) How to prove the completeness?

On the first question, let us first make the task a little easier by noting that, due to Corollary 4.3, we only need to identify the axioms χ_F for the *minimal* frames $\mathsf{F} \in \mathcal{C}$ such that $\chi_\mathsf{F} \in \mathbf{ML}_U^{as}$, where 'minimal' is in sense (cf. Proposition 4.4) that there is no $\mathsf{F}' \in \mathcal{C}$ such that $\chi_{\mathsf{F}'} \in \mathbf{ML}_U^{as}$ and $\mathsf{F} \in \mathrm{BM}^{-1}(\mathsf{F}')$, but $\mathsf{F} \ncong \mathsf{F}'$. Equivalently, we are looking for the maximal sets of frames of the type $\mathrm{BM}^{-1}(\mathsf{F})$ for $\mathsf{F} \in \mathcal{C}$ which have asymptotic measure 0. For that, the membership in $\mathrm{BM}^{-1}(\mathsf{F})$ should almost surely contradict (EXT); equivalently, χ_F should follow from (EXT). Being a $M\Pi_1^1$-condition, by compactness χ_F should then follow from some $(\mathsf{EXT})_n$, hence some extension axiom η_F should fail in all frames in $\mathrm{BM}^{-1}(\mathsf{F})$. To ensure the latter, one should naturally look for η_F that fails in F but is preserved in bounded morphic images, so it must fail in all frames from $\mathrm{BM}^{-1}(\mathsf{F})$. A classic result by van Benthem [35, Thm 15.11] characterises the first-order sentences in the language with $=$ and R that are preserved by bounded morphisms as precisely those that are equivalent to ones constructed from atomic formulae, $\top$, and $\bot$ using $\wedge, \vee, \exists, \forall$, and restricted universal quantification $\forall z(Ryz \to \ldots)$ for $z \neq y$. By looking at the syntactic shape of (EXT), one can see that only few of them satisfy the description above. Still, they generate infinitely many axioms, as the next proposition shows.

Proposition 4.7 *There is a subset Φ of infinitely many axioms in Ξ_U^{as}, none of which follows in terms of $\vDash_{fin}^{fr}$ from all others.*

Proof. Consider the sequence of frames $\{\mathsf{F}_n\}_{n \in \mathbb{N}}$ defined as follows.

Let $\mathsf{F}_n = \langle W_n, R_n \rangle$ where $W_n = \{0, 1, \ldots n\}$, and
$R_n = \{(k, 0) \mid k \in W_n\} \cup \{(0, k) \mid k \in W_n\} \cup \{(k, k+1) \mid k = 1, \ldots, n-1\}$.

Now, let $\Phi = \{\chi_{\mathsf{F}_n} \mid n \in \mathbb{N}\}$.

Clearly, 0 is a central point, so each F_n is in $\mathcal{C}$. Next, each χ_{F_n} is in $\mathbf{ML}_U^{as}$. Indeed, note that $\mathrm{BM}^{-1}(\mathsf{F}_n)$ has an asymptotic measure 0 because $\forall x \exists y(Rxy \wedge \neg Ryy)$ is an instance of the extension axiom $(\mathsf{EXT})_1$ that fails in each F_n, hence in every $\mathsf{G} \in \mathrm{BM}^{-1}(\mathsf{F}_n)$, because it is preserved in bounded morphic images. Lastly, each F_n is minimal in the sense above, as it is easy to

see that neither of them has proper bounded morphic images (different from F_n and F_0). Let $\Phi^{-n} = \{\chi_{\mathsf{F}_m} \mid 0 < m, m \neq n\}$. Then $\Phi^{-n} \not\models^{\mathsf{fr}}_{\mathsf{fin}} \chi_{\mathsf{F}_n}$ for each $n \in \mathbb{N}$, because $\mathsf{F}_n \models \Phi^{-n}$, while $\mathsf{F}_n \not\models \chi_{\mathsf{F}_n}$.   □

The translated set Φ^d provides likewise infinitely many independent axioms in Ξ^{as}. The proposition above makes the following conjecture very likely, but in order to prove it we need either a provably complete infinitary axiomatization of $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ over $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}}$ or a proof that $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ is not recursively axiomatizable.

Conjecture 4.8   *The logic* $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ *is not finitely axiomatizable over* $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}}$. *Respectively,* $\mathbf{ML}^{\mathsf{as}}$ *is not finitely axiomatizable over* $\mathbf{ML}^{\mathsf{r}}$.

It is conceivable that additional axioms from $\Xi^{\mathsf{as}}_{\mathsf{U}}$ may be needed to add to Φ for the axiomatization of $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$, and likewise for $\mathbf{ML}^{\mathsf{as}}$. To speculate a little on these, note first that the extension axioms η that fit van Benthem's syntactic description for preservation under bounded morphisms can have at most one universally quantified variable, i.e., be of the type $\forall x \exists y$. Furthermore, for η to fail in some $\mathrm{BM}^{-1}(\mathsf{F})$ such that $\chi_{\mathsf{F}} \in \Xi^{\mathsf{as}}_{\mathsf{U}}$, there must be a negative atom, which can only be $\neg Ryy$. This restricts the syntactic possibilities for η to just a few, that can be easily described. Thereafter, the frames $\mathsf{F} \in \mathcal{C}$ for which such η fails in $\mathrm{BM}^{-1}(\mathsf{F})$, hence the further axioms $\chi_{\mathsf{F}} \in \Xi^{\mathsf{as}}_{\mathsf{U}}$ that are generated by them, are also easily describable. And, now the big unknown is: are these *all* axioms that are missing, or are there more, that are not identifiable in such way? If these are all, then the logic $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ is recursively (even if not finitely) axiomatizable over $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}}$ and even stands a chance to be decidable, too, like $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}}$ is; likewise for $\mathbf{ML}^{\mathsf{as}}$. Otherwise, the problem with the identification of all missing axioms is very likely going beyond logic. Indeed, the question for which frames $\mathsf{F} \in \mathcal{C}$ it holds that $\chi_{\mathsf{F}} \in \mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ may then hinge on rather difficult combinatorial-probabilistic calculations, as results in [14] and [29], as well as an empirical study in [32], have indicated.

To sum up: it is currently unknown whether the set $\Xi^{\mathsf{as}}_{\mathsf{U}}$ is even recursively enumerable, though I would conjecture that it is. But even if that is the case, the question whether $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}} \cup \Xi^{\mathsf{as}}_{\mathsf{U}}$ axiomatizes $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ remains open. The core of the problem is that we cannot conclude $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}} \cup \Xi^{\mathsf{as}}_{\mathsf{U}} \vdash \phi$ from $\mathbf{ML}^{\mathsf{r}}_{\mathsf{U}} \cup \Xi^{\mathsf{as}}_{\mathsf{U}} \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$, because we have no recursive axiomatization of $\models^{\mathsf{fr}}_{\mathsf{fin}}$ in ML_{U} (and I currently do not know if one exists). It seems a currently open question whether and when $\Gamma \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$ implies derivability over a suitably recursively axiomatized base logic, beyond the special case established in Proposition 4.4. This is currently unknown to me even for the special case when $\chi_{\mathsf{F}} \models^{\mathsf{fr}}_{\mathsf{fin}} \phi$, where $\chi_{\mathsf{F}} \in \mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$. Likewise for $\models^{\mathsf{fr}}_{\mathsf{fin}}$ in ML.

An important related question [11] is whether the logic $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ (resp. $\mathbf{ML}^{\mathsf{as}}$) is *Kripke complete*, i.e. whether it is the modal logic of any class of Kripke frames. If so, it is certainly the modal logic of the class of *all* (not necessarily finite) frames F such that $\mathsf{F} \models \mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$. Equivalently, the question is whether every non-validity of $\mathbf{ML}^{\mathsf{as}}_{\mathsf{U}}$ is refuted in some (finite, or not) frame F such that

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[11] Raised by Evgeny Zolin [37].

$\mathsf{F} \models \mathbf{ML}_U^{as}$; likewise for $\mathbf{ML}^{as}$. While this is rather plausible, it does not seem to follow obviously from what is currently known about $\mathbf{ML}_U^{as}$ (resp. $\mathbf{ML}^{as}$), so I would add it to the list of currently open problems.

Finally, briefly on the question of proving the completeness of the axiomatization of $\mathbf{ML}_U^{as}$ and the respective translation for $\mathbf{ML}^{as}$, if and when it is identified. It is very easy to see that they would be equally complete. This problem seems not less challenging, because – unlike the axioms from the scheme MODEXT – the truly second-order axioms, like those from Ξ_U^{as}, are likely not to be canonical, as the kernel partitions generated in the canonical model by the axioms $\chi_\mathsf{F} \in \Xi_U^{as}$ need not be syntactically definable there. Still, how difficult that problem is can only be assessed when all axioms are explicitly known.

On this note, I leave the question of establishing a provably complete axiomatization of $\mathbf{ML}^{as}$, while better understood now, still open.

5   Concluding remarks and further challenges

Besides the open questions regarding the axiomatization of $\mathbf{ML}^{as}$, stated above, many other related problems arise. To mention just one such generic question: given a class $\mathcal{K}$ of Kripke frames, what is the modal logic of almost sure validities of $\mathcal{K}$? The case when the modal logic of $\mathcal{K}$ satisfies the 0-1 law seems to be considerably easier (though, by no means trivial) than the case of $\mathcal{K} = \mathcal{F}^{fin}$ studied here, as it then boils down to axiomatizing the modal logic of the respective analogue of countable random frame, relativised to the class $\mathcal{K}$, if it exists. Quite promising recent results of that type were announced in [36] for the provability logic and two versions of Grzegorczyk logic.

Further open problems arise when going beyond modal logic, to the full $M\Sigma_1^1$ and $M\Pi_1^1$ on graphs, digraphs, and other important classes of structures. Axiomatizing the almost sure theories of these may very likely lead to quite complicated combinatorial-probabilistic computations proving almost sure existence (resp., non-existence) of kernel partitions. In general, little is known about these so far and the challenge to understand them is wide open.

Acknowledgments

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Appendix

Example. This example illustrates the details of the derivation sketched in the proof of Lemma 2.4. Consider the frames K_2 and K_3 defined in Section 3.

For convenience, I will rename the points in K_3:

$\mathsf{K}_2 = \langle \{x, y\}, \{(x, x), (x, y), (y, x)\} \rangle$ and

$\mathsf{K}_3 = \langle \{u, v_1, v_2\}, \{(u, u), (u, v_1), (u, v_2), (v_1, u)(v_2, u)\} \rangle$.

The (slightly simplified) characteristic formulae of these are as follows:

$$\chi_{\mathsf{K}_2}(p_x, p_y) := \neg[\mathsf{U}]\big(\langle\mathsf{U}\rangle p_x \wedge \langle\mathsf{U}\rangle p_y \wedge (p_x \vee p_y) \wedge (p_x \to \neg p_y) \wedge$$
$$(p_x \to \Diamond p_x) \wedge (p_x \to \Diamond p_y) \wedge (p_y \to \Diamond p_x) \wedge (p_y \to \Box \neg p_y)\big).$$

$$\chi_{\mathsf{K}_3}(q_u, q_{v_1}, q_{v_2}) := \neg[\mathsf{U}]\big(\langle\mathsf{U}\rangle q_u \wedge \langle\mathsf{U}\rangle q_{v_1} \wedge \langle\mathsf{U}\rangle q_{v_2} \wedge (q_u \vee q_{v_1} \vee q_{v_2}) \wedge$$
$$(q_u \to \neg q_{v_1}) \wedge (q_u \to \neg q_{v_2}) \wedge (q_{v_1} \to \neg q_{v_2}) \wedge$$
$$(q_u \to \Diamond q_u) \wedge (q_u \to \Diamond q_{v_1}) \wedge (q_u \to \Diamond q_{v_2}) \wedge (q_{v_1} \to \Diamond q_u) \wedge (q_{v_2} \to \Diamond q_u) \wedge$$
$$(q_{v_1} \to \Box\neg q_{v_1}) \wedge (q_{v_1} \to \Box\neg q_{v_2}) \wedge (q_{v_2} \to \Box\neg q_{v_1}) \wedge (q_{v_2} \to \Box\neg q_{v_2})\big).$$

It is easy to check that the mapping $h : \mathsf{K}_3 \to \mathsf{K}_2$ defined by $h(u) = x, h(v_1) = h(v_2) = y$ is a bounded morphism.

The substitution σ_h defined in the proof of Lemma 2.4 acts as follows:

$$\sigma_h(p_x) := q_u, \quad \sigma_h(p_y) := (q_{v_1} \vee q_{v_2}).$$

Respectively,

$\xi_{\mathsf{K}_3 \to \mathsf{K}_2}(q_u, q_{v_1}, q_{v_2}) = \sigma_h(\chi_{\mathsf{K}_2}(p_x, p_y)) =$

$\neg[\mathsf{U}]\big(\langle\mathsf{U}\rangle q_u \wedge \langle\mathsf{U}\rangle(q_{v_1} \vee q_{v_2}) \wedge (q_u \vee (q_{v_1} \vee q_{v_2})) \wedge (q_u \to \neg(q_{v_1} \vee q_{v_2})) \wedge$

$(q_u \to \Diamond q_u) \wedge (q_u \to \Diamond(q_{v_1} \vee q_{v_2})) \wedge ((q_{v_1} \vee q_{v_2}) \to \Diamond q_u) \wedge ((q_{v_1} \vee q_{v_2}) \to \Box\neg(q_{v_1} \vee q_{v_2}))\big).$

After simple equivalent transformations in $\mathbf{K}_\mathsf{U}$, it is transformed to

$$\xi'_{\mathsf{K}_3 \to \mathsf{K}_2}(q_u, q_{v_1}, q_{v_2}) =$$

$\neg[\mathsf{U}]\big(\langle\mathsf{U}\rangle q_u \wedge (\langle\mathsf{U}\rangle q_{v_1} \vee \langle\mathsf{U}\rangle q_{v_2}) \wedge (q_u \vee q_{v_1} \vee q_{v_2}) \wedge (q_u \to \neg q_{v_1}) \wedge (q_u \to \neg q_{v_2}) \wedge$

$(q_u \to \Diamond q_u) \wedge (q_u \to (\Diamond q_{v_1} \vee \Diamond q_{v_2})) \wedge (q_{v_1} \to \Diamond q_u) \wedge (q_{v_2} \to \Diamond q_u) \wedge$

$(q_{v_1} \to \Box\neg q_{v_1}) \wedge (q_{v_1} \to \Box\neg q_{v_2}) \wedge (q_{v_2} \to \Box\neg q_{v_1}) \wedge (q_{v_2} \to \Box\neg q_{v_2})\big).$

By a direct inspection, one can see that every conjunct inside the scope of $\neg[\mathsf{U}]$ in $\xi'_{\mathsf{K}_3 \to \mathsf{K}_2}(q_u, q_{v_1}, q_{v_2})$ is either identical, or follows propositionally from a conjunct inside the scope of $\neg[\mathsf{U}]$ in $\chi_{\mathsf{K}_3}(q_u, q_{v_1}, q_{v_2})$.

Therefore, $\models \neg\chi_{\mathsf{K}_3}(q_u, q_{v_1}, q_{v_2}) \to \neg\xi'_{\mathsf{K}_3 \to \mathsf{K}_2}(q_u, q_{v_1}, q_{v_2})$,

hence $\models \xi'_{\mathsf{K}_3 \to \mathsf{K}_2}(q_u, q_{v_1}, q_{v_2}) \to \chi_{\mathsf{K}_3}(q_u, q_{v_1}, q_{v_2})$.

Equivalently, $\models \sigma_h(\chi_{\mathsf{K}_2}(p_x, p_y)) \to \chi_{\mathsf{K}_3}(q_u, q_{v_1}, q_{v_2})$.