Actuality in Intuitionistic Logic

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Abstract

In "Empirical Negation", Michael De takes up the challenge of extending intuitionism from mathematical discourse to empirical discourse, and to this end, he introduced an expansion of intuitionistic propositional logic obtained by adding a unary connective called empirical negation. The intuitive reading of empirical negation of A is: it is not the case that there is sufficient evidence at present that A. From a model-theoretic perspective, cashed out in terms of pointed Kripke models for intuitionistic logic, empirical negation of A is forced at a point iff A is not forced at the base point. Then, a simple calculation reveals that double empirical negation of A is forced at a point iff A is forced at the base point. In other words, double empirical negation can be seen as an actuality operator explored by John N. Crossley, Lloyd Humberstone, Martin Davies and more. Based on these, we introduce an expansion of intuitionistic propositional logic obtained by adding actuality. Our main results include sound and strongly complete axiomatization as well as comparisons to closely related systems such as Global Intuitionistic Logic of Satoko Titani as well as LGP of Matthias Baaz.

Keywords: Actuality, Empirical Negation, Global Intuitionistic Logic, Intuitionistic Modal Logic, Completeness, Sequent Calculus.

1 Introduction

In the literature, there are various expansions of intuitionistic logic, based on a number of different motivations. One of the motivations that seems to be popular is to extend intuitionism from mathematical discourse to empirical discourse. To this end, the role played by proof within the mathematical discourse will be played by warrant/evidence/verification/etc. within the empirical discourse.

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The main background of this paper, namely "Empirical Negation" by Michael De, is a contribution within the above motivation, following the discussions led by Michael Dummett and Neil Tennant.[3] De's focus in [16] was on negation, and expanded the language of intuitionistic propositional logic by adding empirical negation. The intuitive reading of empirical negation of A is that "it is not the case that there is sufficient evidence at present that A". Model theoretically, this is formulated with the help of pointed Kripke models for intuitionistic logic. More specifically, empirical negation of A is forced at a point iff A is not forced at the base point. Following De's paper, a Hilbert-style axiomatization was given for the expansion of intuitionistic logic in [17], and in [18], a comparison of empirical negation and classical negation was carried out over subintuitionistic logic, introduced and explored by Greg Restall in [38].

Now, a simple calculation reveals that double empirical negation of A is forced at a point iff A is forced at the base point. In other words, double empirical negation can be seen as an actuality operator explored by John N. Crossley, Lloyd Humberstone, Martin Davies and more. This then gives rise to a natural question of exploring an expansion of intuitionistic logic enriched by actuality operator.[4] The aim of this paper is twofold, and the first aim is to address this question. Although the notion of actuality has been discussed in classical settings (see our brief overview below), no attempts are known, to the best of authors' knowledge, to discuss the notion of actuality based on intuitionistic logic.[5] However, it is of significant interest how we can incorporate the notion along the philosophical foundation of Dummett-Tennant-De. The second aim is to draw some connections to closely related systems. This enables us to uncover links with other logical concepts, such as empirical negation and globality. For this purpose, we shall adopt a language that includes absurdity and therefore negation. Nonetheless we shall also observe how the notion of actuality is independent of that of negation, which is an advantage over an approach that defines actuality in terms of empirical negation. Before moving further, let us briefly review some of the developments in the literature related to our aim.

Actuality The notion of actuality has been studied in modal logic for a long time, and various conceptualizations have been introduced. Even at an early period, Crossley, Humberstone and Davies [14,15] already introduced two different actuality operators, A and $\mathscr{F}$ (read *fixedly*). Each model $\mathcal{M}$ has a distinguished world w^*, and $\mathsf{A}\varphi$ is true at w iff φ is true at w^*. On the other hand, $\mathscr{F}\varphi$ is true at w iff for every model $\mathcal{M}'$, φ is true at $\mathcal{M}'$'s distinguished world w'. These two operators represent different intuitions about whether 'the actual world' is necessarily so or not.

[3] Another interesting direction following Dummett is to discuss not only verification, but also falsification. This path is explored by Andreas Kapsner in great detail in [33].

[4] HO would like to thank Patrick Blackburn for pointing this out and encouraging him to pursue this direction at AiML 2016 in person.

[5] Note that there is a recent work on the notion of actuality based on relevant logics by Shawn Standefer in [42].

Another example for flexible actuality is that of Dominic Gregory [29], whose semantics includes a mapping @, which maps a world w to *its* actual world $@(w)$ in the same model, with a couple of conditions on @. This in particular allows there being more than one actual worlds in a model. [6]

Baaz' LGP and Titatni's GI Recall that Gödel-Dummett logic, introduced in [24] by Dummett, is an extension of intuitionistic logic with the linearity axiom:

$$(A \to B) \lor (B \to A). \tag{Lin}$$

Semantically, this logic is characterised by linear Kripke frames, which enables us to see it as a fuzzy logic in intuitionistic setting.

Then, in [2], Matthias Baaz expanded Gödel-Dummett logic by an additional operator, $\triangle$, which he called a *projection* modality, also later known as Baaz' Delta. The resulting logic is named **LGP**. Semantically, a formula of the form $\triangle A$ attains either the value 1 or 0, and it attains the value 1 iff A has the value 1. [7] In other words, $\triangle A$ is true iff A is valid in the model. Baaz in the same paper also mentions an operator equivalent to empirical negation in the setting of Gödel-Dummett logic (cf. [2, p.33]).

A logic closely related to **LGP** of Baaz is Satoko Titani's *global intuitionistic logic* **GI**, introduced in [46]. This logic, formulated as a sequent calculus, is defined by adding to intuitionistic logic an operator $\Box$ of *globalization*. From a semantic perspective, in terms of algebraic semantics, $\Box$ has the same interpretation as $\triangle$. There is also a fuzzy extension of **GI** called *fuzzy intuitionistic logic with globalization* **GIF** proposed by Gaisi Takeuti and Satoko Titani in [45], whose propositional fragment is equivalent to **LGP** (cf. [13, Remark 3]).

Note here that global intuitionistic logic can be regarded as an instance of intuitionistic modal logics which are equipped with at least two accessibility relations, intuitionistic $\leq$ and modal R. This is studied since 1948 by Frederic B. Fitch in [25], followed by Arthur N. Prior's [37] and R. A. Bull's papers [11,12], and later major developments include [7,8,21,36,39,40,41,48]. Some close connections of global intuitionistic logic to intuitionistic modal logics are studied by Hiroshi Aoyama in [1].

Based on these, this paper is structured as follows. We first introduce intuitionistic logic with actuality operator, called $\mathbf{IPC}^{@}$, both in terms of semantics and proof system, in §2. Then, in §3, we establish the soundness and strong completeness of $\mathbf{IPC}^{@}$. This is followed by a comparison of $\mathbf{IPC}^{@}$ with related systems in §4 and §5. More specifically, $\mathbf{IPC}^{@}$ is compared with intuitionistic logic with empirical negation as well as logic of actuality of Crossley and Humberstone in §4. We then turn to compare $\mathbf{IPC}^{@}$ with **LGP** of Baaz and **GI** of Titani in §5. The paper concludes with a brief summary of our main results and some directions for future research in §6.

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6 For more discussions on actuality, see, for instance, [26,32,43].

7 This condition is closely related to the framework of *simple monadic Heyting algebra* which is explored in detail in [6] by Guram Bezhanishvili. We would like to thank one of the referees for directing our attention to this paper.

2 Semantics and Proof system

After setting up the language, we first present the semantics, and then turn to the proof system.

Definition 2.1 The language $\mathcal{L}_\perp^@$ consists of a finite set $\{@, \perp, \wedge, \vee, \rightarrow\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}_\perp^@$. We denote a formula of $\mathcal{L}_\perp^@$ by A, B, C, etc. and a set of formulas of $\mathcal{L}_\perp^@$ by Γ, Δ, Σ, etc.

2.1 Semantics

Definition 2.2 A model for the language $\mathcal{L}_\perp^@$ is a quadruple $\langle W, g, \leq, V\rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); $\leq$ is a partial order on W with g being the least element; and $V : W \times \mathsf{Prop} \rightarrow \{0, 1\}$ an assignment of truth values to state-variable pairs with the condition that $V(w_1, p) = 1$ and $w_1 \leq w_2$ only if $V(w_2, p) = 1$ for all $p \in \mathsf{Prop}$ and all $w_1, w_2 \in W$. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$;
- $I(w, \perp) = 0$;
- $I(w, @A) = 1$ iff $I(g, A) = 1$;
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$;
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$;
- $I(w, A \rightarrow B) = 1$ iff for all $x \in W$: if $w \leq x$ and $I(x, A) = 1$ then $I(x, B) = 1$.

Semantic consequence is now defined in terms of truth preservation at g: $\Gamma \models A$ iff for all models $\langle W, g, \leq, I\rangle$, $I(g, A) = 1$ if $I(g, B) = 1$ for all $B \in \Gamma$.

2.2 Proof System

Definition 2.3 The system $\mathbf{IPC}^@$ consists of the following axiom schemata and rules of inference:

$$\perp\rightarrow A \quad \text{(Ax0)}$$
$$A\rightarrow(B\rightarrow A) \quad \text{(Ax1)}$$
$$(A\rightarrow(B\rightarrow C))\rightarrow((A\rightarrow B)\rightarrow(A\rightarrow C)) \quad \text{(Ax2)}$$
$$(A \wedge B)\rightarrow A \quad \text{(Ax3)}$$
$$(A \wedge B)\rightarrow B \quad \text{(Ax4)}$$
$$(C\rightarrow A)\rightarrow((C\rightarrow B)\rightarrow(C\rightarrow(A \wedge B))) \quad \text{(Ax5)}$$
$$A\rightarrow(A \vee B) \quad \text{(Ax6)}$$
$$B\rightarrow(A \vee B) \quad \text{(Ax7)}$$
$$(A\rightarrow C)\rightarrow((B\rightarrow C)\rightarrow((A \vee B)\rightarrow C)) \quad \text{(Ax8)}$$
$$@(A\rightarrow B)\rightarrow(@A\rightarrow @B) \quad \text{(Ax9)}$$
$$@A\rightarrow A \quad \text{(Ax10)}$$
$$@A\rightarrow @@A \quad \text{(Ax11)}$$
$$@A \vee (@A\rightarrow B) \quad \text{(Ax12)}$$
$$@(A\vee B)\rightarrow(@A\vee @B) \quad \text{(Ax13)}$$
$$\frac{A}{@A} \quad \text{(RN)}$$
$$\frac{A \quad A\rightarrow B}{B} \quad \text{(MP)}$$

Finally, we write $\Gamma \vdash A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of $\mathbf{IPC}^@$; (iii) is obtained by (MP) or (RN) from formulas preceding it in sequence.

Remark 2.4 We will refer to the subsystem of $\mathbf{IPC}^{@}$ which consists of axiom schemata (Ax1)–(Ax8) and a rule of inference (MP) as $\mathbf{IPC}^{+}$.

Note that the deduction theorem does not hold with respect to $\rightarrow$ in $\mathbf{IPC}^{@}$. However, we do have a deduction theorem in a slightly different form, and our goal now is to prove this. For this purpose, we begin with some preparations.

Fact 2.5 *The following formulas are provable in* $\mathbf{IPC}^{+}$ *and thus in* $\mathbf{IPC}^{@}$.

$$A \rightarrow A \qquad (1)$$
$$(A \vee B) \rightarrow (B \vee A) \qquad (2)$$
$$(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)) \qquad (3)$$
$$(A \vee B) \rightarrow ((B \rightarrow C) \rightarrow (A \vee C)) \qquad (4)$$
$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C) \qquad (5)$$

Now, we can prove one direction of the deduction theorem.

Proposition 2.6 *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, *if* $\Gamma, A \vdash B$ *then* $\Gamma \vdash @A \rightarrow B$.

Proof. By the induction on the length n of the proof of $\Gamma, A \vdash B$. If $n = 1$, then we have the following three cases.

- If B is one of the axioms of $\mathbf{IPC}^{@}$, then we have $\vdash B$. Therefore, by (Ax1), we obtain $\vdash @A \rightarrow B$ which implies the desired result.
- If $B \in \Gamma$, we have $\Gamma \vdash B$, and thus we obtain the desired result by (Ax1).
- If $B = A$, then by (Ax10), we have $@A \rightarrow B$ which implies the desired result.

For $n > 1$, then there are two additional cases to be considered.

- If B is obtained by applying (MP), then we will have $\Gamma, A \vdash C$ and $\Gamma, A \vdash C \rightarrow B$ lengths of the proof of which are less than n. Thus, by induction hypothesis, we have $\Gamma \vdash @A \rightarrow C$ and $\Gamma \vdash @A \rightarrow (C \rightarrow B)$, and by (Ax2) and (MP), we obtain $\Gamma \vdash @A \rightarrow B$ as desired.
- If B is obtained by applying (RN), then $B = @C$ and we will have $\Gamma, A \vdash C$ length of the proof of which is less than n. Thus, by induction hypothesis, we have $\Gamma \vdash @A \rightarrow C$. By (Ax9) and (RN), we have $\Gamma \vdash @@A \rightarrow @C$. Another application of (Ax9) gives us $\Gamma \vdash @A \rightarrow @C$, i.e. $\Gamma \vdash @A \rightarrow B$ as desired.

This completes the proof. □

Proposition 2.7 *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, *if* $\Gamma \vdash @A \rightarrow B$ *then* $\Gamma, A \vdash B$.

Proof. By the assumption $\Gamma \vdash @A \rightarrow B$. Moreover, we have $\Gamma, A \vdash @A$ by (RN). Thus, we obtain the desired result by (MP). □

By combining Propositions 2.6 and 2.7, we obtain the following theorem.

Theorem 2.8 *For all* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, $\Gamma, A \vdash B$ *iff* $\Gamma \vdash @A \rightarrow B$.

Let us mention a corollary of the deduction theorem which shall prove vital for the completeness theorem.

Corollary 2.9 *If* $A \vdash C$ *and* $B \vdash C$, *then* $A \vee B \vdash C$.

Proof. If $A \vdash C$ and $B \vdash C$, then by deduction theorem $\vdash @A \rightarrow C$ and $\vdash @B \rightarrow C$. Thus $\vdash (@A \vee @B) \rightarrow C$; now use (Ax13) to deduce $\vdash @(A \vee B) \rightarrow C$. By deduction theorem again, we conclude $A \vee B \vdash C$. □

3 Soundness and Completeness

We now turn to prove the soundness and the strong completeness. The proofs are in large part analogous to those of [17,18] which build on [38].

3.1 Soundness

Theorem 3.1 *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \vdash A$ *then* $\Gamma \models A$.

Proof. By induction on the length of the proof. □

3.2 Key notions for completeness

In below we introduce some concepts used in the argument for completeness.

(i) $\Sigma \vdash_\pi A$ iff $\Sigma \cup \Pi \vdash A$.
(ii) Σ is a Π-*theory* iff:
  (a) if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$.
  (b) if $\vdash_\pi A \to B$ then (if $A \in \Sigma$ then $B \in \Sigma$).
(iii) Σ is *prime* iff (if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$).
(iv) $\Sigma \vdash_\pi \Delta$ iff for some $D_1, \ldots, D_n \in \Delta$, $\Sigma \vdash_\pi D_1, \ldots, D_n$.
(v) $\vdash_\pi \Sigma \to \Delta$ iff for some $C_1, \ldots, C_n \in \Sigma$ and $D_1, \ldots, D_m \in \Delta$:

$$\vdash_\pi C_1 \wedge \cdots \wedge C_n \to D_1 \vee \cdots \vee D_m.$$

(vi) Σ is Π-*deductively closed* iff (if $\Sigma \vdash_\pi A$ then $A \in \Sigma$).
(vii) $\langle \Sigma, \Delta \rangle$ is a Π-*partition* iff:
  (a) $\Sigma \cup \Delta = \mathsf{Form}$
  (b) $\nvdash_\pi \Sigma \to \Delta$
(viii) Σ is *non-trivial* iff $A \notin \Sigma$ for some formula A.

Lemma 3.2 *If* Γ *is a non-empty* Π-*theory, then* $\Pi \subseteq \Gamma$.

Proof. Take $A \in \Pi$. Then, we have $\Pi \vdash A$. Now since Γ is non-empty, take any $C \in \Gamma$. Then, by (Ax1), we obtain $\Pi \vdash C \to A$, i.e. $\vdash_\pi C \to A$. Thus, combining this together with $C \in \Gamma$ and the assumption that Γ is Π-theory, we conclude that $A \in \Gamma$. □

3.3 Extension lemmas

We now introduce a number of lemmas concerning extensions of sets with various properties. For the proofs, cf. [17, §2] which are based on [38].

Lemma 3.3 *If* $\langle \Sigma, \Delta \rangle$ *is a* Π-*partition then* Σ *is a prime* Π-*theory.*

Lemma 3.4 *If* $\nvdash_\pi \Sigma \to \Delta$ *then there are* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* $\langle \Sigma', \Delta' \rangle$ *is a* Π-*partition.*

Corollary 3.5 *Let* Σ *be a non-empty* Π-*theory,* Δ *be closed under disjunction, and* $\Sigma \cap \Delta = \emptyset$. *Then there is* $\Sigma' \supseteq \Sigma$ *such that* $\Sigma' \cap \Delta = \emptyset$ *and* Σ' *is a prime* Π-*theory.*

Lemma 3.6 *If* $\Sigma \nvdash \Delta$ *then there are* $\Sigma' \supseteq \Sigma$ *and* $\Delta' \supseteq \Delta$ *such that* $\langle \Sigma', \Delta' \rangle$ *is a partition, and* Σ' *is deductively closed.*

We shall mention that the proof of this lemma relies on Corollary 2.9, and consequently on (Ax13). Hence the same argument cannot be directly imitated by a logic lacking this axiom, such as **GIPC** in §5.

Corollary 3.7 *If* $\Sigma \nvdash A$ *then there are* $\Pi \supseteq \Sigma$ *such that* $A \notin \Pi$*,* Π *is a prime* Π*-theory and is* Π*-deductively closed.*

3.4 Counter-example lemma

Lemma 3.8 *If* Δ *is a* Π*-theory and* $A \to B \notin \Delta$*, then there is a prime* Π*-theory* Γ*, such that* $A \in \Gamma$ *and* $B \notin \Gamma$*.*

Proof. Let $\Sigma = \{C : A \to C \in \Delta\}$. We check that Σ is a Π-theory. First, if $C_1, C_2 \in \Sigma$ then $A \to C_1, A \to C_2 \in \Delta$. Since $\vdash (A \to C_1 \wedge A \to C_2) \to (A \to (C_1 \wedge C_2))$ and Δ a Π-theory, we have $A \to (C_1 \wedge C_2) \in \Delta$. Thus $C_1 \wedge C_2 \in \Sigma$. Now suppose that $\vdash_\pi C \to D$ and $C \in \Sigma$. Then $\vdash_\pi (A \to C) \to (A \to D)$ and $A \to C \in \Delta$; so $A \to D \in \Delta$ and hence $D \in \Sigma$.

Clearly $A \in \Sigma$ and $B \vee \cdots \vee B \notin \Sigma$. Based on this, let Δ' be the closure of $\{B\}$ under disjunction. Then $\Sigma \cap \Delta' = \emptyset$, and the result follows from Corollary 3.5.□

Note that, since Σ is non-trivial, the obtained Γ is non-trivial as well.

3.5 Completeness

We are now ready to prove the completeness.

Theorem 3.9 *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \models A$ *then* $\Gamma \vdash A$*.*

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash A$. Then, by Corollary 3.7, there is a $\Pi \supseteq \Gamma$ such that Π is a prime Π-theory, Π-deductively closed and $A \notin \Pi$. Define the interpretation $\mathfrak{A} = \langle X, \Pi, \leq, I \rangle$, where $X = \{\Delta : \Delta \text{ is a non-trivial prime } \Pi\text{-theory}\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and I is defined thus. For every state Σ and propositional parameter p:

$$I(\Sigma, p) = 1 \text{ iff } p \in \Sigma$$

We show by induction on B that $I(\Sigma, B) = 1$ iff $B \in \Sigma$. We concentrate on the cases where B has the form $@C$ and $C \to D$.

When $B \equiv @C$, if $I(\Sigma, @C) = 1$ then by definition $I(\Pi, C) = 1$. By IH this is equivalent to $C \in \Pi$. Then $C \in \Sigma$ as $\Pi \subseteq \Sigma$ and also $\vdash_\pi @C$ by (RN). Hence $\vdash_\pi C \to @C$ by (Ax1). Now as Σ is a Π-theory, $C \in \Sigma$ implies $@C \in \Sigma$. For the other direction, it suffices to show $@C \in \Sigma$ implies $C \in \Pi$. First note $@C \vee @C \to D \in \Pi$ for all D because Π is Π-deductively closed. Then as Π is a prime theory, for each D either $@C \in \Pi$ or $@C \to D \in \Pi$. That is, either $@C \in \Pi$ or for all D, $@C \to D \in \Pi$. But if the latter, because Σ is a Π-theory, that $\Pi \subseteq \Sigma$ and $\vdash (@C \wedge (@C \to D)) \to D$ imply $D \in \Sigma$ for all D. This contradicts the non-triviality of Σ, so it must be that $@C \in \Pi$. But then $C \in \Pi$ by (Ax10) and Π being a Π-theory.

When $B \equiv C \to D$, by IH $I(\Sigma, C \to D) = 1$ iff for all Δ s.t. $\Sigma \subseteq \Delta$, if $C \in \Delta$ then $D \in \Delta$. Hence it suffices to show that this latter condition is equivalent to $C \to D \in \Sigma$. For the forward direction, we argue by contraposition; so assume $C \to D \notin \Sigma$. Then by Lemma 3.8 we can find find a non-trivial prime

Π-theory Σ' such that $C \in \Sigma'$ but $D \notin \Sigma'$. For the backward direction, assume $C \to D \in \Sigma$ and $C \in \Delta$ for any Δ s.t. $\Sigma \subseteq \Delta$. Then $C \to D \in \Delta$ as well, and so $D \in \Delta$ since Δ is a Π-theory.

It now suffices to observe that $B \in \Pi$ for all $B \in \Gamma$ and $A \notin \Pi$, which in view of the above means $\Gamma \not\models A$. This completes the proof. □

4 Comparison (I)

In this section, we give some comparisons of $\mathbf{IPC}^{@}$ with $\mathbf{IPC}^{\sim}$, as given in [16,17], and **S5A** of Crossley and Humberstone, as given in [14].

4.1 Empirical negation and actuality

$\mathbf{IPC}^{\sim}$ employs the language $\mathcal{L}^{\sim} = \{\sim, \wedge, \vee, \to\}$, and is axiomatized as follows.

Definition 4.1 The system $\mathbf{IPC}^{\sim}$ consists of (Ax1)-(Ax8), (MP) and the following axiom schemata and a rule of inference:

$$A \vee {\sim}A \quad \text{(N1)}$$
$${\sim}A \to ({\sim}{\sim}A \to B) \quad \text{(N2)}$$
$$\frac{A \vee B}{{\sim}A \to B} \quad \text{(RP)}$$

We shall denote the deducibility in $\mathbf{IPC}^{\sim}$ by $\vdash_{\sim}$. The deduction theorem holds in the form $\Gamma, A \vdash_{\sim} B$ iff $\Gamma \vdash_{\sim} {\sim}{\sim}A \to B$ (cf. [17, Theorem 2.1]). The corresponding semantics for $\mathbf{IPC}^{\sim}$ is almost identical to that of $\mathbf{IPC}^{@}$, except for the valuation of formulas of the form ${\sim}A$, which is given by:

$$I(w, {\sim}A) = 1 \text{ iff } I(g, A) = 0.$$

Remark 4.2 Note that Kosta Došen, in papers [20,22,23], considered negative modalities in models with two relations between worlds, like the models for intuitionistic modal logics, and one of them has the following condition:

$$w \Vdash {\sim}A \text{ iff for some } w' \in W, wRw' \text{ and } w' \nVdash A.$$

Although the modal relation R is absorbed by the intuitionistic relation $\leq$, empirical negation can be seen as having this type of valuation. Interestingly, Došen considered this sort of absorption is a necessary condition for a negative modality to be deemed a 'negation' (cf. [23, p.85]). For a recent discussion on negation understood as negative modality, see [4,5,19]. See also [31] for an up-to-date survey on negation, as well as negative modalities, in general.

Remark 4.3 There are two more things to note with this valuation. First, intuitionistic $\perp$ and consequently the intuitionistic negation $\neg$ is definable in $\mathbf{IPC}^{\sim}$ by setting $\perp := {\sim}(A \to A)$. Second, since $I(w, {\sim}{\sim}A)$=1 iff $I(g, A)$=1, we see @ is also definable in $\mathbf{IPC}^{\sim}$ by $@A := {\sim}{\sim}A$.

A natural question then would be whether we can go the opposite direction, namely, is $\sim$ definable in $\mathbf{IPC}^{@}$? It turns out that this also holds. Since we have $\perp$ in $\mathcal{L}^{@}_{\perp}$, we readily see: $I(w, \neg@A)$=1 iff $I(g, A)$=0. The situation changes once we drop $\perp$ from the language. Let $\mathbf{IPC}^{@+}$ be defined in the language $\mathcal{L}^{@} = \{@, \wedge, \vee, \to\}$ with (Ax1)-(Ax13), (RN) and (MP). The completeness for $\mathbf{IPC}^{@+}$ with respect to Kripke models with the base state is readily obtainable by an analogous means to that of $\mathbf{IPC}^{@}$.

Proposition 4.4 *$\sim$ is not definable in* $\mathbf{IPC}^{@+}$.

Proof. If $\sim$ is definable in $\mathbf{IPC}^{@+}$, then as we have seen $\perp$ is also definable as $\sim(A\to A)$. Let F be such a formula. Now choose a model such that $V(w,p)=1$ for all p and $w\in W$. Then by induction on formula we can establish $I(w,A)=1$ for all A and $w\in W$. So in particular, $I(w,F)=1$ for all $w\in W$, a contradiction.□

Therefore $\mathbf{IPC}^{@+}$ may be seen as an intuitionistic system with actuality operator that is independent of negation. This system consequently has an advantage over $\mathbf{IPC}^{@}$ and $\mathbf{IPC}^{\sim}$ when a non-standard notion of negation is espoused. Moreover it offers a suitable starting point for combining intuitionism in empirical discourse and the school of intuitionism which eschews negation altogether, as a result of scepticism towards unrealised concepts (cf. [30]).

4.2 Classical actuality and constructive actuality

We now turn to compare $\mathbf{IPC}^{@}$ to **S5A** of Crossley and Humberstone. To this end, we first review the basics of **S5A**, with a slightly difference in the notation to replace A, for actuality, by @. Then the system is described by the language $\mathcal{L}_m^{@} = \{@, \Box, \perp, \wedge, \vee, \to\}$.

Definition 4.5 [Crossley & Humberstone] An **S5A**-model for the language $\mathcal{L}_m^{@}$ is a triple $\langle W, g, V\rangle$, where W is a non-empty set (of states); $g \in W$ (the base state); and $V : W \times \mathsf{Prop} \to \{0,1\}$ an assignment of truth values to state-variable pairs. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w,p) = V(w,p)$;
- $I(w,\perp) = 0$;
- $I(w,\Box A) = 1$ iff for all $w \in W$, $I(w,A) = 1$;
- $I(w,@A) = 1$ iff $I(g,A) = 1$;
- $I(w,A\wedge B) = 1$ iff $I(w,A) = 1$ and $I(w,B) = 1$;
- $I(w,A\vee B) = 1$ iff $I(w,A) = 1$ or $I(w,B) = 1$;
- $I(w,A\to B) = 1$ iff $I(w,A) \neq 1$ or $I(w,B) = 1$.

Then, **S5A**-validity is defined in terms of truth at all $w \in W$: $\models_{\mathbf{S5A}} A$ iff for all **S5A**-models $\langle W, g, I\rangle$, $I(w,A) = 1$ for all $w \in W$.

Definition 4.6 [Crossley and Humberstone] The axiomatic proof system for **S5A** consists of the following axioms in addition to any axiomatization of **S5**:

$$@(@A \to A) \quad \text{(A1)} \qquad @A \leftrightarrow \neg@\neg A \quad \text{(A3)} \qquad @A \to \Box@A \quad \text{(A5)}$$
$$@(A \to B) \to (@A \to @B) \quad \text{(A2)} \qquad \Box A \to @A \quad \text{(A4)}$$

We refer to the derivability in **S5A** as $\vdash_{\mathbf{S5A}}$.

Based on these, Crossley and Humberstone established the following result.

Theorem 4.7 (Crossley and Humberstone) *For all* $A \in \mathsf{Form}_m^{@}$, $\models_{\mathbf{S5A}} A$ *iff* $\vdash_{\mathbf{S5A}} A$.

The above axiomatization seen in view of $\mathbf{IPC}^{@}$ is problematic since the right-to-left direction of (A3) is *not* valid/derivable. However, a slightly different axiomatization will allow us to compare **S5A** and $\mathbf{IPC}^{@}$ more easily.

Proposition 4.8 *Let* $\vdash_{\mathbf{S5A'}}$ *be the derivability in a system obtained from the axiomatic proof system for* **S5A** *by replacing* (A3) *by the following two axioms:*

$$@A \to \neg @\neg A \qquad \text{(A3.1)} \qquad @(A \vee B) \to (@A \vee @B) \qquad \text{(A3.2)}$$

Then, for all $A \in \mathsf{Form}^{@}_{m}$, $\vdash_{\mathbf{S5A'}} A$ *iff* $\vdash_{\mathbf{S5A}} A$.

Proof. For the left-to-right direction, it suffices to check that (A3.2) is derivable in **S5A**. In view of (A3), (A3.2) is derivable iff $\vdash_{\mathbf{S5A}} (@\neg A \wedge @\neg B) \to @(\neg A \wedge \neg B)$. But this is obvious since @ is an extension of **K**-modality.

For the other way around, it suffices to prove $\vdash_{\mathbf{S5A'}} @A \vee @\neg A$. Since we have classical tautologies, we have $\vdash_{\mathbf{S5A'}} A \vee \neg A$, and by the rule of necessitation, we have $\vdash_{\mathbf{S5A'}} \Box(A \vee \neg A)$. This implies $\vdash_{\mathbf{S5A'}} @(A \vee \neg A)$ in view of (A4), and finally we make use of (A3.2) to obtain the desired result. □

Remark 4.9 Note first that even though we do not have the necessity operator in $\mathbf{IPC}^{@}$, the actuality operator also enjoys the following condition:

$$I(w, @A) = 1 \text{ iff for all } w \in W, I(w, A) = 1$$

This is because the base point is the root. Thus, if we regard □ as @ in the above axiomatization of **S5A**, then we can see that all the axiom schemata and rules of inference related to □ and @ in **S5A** are derivable in $\mathbf{IPC}^{@}$.

Therefore, there is a sense in which $\mathbf{IPC}^{@}$ is a generalization of **S5A**. But there is also a sense in which this generalization is not simple. More specifically, we obtain the following result.

Proposition 4.10 $\mathbf{IPC}^{@}$ *plus Peirce's law collapses into* **Triv** *based on* **CL**.

Proof. In view of (Ax10), it suffices to prove $A \to @A$ in the extension. Note first that $A \vee (A \to B)$ is still derivable from an instance of Peirce's law, namely $(((A \vee (A \to B)) \to A) \to (A \vee (A \to B))) \to (A \vee (A \to B))$. Then as before we obtain $@A \vee @(A \to B)$, which entails $(@A \to @B) \to @(A \to B)$. Take $B \equiv @A$ and we have $(@A \to @@A) \to @(A \to @A)$. By (Ax11) and (Ax10), we obtain $A \to @A$. □

Remark 4.11 The above proof does not rely on the existence of $\perp$ in the language, and thus also applies to $\mathbf{IPC}^{@+}$.

5 Comparison (II)

In this section, we offer further comparisons of $\mathbf{IPC}^{@}$ with **LGP** of Baaz, as given in [2], and **GIPC** of Titani, as given in [46].

5.1 Baaz Delta and actuality

As we mentioned in the introduction, Baaz' logic **LGP** is Gödel-Dummet logic equipped with a projection modality △. Let us first look at the precise formulation in [2]. (For the sake of simplicity, we shall hereafter use $\mathcal{L}^{@}_{\perp}$ to describe the system, so @ will be used instead of △.)

Definition 5.1 [Baaz] Let $V \subseteq [0, 1]$ be a set of *truth values* containing 0 and 1. A *valuation* $\mathfrak{V}$ based on V assigns a truth value in V to each propositional variable. $\mathfrak{V}$ is extended to all propositions by the clauses:

- $\mathfrak{V}(\bot) = 0$
- $\mathfrak{V}(A{\wedge}B) = min(\mathfrak{V}(A), \mathfrak{V}(B))$
- $\mathfrak{V}(A{\vee}B) = max(\mathfrak{V}(A), \mathfrak{V}(B))$
- $\mathfrak{V}(A{\to}B) = \begin{cases} \mathfrak{V}(B) & \text{if } \mathfrak{V}(A) > \mathfrak{V}(B) \\ 1 & \text{if } \mathfrak{V}(A) \leq \mathfrak{V}(B) \end{cases}$
- $\mathfrak{V}(@A) = \begin{cases} 1 & \text{if } \mathfrak{V}(A) = 1 \\ 0 & \text{if } \mathfrak{V}(A) \neq 1 \end{cases}$

Then $\mathbf{GP}(V) := \{A : \mathfrak{V}(A) = 1 \text{ for every } \mathfrak{V} \text{ based on } V\}$.

Definition 5.2 **LGP** is axiomatized by adding the following axiom to $\mathbf{IPC}^{@}$.

$$(A \to B) \vee (B \to A) \tag{Lin}$$

Let V be infinite. Baaz showed the following weak completeness for **LGP**.

Theorem 5.3 (Baaz) *For all* $A \in \mathsf{Form}$*,* $\mathbf{LGP} \vdash A$ *iff* $A \in \mathbf{GP}(V)$*.*

As is well-known (e.g. [27, Theorem 19, Chapter 4]), Kripke-semantically (Lin) corresponds to lineally ordered Kripke frames. Thus as an improvement, we obtain a *strong* completeness proof for **LGP**, in view of Theorem 3.9. More specifically, let us denote $\vdash_l$ and $\models_l$ for the derivability in **LGP** and semantic consequence with respect to the class of linearly ordered models, respectively.

Proposition 5.4 *For all* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*,* $\Gamma \vdash_l A$ *iff* $\Gamma \models_l A$*.*

Proof. For soundness, we have to check that (Lin) holds in any linearly ordered model. Given a linearly ordered model $\langle W, g, \leq, I\rangle$ and formulas A and B, let us denote $V(A) = \{w : I(w, A) = 1\}$ and $V(B) = \{w : I(w, B) = 1\}$. Then we have $V(A) \subseteq V(B)$ or $V(B) \subseteq V(A)$. Hence $I(g, A \to B \vee B \to A) = 1$.

For completeness, we have to check that the counter-model construction of Theorem 3.9 creates a linearly ordered model. Suppose otherwise. Then there are states Σ_1 and Σ_2 such that neither $\Sigma_1 \subseteq \Sigma_2$ nor $\Sigma_2 \subseteq \Sigma_1$. Then we can find a formula A_1 in Σ_1 not in Σ_2, and A_2 in Σ_2 not in Σ_1. Now as the base state Π is a prime Π-theory, $A_1 \to A_2 \vee A_2 \to A_1 \in \Pi$, and so $A_1 \to A_2 \in \Pi$ or $A_2 \to A_1 \in \Pi$. Without loss of generality, assume the former. Then because Σ_1 is a Π-theory, $A_1 \wedge (A_1 \to A_2) \in \Sigma_1$; thus $A_2 \in \Sigma_1$, a contradiction. Therefore the counter-model has to be linearly ordered. This completes the proof. □

Remark 5.5 The above result clarifies that $\mathbf{IPC}^{@}$ is a generalization of **LGP** to include non-linearly ordered models. To give a further comparison, for **LGP** it is observed in [2] that $\neg\neg A$ is a dual projection operator of $@A$, attaining 1 if $A \neq 0$ and 0 otherwise. In the setting of $\mathbf{IPC}^{@}$, this true-if-not-false type of operator is perhaps better captured by $\neg@\neg A$ (i.e. $\sim\neg A$). $I(w, \neg@\neg A)$=1 iff for some $u{\in}W, I(u, A)$=1; so while $\neg\neg A{\to}\neg@\neg A$ holds in general, $\neg@\neg A{\to}\neg\neg A$ does not. One may readily check that this latter implication is equivalent to *the weak excluded middle* $\neg A{\vee}\neg\neg A$ as an axiom; in particular $\neg@\neg A$ and $\neg\neg A$ becomes equivalent in **LGP**, because (Lin) implies the weak excluded middle.

5.2 A reformulation of global intuitionistic logic

Next we shall consider propositional global intuitionistic logic (to be called **GIPC**). Let us first look at the formulation of the logic in sequent calculus as given in [46,1]. The system will be described in the language $\mathcal{L}^{@}_{\perp}$. Originally, however, $\Box$ was used in place of @, and $\neg$ was taken as primitive, rather than $\perp$. We shall call the calculus **LGJ** and the derivability by $\vdash_{gGI}$.

Definition 5.6 [Titani & Aoyama] The rule of the calculus **LGJ** are as follows.

$$A \Rightarrow A \text{ [Ax]} \qquad \perp \Rightarrow \text{ [L}\perp\text{]}$$

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\text{[LW]} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\text{[RW]}$$

$$\frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\text{[LC]} \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\text{[RC]}$$

$$\frac{\Gamma, A, B, \Pi \Rightarrow \Delta}{\Gamma, B, A, \Pi \Rightarrow \Delta}\text{[LE]} \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}\text{[RE]}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}\text{[Cut]}$$

$$\frac{A_i, \Gamma \Rightarrow \Delta}{A_1 \wedge A_2, \Gamma \Rightarrow \Delta}\text{[L}\wedge\text{]} \qquad \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}\text{[R}\wedge\text{]}$$

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\text{[L}\vee\text{]} \qquad \frac{\Gamma \Rightarrow \Delta, A_i}{\Gamma \Rightarrow \Delta, A_1 \vee A_2}\text{[R}\vee\text{]}$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Pi \Rightarrow \Lambda}{A \to B, \Gamma, \Pi \Rightarrow \Delta, \Lambda}\text{[L}\to\text{]} \qquad \frac{A, \Gamma \Rightarrow \bar{\Delta}, B}{\Gamma \Rightarrow \bar{\Delta}, A \to B}\text{[R}\to\text{]}$$

$$\frac{A, \Gamma \Rightarrow \Delta}{@A, \Gamma \Rightarrow \Delta}\text{[L@]} \qquad \frac{\bar{\Gamma} \Rightarrow \bar{\Delta}, A}{\bar{\Gamma} \Rightarrow \bar{\Delta}, @A}\text{[R@]}$$

In the above, $i \in \{1, 2\}$ and $\bar{\Gamma}$ and $\bar{\Delta}$ are finite sequences of @-*closed formulas*, which are formulas built from $\perp$ and formulas of the form $@A$, by the connectives $\wedge, \vee, \to$. For example, $@@A, @A \wedge @(\perp \to C), \neg@(\neg A \vee B)$ are all @-closed formulas. We shall denote @-closed formulas by $\bar{A}, \bar{B}$ and so on.

We wish to compare **GIPC** with **IPC**$^{@}$. For this purpose it is preferable to have at hand a Hilbert-style axiomatization. This we claim to be the following.

Definition 5.7 The system **GIPC** consists of (Ax0)-(Ax12), (MP),(RN) and the following axiom scheme:

$$(@A \to @B) \to @(@A \to B) \tag{Ax14}$$

The derivability in **GIPC** will be denoted by $\vdash_{GI}$.

Remark 5.8 Note that the deduction theorem, in the form of Theorem 2.8, holds for **GIPC** as well, by the same argument.

We now show a lemma before proving that **LGJ** and **GIPC** are equivalent.

Lemma 5.9 *Let $\bar{A}$ be @-closed. Then,* (i) $\vdash_{GI} \bar{A} \vee \bar{A} \to B$*, and* (ii) $\vdash_{GI} \bar{A} \to @\bar{A}$.

Proof. For (i), we argue by induction on the complexity of A.

- If $\bar{A} \equiv \bot$, then $\vdash_{GI} \bot \vee \bot \to B$.
- If $\bar{A} \equiv @A$, then $@A \vee @A \to B$ is an instance of (Ax12).
- If $\bar{A} \equiv \bar{C} \wedge \bar{D}$, then by IH $\vdash_{GI} \bar{C} \vee \bar{C} \to B$ and $\vdash_{GI} \bar{D} \vee \bar{D} \to B$. So $\vdash_{GI} (\bar{C} \wedge \bar{D}) \vee (\bar{C} \wedge \bar{D}) \to B$.
- If $\bar{A} \equiv \bar{C} \vee \bar{D}$, similarly $\vdash_{GI} (\bar{C} \vee \bar{D}) \vee (\bar{C} \vee \bar{D}) \to B$.
- If $\bar{A} \equiv \bar{C} \to \bar{D}$, by IH $\vdash_{GI} \bar{C} \vee \bar{C} \to \bar{D}$ and $\vdash_{GI} \bar{D} \vee \bar{D} \to B$. So $\vdash_{GI} (\bar{C} \to \bar{D}) \vee (\bar{C} \to \bar{D}) \to B$.

For (ii), we similarly argue by induction on A.

- If $\bar{A} \equiv \bot$, then $\bot \to @\bot$ is an instance of (Ax0).
- If $\bar{A} \equiv @A$, then $@A \to @@A$ is an instance of (Ax11).
- If $\bar{A} \equiv \bar{B} \wedge \bar{C}$, then by IH $\vdash_{GI} \bar{B} \to @\bar{B}$ and $\vdash_{GI} \bar{C} \to @\bar{C}$. Thus $\vdash_{GI} \bar{B} \wedge \bar{C} \to @\bar{B} \wedge @\bar{C}$. Now it is easy to check via the deduction theorem that $\vdash_{GI} @\bar{B} \wedge @\bar{C} \to @(\bar{B} \wedge \bar{C})$. Hence $\vdash_{GI} \bar{B} \wedge \bar{C} \to @(\bar{B} \wedge \bar{C})$.
- If $\bar{A} \equiv \bar{B} \vee \bar{C}$, then using the same IH as above, we see $\vdash_{GI} \bar{B} \vee \bar{C} \to @\bar{B} \vee @\bar{C}$. Again it is an easy consequence of the deduction theorem that $\vdash_{GI} @\bar{B} \to @(\bar{B} \vee \bar{C})$ and $\vdash_{GI} @\bar{C} \to @(\bar{B} \vee \bar{C})$. Hence $\vdash_{GI} \bar{B} \vee \bar{C} \to @(\bar{B} \vee \bar{C})$.
- If $\bar{A} \equiv \bar{B} \to \bar{C}$, then using (Ax10) and the IH that $\vdash_{GI} \bar{C} \to @\bar{C}$ we infer $\vdash_{GI} (\bar{B} \to \bar{C}) \to (@\bar{B} \to @\bar{C})$. Thus by (Ax14) $\vdash_{GI} (\bar{B} \to \bar{C}) \to @(@\bar{B} \to \bar{C})$. Also by the IH that $\vdash_{GI} \bar{B} \to @\bar{B}$ we have $\vdash_{GI} (@\bar{B} \to \bar{C}) \to (\bar{B} \to \bar{C})$. So by (RN) and (Ax9), $\vdash_{GI} @(@\bar{B} \to \bar{C}) \to @(\bar{B} \to \bar{C})$. Combining the above two observations, we conclude $\vdash_{GI} (\bar{B} \to \bar{C}) \to @(\bar{B} \to \bar{C})$.

This completes the proof. □

Proposition 5.10 *The following equivalence hold between* **LGP** *and* **GIPC***.*
(i) *For all* $A \in \mathsf{Form}$*, if* $\vdash_{GI} A$ *then* $\vdash_{gGI} \Rightarrow A$*.*
(ii) *For all* $\Gamma, \Delta \subseteq \mathsf{Form}$*, if* $\vdash_{gGI} \Gamma \Rightarrow \Delta$ *then* $\vdash_{GI} \bigwedge \Gamma \to \bigvee \Delta$*.*

Proof. For (i), given the correspondence in intuitionistic logic, it suffices to consider axioms involving @ and (RN). Here we show cases for (Ax12) and (Ax14), which are stated but not shown in [1, Proposition 2.1]; other cases are immediate.

Ax12

$$\dfrac{\dfrac{\dfrac{@A \Rightarrow @A}{@A \Rightarrow @A, B}\,[\mathrm{RW}]}{\Rightarrow @A, @A \to B}\,[\mathrm{R}{\to}]}{\Rightarrow @A \vee @A \to B}\,[\mathrm{R}\vee],[\mathrm{RC}]$$

Ax14

$$\dfrac{\dfrac{\dfrac{\dfrac{@A \Rightarrow @A \qquad \dfrac{B \Rightarrow B}{@B \Rightarrow B}\,[\mathrm{L}@]}{@A \to @B, @A \Rightarrow B}\,[\mathrm{L}{\to}]}{@A \to @B \Rightarrow @A \to B}\,[\mathrm{R}{\to}]}{@A \to @B \Rightarrow @(@A \to B)}\,[\mathrm{R}@]}{\Rightarrow (@A \to @B) \to @(@A \to B)}\,[\mathrm{R}{\to}]$$

For (ii), we treat here the cases for [R→], [L@] and [R@].

- For [R→], by IH $\vdash_{GI} (\bigwedge \Gamma \wedge A) \to (\bigvee \bar{\Delta} \vee B)$. So $\vdash_{GI} \bigwedge \Gamma \to (A \to (\bigvee \bar{\Delta} \vee B))$. Now by Lemma 5.9 (i), $\vdash_{GI} \bigvee \bar{\Delta} \vee \bigvee \bar{\Delta} \to B$. Thus $\vdash_{GI} \bigwedge \Gamma \to (\bigvee \bar{\Delta} \vee A \to B)$.
- For [L@], by IH $\vdash_{GI} (A \wedge \bigwedge \Gamma) \to \bigvee \Delta$. Then $\vdash_{GI} A \to (\bigwedge \Gamma \to \bigvee \Delta)$. So by

(Ax10) $\vdash_{GI} @A \to (\bigwedge \Gamma \to \bigvee \Delta)$. Hence $\vdash_{GI} (@A \wedge \bigwedge \Gamma) \to \bigvee \Delta$.

- For [R@], by IH $\vdash_{GI} \bigwedge \bar{\Gamma} \to (\bigvee \bar{\Delta} \vee A)$. Then $\vdash_{GI} (\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A)) \to A$. Thus by (RN) and (Ax9), $\vdash_{GI} @(\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A)) \to @A$. Here we note $@(\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A))$ is @-closed. So by Lemma 5.9 (ii), $\vdash_{GI} (\bigwedge \bar{\Gamma} \wedge (\bigvee \bar{\Delta} \to @A)) \to @A$. Also by Lemma 5.9 (i), $\vdash_{GI} \bigvee \bar{\Delta} \vee \bigvee \bar{\Delta} \to @A$. From these we deduce $\vdash_{GI} \bigwedge \bar{\Gamma} \to (\bigvee \bar{\Delta} \vee @A)$.

This completes the proof. □

5.3 Globalization and actuality

We are now ready to compare $\mathbf{IPC}^{@}$ and **GIPC**. We first observe that the former logic contains the latter.

Proposition 5.11 $\mathbf{IPC}^{@} \supseteq \mathbf{GIPC}$.

Proof. It suffices to observe that (Ax14) is derivable in $\mathbf{IPC}^{@}$. Applying (RN) and (Ax13) to (Ax12), we obtain $\vdash @A \vee @(@A \to B)$. Then on one hand, since $\vdash @A \to ((@A \to @B) \to @B)$ and $\vdash @B \to @(@A \to B)$ (the latter by (Ax1), (RN) and (Ax9)), we have $\vdash @A \to ((@A \to @B) \to @(@A \to B))$. On the other hand, it is immediate that $\vdash @(@A \to B) \to ((@A \to @B) \to @(@A \to B))$. Therefore $\vdash (@A \to @B) \to @(@A \to B)$. □

Remark 5.12 Baaz, in [2], states sequent rules for △ of **LGP**. It turns out that the same rules can be used to formulate a calculus for $\mathbf{IPC}^{@}$. It is obtained from **LGJ** by relaxing [R@] to

$$\frac{\bar{\Gamma} \Rightarrow \Delta, A}{\bar{\Gamma} \Rightarrow \Delta, @A}\ [\mathrm{R}@]$$

By Proposition 5.11, we can use Lemma 5.9 for $\mathbf{IPC}^{@}$ as well. Then we can argue analogously to Proposition 5.10; the treatments of cases for the new [R@] and (Ax13) are straightforward.

To show that the inclusion of the above proposition is strict, we shall turn to a closely related logic called $\mathbf{TCC}_\omega$. This is a subsystem of $\mathbf{IPC}^{\sim}$ introduced by A. B. Gordienko in [28] as an extension of Richard Sylvan's logic $\mathbf{CC}_\omega$ (cf. [44]). Its axiomatization is that of $\mathbf{IPC}^{\sim}$, except (RP) is replaced with

$$\frac{A \to B}{\sim B \to \sim A}. \tag{RC}$$

The deducibility in $\mathbf{TCC}_\omega$ will be denoted $\vdash_t$. It is easy to check that formulas and rules derivable in $\mathbf{IPC}^{\sim}$ listed in [17, Lemma 2.6, Lemma 2.8] are also derivable in $\mathbf{TCC}_\omega$. In particular, the following formulas and rule are derivable.

$$(\sim A \to A) \to A \qquad \text{(t1)} \qquad\qquad \sim\sim A \to A \qquad \text{(t3)}$$

$$\sim(A \to A) \to B \qquad \text{(t2)} \qquad\qquad \frac{A}{\sim\sim A} \qquad \text{(t4)}$$

Moreover, the same form of the deduction theorem as $\mathbf{IPC}^{\sim}$ holds in $\mathbf{TCC}_\omega$.

Quite similarly to the situation with $\mathbf{IPC}^{@}$ and $\mathbf{IPC}^{\sim}$, we have the following translations between **GIPC** and $\mathbf{TCC}_\omega$.

Definition 5.13 Let $()^{\sim}$ and $()^{@}$ be translations between $\mathcal{L}^{@}_{\perp}$ and $\mathcal{L}^{\sim}$ such that:

$$\begin{array}{rlrl} p^{\sim} &= p & p^{@} &= p \\ (A \circ B)^{\sim} &= A^{\sim} \circ B^{\sim} & (A \circ B)^{@} &= A^{@} \circ B^{@} \\ (@A)^{\sim} &= {\sim}{\sim}A^{\sim} & ({\sim}A)^{@} &= \neg @ A^{@} \\ \perp^{\sim} &= {\sim}(p_0 \to p_0) & & \end{array}$$

where p_0 is a fixed propositional variable, and $\circ \in \{\wedge, \vee \to\}$.

Lemma 5.14 *For all* $A \in \mathsf{Form}$, $\vdash_{GI} A \leftrightarrow (A^{\sim})^{@}$ *and for all* $A \in \mathsf{Form}^{\sim}$, $\vdash_t A \leftrightarrow (A^{@})^{\sim}$.

Proof. By induction on A. Here we look at the cases $A \equiv @B$ and $A \equiv {\sim}B$.

For the former, we need to show $\vdash_{GI} @B \leftrightarrow \neg @ \neg @ (B^{\sim})^{@}$. By IH $\vdash_{GI} B \leftrightarrow (B^{\sim})^{@}$, so its suffices to show $\vdash_{GI} @B \leftrightarrow \neg @ \neg @ B$. We first note $\neg @ B$ is @-closed, thus $\vdash_{GI} \neg @ \neg @ B \leftrightarrow \neg\neg @ B$. Also $\vdash_{GI} \neg\neg @ B \leftrightarrow @B$ from (Ax12). Therefore we conclude $\vdash_{GI} @B \leftrightarrow \neg @ \neg @ B$ as desired.

For the latter, we need $\vdash_t {\sim}B \leftrightarrow ({\sim}{\sim}(B^{@})^{\sim} \to {\sim}(p_0 \to p_0))$. Again by IH $\vdash_t B \leftrightarrow (B^{@})^{\sim}$. Then the equivalence follows by (N2), (t1) and (t2). □

Proposition 5.15 *We have that* (i) *for all* $A \in \mathsf{Form}$, $\vdash_{GI} A$ *iff* $\vdash_t A^{\sim}$, *and* (ii) *for all* $A \in \mathsf{Form}^{\sim}$, $\vdash_t A$ *iff* $\vdash_{GI} A^{@}$.

Proof. By Lemma 5.14, it suffices to show the left-to-right direction.

For (i), we need to check the translations of (Ax9)-(Ax12), (Ax14) and (RN) hold in $\mathbf{TCC}_{\omega}$.

- (Ax9) is translated as ${\sim}{\sim}(A^{\sim} \to B^{\sim}) \to ({\sim}{\sim}A^{\sim} \to {\sim}{\sim}B^{\sim})$, the derivability of which is immediate from the deduction theorem and (RC).
- (Ax10) is translated as ${\sim}{\sim}A^{\sim} \to A^{\sim}$, which is an instance of (t3).
- (Ax11) is translated as ${\sim}{\sim}A^{\sim} \to {\sim}{\sim}{\sim}{\sim}A^{\sim}$. This follows from (N2) and (t1), which imply ${\sim}{\sim}{\sim}A^{\sim} \to {\sim}A^{\sim}$; then use (RC).
- (Ax12) becomes ${\sim}{\sim}A^{\sim} \vee {\sim}{\sim}A^{\sim} \to B^{\sim}$, a consequence of (N1) and (N2).
- For (Ax14), we need to show $\vdash_t ({\sim}{\sim}A^{\sim} \to {\sim}{\sim}B^{\sim}) \to {\sim}{\sim}({\sim}{\sim}A^{\sim} \to B^{\sim})$. First $\vdash_t {\sim}A^{\sim} \vee {\sim}{\sim}A^{\sim}$ from (N1) and ${\sim}{\sim}{\sim}A^{\sim} \to {\sim}A^{\sim}$ as seen above. So $\vdash_t ({\sim}{\sim}A^{\sim} \to {\sim}{\sim}B^{\sim}) \to ({\sim}A^{\sim} \vee {\sim}{\sim}B^{\sim})$. We shall show $\vdash_t ({\sim}A^{\sim} \vee {\sim}{\sim}B^{\sim}) \to {\sim}{\sim}({\sim}{\sim}A^{\sim} \to B^{\sim})$. On one hand, $\vdash_t {\sim}A^{\sim} \to {\sim}{\sim}({\sim}{\sim}A^{\sim} \to B^{\sim})$ from (N2), (t3) and (RC). On the other hand, $\vdash_t {\sim}{\sim}B^{\sim} \to {\sim}{\sim}({\sim}{\sim}A^{\sim} \to B^{\sim})$ from (Ax1) and (RC). Thus $\vdash_t ({\sim}A^{\sim} \vee {\sim}{\sim}B^{\sim}) \to {\sim}{\sim}({\sim}{\sim}A^{\sim} \to B^{\sim})$ as required.
- Finally, (RN) is replicable by (t4).

For (ii), we need to check (N1),(N2) and (RC).

- (N1) is translated into $A^{@} \vee \neg @ A^{@}$, which is an instance of (Ax12).
- (N2) is translated into $\neg @ A^{@} \to (\neg @ \neg @ A^{@} \to B^{@})$. As we observed in Lemma 5.14, $\neg @ \neg @ A^{@}$ is equivalent to $\neg\neg @ A^{@}$; so it follows from (Ax0).
- For (RC), we need to derive $\neg @ B \to \neg @ A$ from $A \to B$. This is possible with (RN),(Ax9) and by contraposition.

This completes the proof. □

The translation allows us to use the Kripke semantics for $\mathbf{TCC}_\omega$.

Definition 5.16 [Gordienko] A $\mathbf{TCC}_\omega$-model for $\mathcal{L}^{\sim}$ is a triple $\langle W, \leq, V\rangle$ with each component as in $\mathbf{IPC}^{@}$. V is extended to interpretation I analogously, except for the interpretation of $\sim A$, which is given by:

$$I(w, \sim A) = 1 \text{ iff } I(w', A) = 0 \text{ for some } w' \in W.$$

We shall use $\models_t$ for the semantic consequence, defined as follows: $\models_t A$ iff for all $\mathbf{TCC}_\omega$-models $\langle W, \leq, V\rangle$, $I(w, A) = 1$ for all $w \in W$.

Remark 5.17 Note in particular that a model of $\mathbf{TCC}_\omega$ does not necessarily have a base state. If it does, then the interpretation coincides with that of $\mathbf{IPC}^{@}$.

We are now ready to separate the two systems.

Theorem 5.18 (Gordienko) *For all* $A \in \mathsf{Form}^{\sim}$, $\vdash_t A$ *iff* $\models_t A$.

Corollary 5.19 $\mathbf{IPC}^{@} \supsetneq \mathbf{GIPC}$.

Proof. First, observe that we have the following valuation for $\sim\sim A$.

$$I(w, \sim\sim A) = 1 \text{ iff } I(w', A) = 1 \text{ for all } w'.$$

Now, if **GIPC** proves (Ax13), then by Proposition 5.15 $\sim\sim(p \vee q) \to \sim\sim p \vee \sim\sim q$ is provable in $\mathbf{TCC}_\omega$. On the other hand, if we consider a model where $W = \{w, w'\}$, $\leq\, = \{(w, w), (w', w')\}$, $V(p) = \{w\}$ and $V(q) = \{w'\}$, then $I(w, \sim\sim(p \vee q)) = 1$, but $I(w, \sim\sim p) = I(w, \sim\sim q) = 0$. Hence this is a countermodel for $\sim\sim(p \vee q) \to \sim\sim p \vee \sim\sim q$. So by the previous theorem, $\nvdash_t \sim\sim(p \vee q) \to \sim\sim p \vee \sim\sim q$. A contradiction. Therefore **GIPC** does not prove (Ax13). □

Remark 5.20 Note that given a model of $\mathbf{TCC}_\omega$, we can define a model for $\mathcal{L}^{@}_{\perp}$ with the interpretation I such that

$$I(w, @A) = 1 \text{ iff } I(w', A) = 1 \text{ for all } w'.$$

Then, it is not difficult to see that each such model corresponds to the original model similarly to Lemma 5.14 and Proposition 5.15. Therefore, it is an immediate consequence of Theorem 5.18 that this gives a sound and weakly complete Kripke semantics for **GIPC**. (This semantics can be also obtained from Ono's semantics via Gordienko's technique; see below.)

We offer a few more words about **GIPC**. In [36], Hiroakira Ono extensively discussed intutitionistic modal systems which are defined by axioms that classically define **S5** when added to **S4**. Aoyama [1] compared some of these systems with **GIPC**,[8] but he did not compare with the strongest of Ono's systems, $\mathbf{L}_4$. It is defined by (Ax0)-(Ax11), $@A \vee @\neg@A$, (MP) and (RP). The Kripke semantics for $\mathbf{L}_4$ in [36] is characterised by modal relation R that is an equivalence relation; this corresponds to the original semantics of $\mathbf{TCC}_\omega$, from which Gordienko derived [28, Lemma 4.4] the semantics of Definition 5.16. This observation and Proposition 5.15 suggest a close relationship between **GIPC** and $\mathbf{L}_4$. In fact, the two systems turn out to coincide.

[8] Some of the comparisons offered in [1] are also observed by Hidenori Kurokawa in [34].

Proposition 5.21 **GIPC** = $\mathbf{L}_4$

Proof. On one hand, $\neg @A$ is @-closed, so by Lemma 5.9 (ii) $\neg @A \to @\neg @A$ is derivable in **GIPC**. Thus with (Ax12), $@A \vee @\neg @A$ is derivable in **GIPC**. Consequently **GIPC** contains $\mathbf{L}_4$. On the other hand, $@A \vee @\neg @A$ implies (Ax12) with (Ax0) and (Ax10). Moreover, $(@A \to @B) \to @(@A \to @B)$ is known to be derivable in $\mathbf{L}_4$ (cf. [36, Figure 2.1]), and it is a consequence of (Ax10), (RN) and (Ax9) that $@(@A \to @B) \to @(@A \to B)$ holds, so (Ax14) is also derivable in $\mathbf{L}_4$. Thus $\mathbf{L}_4$ contains **GIPC** as well. □

5.4 Sequent calculi for $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^\sim$

Finally, we shall use the results obtained so far to formulate sequent calculi for $\mathbf{TCC}_\omega$ and $\mathbf{IPC}^\sim$. We begin with introducing an analogue of @-closed for formulas in $\mathcal{L}^\sim$.

Definition 5.22 We define the class of $\sim$*-closed formulas* by the next clauses.
(i) $\bot$, $\sim A$ are $\sim$-closed.
(ii) If $\bar{B}$ and $\bar{C}$ are $\sim$-closed, then $\bar{B} \circ \bar{C}$ is $\sim$-closed, where $\circ \in \{\wedge, \vee, \to\}$.

It is straightforward to check that if $\bar{A}$ is $\sim$-closed, then $\bar{A}^@$ is @-closed.

Lemma 5.23 *For all* $A \in \mathsf{Form}^\sim$, $\vdash_t \bar{A} \to \sim\sim\bar{A}$.

Proof. By the above observation and Lemma 5.9 (ii), we have $\vdash_{GI} \bar{A}^@ \to @\bar{A}^@$. Thus by Proposition 5.15 (i) and Lemma 5.14, $\vdash_t \bar{A} \to \sim\sim\bar{A}$. □

The sequent rules for $\sim$ corresponding to $\mathbf{TCC}_\omega$ is obtained by the following

$$\frac{\bar{\Gamma} \Rightarrow \bar{\Delta}, A}{\sim A, \bar{\Gamma} \Rightarrow \bar{\Delta}}\,[\mathrm{L}\sim] \qquad \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim A}\,[\mathrm{R}\sim]$$

where $\bar{\Gamma}, \bar{\Delta}$ are $\sim$-closed. The sequent calculus **LT** for $\mathbf{TCC}_\omega$ is obtained by adding the above rules to the positive and non-modal fragment of **LGJ** (derivability denoted by $\vdash_{gT}$).

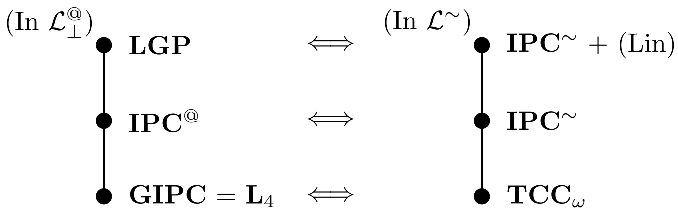
Theorem 5.24 *For all* $\Gamma, \Delta \subseteq \mathsf{Form}^\sim$, $\vdash_{gT} \Gamma \Rightarrow \Delta$ *iff* $\vdash_t \bigwedge\Gamma \to \bigvee\Delta$.

Proof. For the right-to-left direction, we need to check the cases for (N1),(N2) and (RC). Each case is straightforward. For the right-to-left direction, we must check the cases for [L$\sim$] and [R$\sim$]. The latter case is simple; for the former case, $\vdash_t \bar{\Gamma} \to (\bigvee\bar{\Delta} \vee A)$ by IH. Then by (MP) and Lemma 5.23, $\bar{\Gamma} \vdash_t \sim\sim\bigvee\bar{\Delta} \vee A$. So by (N2), (RC) and (t3), we obtain $\bar{\Gamma} \vdash_t \sim A \to \bigvee\bar{\Delta}$. Hence by deduction theorem and Lemma 5.23 again, we conclude $\vdash_t (\bar{\Gamma} \wedge \sim A) \to \bigvee\bar{\Delta}$. □

A sequent calculus for $\mathbf{IPC}^\sim$ has not been considered before. We can now obtain one by removing the condition that $\bar{\Delta}$ is $\sim$-closed in [L$\sim$]. The correspondence with the Hilbert-style system is straightforwardly demonstrable.

6 Concluding remarks

In this article, we introduced $\mathbf{IPC}^@$, an expansion of **IPC**, obtained by adding actuality operator, and compared with systems including **LGP** of Baaz, **GIPC** of Titani and $\mathbf{IPC}^{\sim}$ of De, obtained by adding projection operator, globalization operator and empirical negation respectively. What emerged is the following hierarchy of systems in $\mathcal{L}_{\perp}^{@}$, each corresponding to a system in $\mathcal{L}^{\sim}$.



With respect to these systems, we make some additional observations and mention a few future directions.

Hybrid logic Since there are clear connections between hybrid logics and logics with actuality operator, and in particular there are some results on hybrid logics based on intuitionistic logic (cf. [9,10]), a comparison of $\mathbf{IPC}^{@}$ to these systems will be of great interest.

Kripke semantics vs. Beth semantics We observed that @ in $\mathbf{IPC}^{@}$ and $\sim$ in $\mathbf{IPC}^{\sim}$ are inter-definable (in the presence of $\perp$ in the language), and similarly for **GIPC** and $\mathbf{TCC}_{\omega}$. As we have noted, a crucial difference between the semantics of $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_{\omega}$ (hence the interpretation of @) is that models in the former always has a base state, while the latter in general does not. As a result, Kripke-semantically, even though both @ can be understood as a globalization operator (i.e. true iff true everywhere), only the former can be interpreted as an actuality operator. Yet one may wonder whether one could view @ in **GIPC** as a sort of actuality operator.

Beth semantics offers a possibility for this alternative interpretation. It is a semantics similar to Kripke semantics, but crucially different in that (i) all models have a base state, and (ii) the valuation of disjunction does not require one of the disjuncts to hold in the same world. [9] If we define the clause for $\sim$ as in the Kripke semantics of $\mathbf{IPC}^{\sim}$, we obtain Beth semantics with empirical negation (cf. Appendix). One of the present authors have shown elsewhere in [35] that (RP) is not valid, but $\mathbf{TCC}_{\omega}$ is sound and complete with respect to this semantics. This means that @ in **GIPC** can be understood as actuality operator *with respect to Beth semantics.* Thus there are two types of actuality operator/empirical negation in intuitionistic logic, Kripke-type and Beth-type.

With this kind of perspective, we can connect results related to **GIPC** with empirical negation. For instance, Titani's global intutionistic set theory can be seen as a mathematical theory with Beth-type empirical negation, by reading $\neg\Box$ as $\sim$. This could then encourage the investigation of intuitionistic set theory with Kripke-type empirical negation, as a possible future direction.

Quantifiers Global intuitionistic logic was originally formulated in a first-order language. Moreover, quantification for **LGP** has been investigated in

[9] For more information, cf. [47, Chapter 13].

[2,3]. From this perspective, it seems to be a natural direction to consider first-order systems for $\mathbf{IPC}^{@}$. This can be particularly interesting because like disjunction, existential quantifier has differing interpretations in Kripke and Beth models. Therefore we might be able to find an interesting interaction between quantifiers and modal operators. Moreover, for the purpose of comparing $\mathbf{IPC}^{@}$ to **S5A** of Crossley and Humberstone, we also need quantifiers, and this will be yet another motivation for adding quantifiers.

Hypersequent calculi The sequent calculus for global intuitionistic logic **GI** defined by Titani and Aoyama is not cut-eliminatable, as observed by Agata Ciabattoni in [13, p.437]. She instead formulated a cut-free hypersequent calculus for **GI** and for **GIF**. We may then expect a similar approach to be quite beneficial in pursuing cut-free sequent calculi for the systems we have considered, namely $\mathbf{IPC}^{@}$, $\mathbf{IPC}^{\sim}$ and $\mathbf{TCC}_{\omega}$.

Appendix

Beth semantics for $\mathbf{TCC}_{\omega}$ We shall employ the following notations for sequences and related notions.

- $\alpha, \beta, \ldots$: infinite sequences of the form $\langle \alpha_1, \alpha_2, \ldots \rangle$ of natural numbers.
- $\langle \rangle$: the empty sequence.
- $b, b', \ldots$: finite sequences of the form $\langle b_1, \ldots, b_n \rangle$ of natural numbers.
- $b * b'$: b concatenated with b'.
- $lh(b)$: the length of b.
- $b \preceq b'$: $b * b'' = b'$ for some b''.
- $b \prec b'$: $b \preceq b'$ and $b \neq b'$.
- $\bar{\alpha}n$: α's initial segment up to the nth element.
- $\alpha \in b$: b is α's initial segment.

We define a *tree* to be a set T of finite sequences of natural number such that $\langle \rangle \in T$, $b \in T \vee b \notin T$ and $b \in T \wedge b' \prec b \rightarrow b' \in T$. We call each finite sequence in T as a *node* and $\langle \rangle$ as the *root*. A *successor* of a node b is a node of the form $b * \langle x \rangle$. By *leaves* of T, we mean the nodes of T which do not have a successor, i.e. nodes b such that $\neg\exists x(b * \langle x \rangle) \in T$. A *spread* then is a tree whose nodes always have a successor, i.e. $\forall b \in T \exists x(b * \langle x \rangle \in T)$.

A *Beth model* then is a triple $(W, \preceq, V)$, where $(W, \preceq)$ defines a spread and $V : W \times \mathsf{Prop} \rightarrow \{0, 1\}$ an assignment of truth values to state-variable pairs with the condition that:

$V(b, p) = 1$ iff for all $\alpha \in b$ there is m such that $(V(\bar{\alpha}m, p) = 1)$. [covering]

An interpretation I for Beth model is defined by the following clauses.

- $I(b, p) = V(b, p)$;
- $I(b, A \wedge B)=1$ iff $I(b, A)=1$ and $I(b, B)=1$;
- $I(b, A \vee B)=1$ iff for all $\alpha \in b$ there is n such that $I(\bar{\alpha}n, A)=1$ or $I(\bar{\alpha}n, B)=1$;
- $I(b, A \rightarrow B) = 1$ iff for all $b \in W$: if $b \preceq b'$ and $I(b', A)=1$ then $I(b', B)=1$;
- $I(b, {\sim}A) = 1$ iff $I(\langle \rangle, A) = 0$.

The semantic consequence is then defined as in Kripke semantics.

References

[1] Aoyama, H., *The semantic completeness of a global intuitionistic logic*, Mathematical Logic Quarterly **44** (1998), pp. 167–175.

[2] Baaz, M., *Infinite-valued Gödel logics with 0-1-projections and relativizations*, in: P. Hájek, editor, *Gödel '96*, Springer, 1996 pp. 23–33.

[3] Baaz, M., N. Preining and R. Zach, *Completeness of a hypersequent calculus for some first-order godel logics with delta*, in: *36th International Symposium on Multiple-Valued Logic (ISMVL'06)*, IEEE, 2006, pp. 9–9.

[4] Berto, F., *A modality called 'negation'*, Mind **124** (2015), pp. 761–793.

[5] Berto, F. and G. Restall, *Negation on the Australian Plan*, Journal of Philosophical Logic **48** (2019), pp. 1119–1144.

[6] Bezhanishvili, G., *Varieties of monadic Heyting algebras. Part I*, Studia Logica **61** (1998), pp. 367–402.

[7] Bierman, G. M. and V. C. de Paiva, *On an intuitionistic modal logic*, Studia Logica **65** (2000), pp. 383–416.

[8] Božić, M. and K. Došen, *Models for normal intuitionistic modal logics*, Studia Logica **43** (1984), pp. 217–245.

[9] Braüner, T., *Axioms for classical, intuitionistic, and paraconsistent hybrid logic*, Journal of Logic, Language and Information **15** (2006), pp. 179–194.

[10] Braüner, T., *Intuitionistic hybrid logic: Introduction and survey*, Information and Computation **209** (2011), pp. 1437–1446.

[11] Bull, R. A., *A modal extension of intuitionist logic.*, Notre Dame Journal of Formal Logic **6** (1965), pp. 142–146.

[12] Bull, R. A., *MIPC as the formalisation of an intuitionist concept of modality*, The Journal of Symbolic Logic **31** (1966), pp. 609–616.

[13] Ciabattoni, A., *A proof-theoretical investigation of global intuitionistic (fuzzy) logic*, Archive for Mathematical Logic **44** (2005), pp. 435–457.

[14] Crossley, J. N. and L. Humberstone, *The logic of "actually"*, Reports on Mathematical Logic **8** (1977), pp. 1–29.

[15] Davies, M. and L. Humberstone, *Two notions of necessity*, Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition **38** (1980), pp. 1–30.

[16] De, M., *Empirical Negation*, Acta Analytica **28** (2013), pp. 49–69.

[17] De, M. and H. Omori, *More on empirical negation.*, in: *Advances in modal logic*, 2014, pp. 114–133.

[18] De, M. and H. Omori, *Classical and empirical negation in subintuitionistic logic.*, in: *Advances in Modal Logic*, 2016, pp. 217–235.

[19] De, M. and H. Omori, *There is more to negation than modality*, Journal of Philosophical logic **47** (2018), pp. 281–299.

[20] Došen, K., *Negative modal operators in intuitionistic logic*, Publication de l'Instutute Mathematique, Nouv. Ser **35** (1984), pp. 3–14.

[21] Došen, K., *Models for stronger normal intuitionistic modal logics*, Studia Logica **44** (1985), pp. 39–70.

[22] Došen, K., *Negation as a modal operator*, Reports on Mathematical Logic **20** (1986), pp. 15–27.

[23] Došen, K., *Negation in the light of modal logic*, in: *What is Negation?*, Springer, 1999 pp. 77–86.

[24] Dummett, M., *A propositional calculus with denumerable matrix*, The Journal of Symbolic Logic **24** (1959), pp. 97–106.

[25] Fitch, F. B., *Intuitionistic modal logic with quantifiers*, Portugaliae mathematica **7** (1948), pp. 113–118.

[26] Fritz, P., *What is the correct logic of necessity, actuality and apriority?*, The Review of Symbolic Logic **7** (2014), pp. 385–414.

[27] Gabbay, D. M., "Semantical investigations in Heyting's intuitionistic logic," Synthese Library **148**, Springer, 1981.

[28] Gordienko, A. B., *A Paraconsistent Extension of Sylvan's Logic*, Algebra and Logic **46** (2007), pp. 289–296.

[29] Gregory, D., *Completeness and decidability results for some propositional modal logics containing "actually" operators*, Journal of Philosophical Logic **30** (2001), pp. 57–78.

[30] Heyting, A., *G.F.C. Griss and his negationless intuitionistic mathematics*, Synthese (1953), pp. 91–96.

[31] Horn, L. and H. Wansing, *Negation*, in: E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*, 2020, Spring 2020 edition .

[32] Humberstone, L., *Two-dimensional adventures*, Philosophical Studies **118** (2004), pp. 17–65.

[33] Kapsner, A., "Logics and falsifications," Trends in logic **40**, Springer, 2014.

[34] Kurokawa, H., *Hypersequent calculi for intuitionistic logic with classical atoms*, Annals of Pure and Applied Logic **161** (2009), pp. 427–446.

[35] Niki, S., *Empirical negation, co-negation and the contraposition rule I: Semantical investigations*, submitted.

[36] Ono, H., *On some intuitionistic modal logics*, Publications of the Research Institute for Mathematical Sciences **13** (1977), pp. 687–722.

[37] Prior, A. N., "Time and modality," Oxford University Press, 1957.

[38] Restall, G., *Subintuitionistic Logics*, Notre Dame Journal of Formal Logic **35** (1994), pp. 116–126.

[39] Servi, G. F., *On modal logic with an intuitionistic base*, Studia Logica **36** (1977), pp. 141–149.

[40] Simpson, A. K., "The proof theory and semantics of intuitionistic modal logic," Ph.D. thesis, University of Edinburgh (1994).

[41] Sotirov, V. H., *Modal theories with intuitionistic logic*, in: *Proceedings of the Conference on Mathematical Logic, Sofia*, 1980, pp. 139–171.

[42] Standefer, S., *Actual issues for relevant logics*, Ergo (forthcoming).

[43] Stephanou, Y., *The meaning of 'actually'*, dialectica **64** (2010), pp. 153–185.

[44] Sylvan, R., *Variations on da Costa C Systems and Dual-Intuitionistic Logics I. Analyses of C_ω and CC_ω*, Studia Logica **49** (1990), pp. 47–65.

[45] Takeuti, G. and S. Titani, *Global intuitionistic fuzzy set theory*, in: P. Hájek, editor, *The Mathematics of Fuzzy Systems*, TUV-Verlag, 1986 pp. 291–301.

[46] Titani, S., *Completeness of global intuitionistic set theory*, The Journal of Symbolic Logic **62** (1997), pp. 506–528.

[47] Troelstra, A. S. and D. van Dalen, "Constructivism in Mathematics: An introduction, volume II," Elsevier, 1988.

[48] Ursini, A., *A modal calculus analogous to K4W, based on intuitionistic propositional logic, I^o*, Studia Logica **38** (1979), pp. 297–311.