An Extension of Connexive Logic C

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Abstract

In [38], one of the present authors introduced a system of connexive logic, called **C**, as a simple variant of Nelson's Logic **N4**, obtained by making a small change in the falsification clause for the conditional. This was an important step marked in the field of connexive logic since **C** can be seen as the first system of connexive logic with an intuitively plausible semantics. The aim of this article is to consider an extension of **C** obtained by adding the law of excluded middle with respect to the strong negation. The extension of **C** is motivated by three questions. The first question comes from a system **CN** devised by John Cantwell. The second question concerns how many more connexive theses, beside the basic theses of Aristotle and Boethius, can be captured within the framework suggested in the above paper. The third question addresses the relation between constructivity and the law of excluded middle. We will show that the quantified version of our extension of **C** satisfies the Existence Property and its dual, but fails to satisfy the Disjunction Property and its dual when the law of excluded middle is restricted to atomic formulas. We will also mention some open problems related to the new system introduced in this article.

Keywords: Connexive logic, Constructive logic, Law of excluded middle, Nelson logic, Many-valued logic, **FDE**, Disjunction Property, Existence Property.

1 Introduction

Connexive logics are traditionally characterized as systems validating the theses of Aristotle and Boethius, namely the following theses (cf. [20,39] for surveys):

Aristotle $\sim(\sim A \to A)$, $\sim(A \to \sim A)$;

Boethius $(A \to B) \to \sim(A \to \sim B)$, $(A \to \sim B) \to \sim(A \to B)$.

Moreover, we require that $(A \to B) \to (B \to A)$ fails to be a theorem.

As one can easily observe, these characteristic theses are *not* valid in classical logic. In other words, connexive logics belong to a larger family of nonclassical logics known as contra-classical logics (cf. [15]). Thus it remained

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as a non-trivial task to find an intuitive system of connexive logic. An important progress was marked when one of the present authors, in [38], suggested to capture connexive theses through a different falsity condition for the conditional by building on the elegant framework of the four-valued logic known as **FDE**, or Belnap-Dunn logic. [3] More specifically, a system of connexive logic called **C** was introduced as a variant of Nelson's logic **N4** (cf. [37,23,16] and references therein) by making a small change to the falsity condition for the conditional (cf. Remark 2.2). [4] As Graham Priest notes in [31, p. 178], the system **C** is most likely to be "one of the simplest and most natural."

The aim of this article is to consider an extension of **C** obtained by adding the law of excluded middle (LEM hereafter) with respect to the strong negation. The extension of **C** is motivated by the following three questions.

Question 1: Can we improve Cantwell's CN? In [6], John Cantwell addresses the question of how to negate indicative conditionals, and defends the following three-valued truth table, suggested by Nuel D. Belnap in [5]:

A	$\sim A$
t	**f**
–	–
f	**t**

$A \to B$	**t**	–	**f**
t	**t**	–	**f**
–	**t**	–	**f**
f	–	–	–

A notable feature here is that the third value is meant to be a gappy value, and that when a conditional has a false antecedent it lacks a truth value. As a byproduct, two formulas $\sim(A \to B)$ and $A \to \sim B$ receive exactly the same value for every assignment, and thus these two formulas are equivalent. [5]

Cantwell then defined the consequence relation by designating both **t** and –, which can be seen as preserving non-false values. As a consequence of this, Cantwell's logic **CN** includes both Aristotle's and Boethius' theses, and thus is connexive, though connexivity is not mentioned at all.

However, **CN** also has the feature of the pure $\to$-fragment of the resulting logic being classical. This implies that formulas such as $(A \to B) \vee (B \to C)$ hold for arbitrary A, B and C. We are fully aware that there are attempts aiming at making sense of the material conditional as an indicative conditional (cf. [32] and references therein). But, it also seems to be a natural question if we can replace the classical conditional by a better conditional, such as a constructive conditional. And this is precisely the first question that motivates us to explore the extension of **C** by LEM.

Question 2: How many desiderata, listed by Estrada-González and Ramírez-Cámara, can be met by connexive logics à la C? In [12], Luis Estrada-González and Elisángela Ramírez-Cámara offer a list of desiderata for connexive logics which contains more than two characteristic theses of

[3] For an overview of systems related to **FDE**, cf. [29].

[4] Tweaking the falsity condition for other connectives seems to be interesting from the perspective of contra-classical logics (cf. [30]).

[5] See [10,11] for a recent discussion on **CN**. See also [9,28] for negated indicative conditionals being equivalent to formulas involving a modality.

connexive logics. In particular, the list contains the following formulas:

Aristotle's second thesis $\sim((A \to B) \wedge (\sim A \to B))$;

Abelard's thesis $\sim((A \to B) \wedge (A \to \sim B))$. [6]

Note that on the one hand, in [40], it has been argued that the above theses should not be considered as defining principles of connexive logic because they are motivated by the idea of negation as cancellation, which is said to be unsuitable as a basis for any validity claims. On the other hand, however, the above formulas are included, for example, in probabilistic approaches to conditionals, and thus interests into these formulas go beyond technical curiosity. They rather seem to capture certain intuitions on conditionals.

Regardless of one's opinion on the above two theses, it is mentioned already in [40,41] that the system **C** fails to include Aristotle's second thesis and Abelard's thesis. But, this does not mean that variants of **C** will fail both or at least one of the above theses. And this is precisely the second question that motivates us to explore the extension of **C** by LEM.

Question 3: What is the relation between constructivity and LEM? As per Jeremy Avigad [1, p. 10], "the words "constructive" and "intuitionistic" are used today almost interchangeably." Yves Lafont [14, p. 149] explains that "[i]ntuitionistic logic is called constructive because of the correspondence between proofs and algorithms," so that intuitionistic implication is essential for regarding intuitonistic logic as constructive. According to Paul Gilmore [13, p. *xiv*], the Disjunction Property and the Existence Property together are "the hallmark property of an intuitionistic/constructive logic." This conception of constructivity (or constructiveness) has been challenged by David Nelson, who in addition to keeping the intuitionistic conditional and desiring the disjunction and existence properties suggested to require also the Constructible Falsity Property (if $\sim(A \wedge B)$ is provable, then $\sim A$ or $\sim B$ is provable) and the dual of the Existence Property, cf. Section 6.

Our third question does not make any sense if we assume intuitionistic logic as the logic in question since the addition of LEM will collapse the logic into classical logic, which is admittedly non-constructive. The situation is similar in the case of Nelson's constructive logics with strong negation. Indeed, for the case with his **N3**, the addition of LEM with respect to the strong negation will again collapse the logic into classical logic. Moreover, for the case with **N4**, the addition of LEM with respect to the strong negation will *not* collapse the logic into classical logic, but instead the resulting logic will be the three-valued logic, for example known as **CLuNs** (cf. [4]), obtained by expanding the well-known three-valued paraconsistent logic **LP**, by the following truth table:

$A{\to}B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	**f**
f	**t**	**t**	**t**

[6] This thesis is also known as Strawson's thesis.

In particular, the ~-free fragment of **CLuNs** is classical, and thus the addition of LEM indeed destroys the constructive features, including the intuitionistic conditional, the Disjunction Property, and the Existence Property.

Note, however, that the proof of the collapse of **N4** into **CLuNs** heavily relies on the falsity condition for the conditional, and thus motivates our third question to explore the effect of adding LEM to **C**, which has a falsity condition different from that of **N4**. As we will show, the addition of LEM to a logic with a constructive implication need not collapse into classical logic. Moreover, the addition of LEM for atomic formulas need not prevent the Existence Property and its dual form holding.

Based on these considerations, the paper is structured as follows. We first revisit connexive logic **C** in §2. This is followed by §3 in which we introduce the extension of **C** by LEM. We then discuss the first two questions in §4. After these discussions at the level of propositional logic, we introduce, in §5, the extension of **QC**, quantified **C**, obtained by adding LEM. We then discuss the proof theory for **QC** and its extension in §6, and conclude the paper by a summary and remarks on future directions in §7.

2 Revisiting C

The language $\mathcal{L}$ consists of a finite set $\{\sim, \wedge, \vee, \rightarrow\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote the set of formulas defined as usual in $\mathcal{L}$ by Form, a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

2.1 Semantics

The following semantics, introduced in [38], is obtained by making a simple change to the standard semantics for Nelson's logic **N4**.

Definition 2.1 A **C**-model for the language $\mathcal{L}$ is a triple $\langle W, \leq, V\rangle$, where W is a non-empty set (of states); $\leq$ is a partial order on W; and $V : W \times \mathsf{Prop} \longrightarrow \{\varnothing, \{0\}, \{1\}, \{0,1\}\}$ is an assignment of truth values to state-variable pairs with the condition that $i \in V(w_1, p)$ and $w_1 \leq w_2$ only if $i \in V(w_2, p)$ for all $p \in \mathsf{Prop}$, all $w_1, w_2 \in W$ and $i \in \{0,1\}$. Valuations V are then extended to interpretations I of state-formula pairs by the following conditions:

- $I(w,p) = V(w,p)$,
- $1 \in I(w, {\sim}A)$ iff $0 \in I(w,A)$,
- $0 \in I(w, {\sim}A)$ iff $1 \in I(w,A)$,
- $1 \in I(w, A \wedge B)$ iff $1 \in I(w,A)$ and $1 \in I(w,B)$,
- $0 \in I(w, A \wedge B)$ iff $0 \in I(w,A)$ or $0 \in I(w,B)$,
- $1 \in I(w, A \vee B)$ iff $1 \in I(w,A)$ or $1 \in I(w,B)$,
- $0 \in I(w, A \vee B)$ iff $0 \in I(w,A)$ and $0 \in I(w,B)$,
- $1 \in I(w, A \rightarrow B)$ iff for all $w_1 \in W$: if $w \leq w_1$ and $1 \in I(w_1, A)$ then $1 \in I(w_1, B)$,
- $0 \in I(w, A \rightarrow B)$ iff for all $w_1 \in W$: if $w \leq w_1$ and $1 \in I(w_1, A)$ then $0 \in I(w_1, B)$.

Finally, the semantic consequence is now defined as follows: $\Gamma \models_{\mathbf{C}} A$ iff for all **C**-models $\langle W, \leq, I\rangle$, and for all $w \in W$: $1 \in I(w,A)$ if $1 \in I(w,B)$ for all $B \in \Gamma$.

Remark 2.2 Note that Nelson's logic **N4** is obtained by replacing the falsity condition for implication by the following condition:

$$0 \in I(w, A\to B) \text{ iff } 1 \in I(w, A) \text{ and } 0 \in I(w, B).$$

2.2 Proof System

We now turn to the proof system.

Definition 2.3 The axiomatic proof system **C** consists of the following axiom schemata and a rule of inference, where $A \leftrightarrow B$ abbreviates $(A\to B) \wedge (B\to A)$:

$$\begin{array}{cl}
A\to(B\to A) & \text{(Ax1)} \\
(A\to(B\to C))\to((A\to B)\to(A\to C)) & \text{(Ax2)} \\
(A \wedge B)\to A & \text{(Ax3)} \\
(A \wedge B)\to B & \text{(Ax4)} \\
(C\to A)\to((C\to B)\to(C\to(A\wedge B))) & \text{(Ax5)} \\
A\to(A \vee B) & \text{(Ax6)} \\
B\to(A \vee B) & \text{(Ax7)} \\
(A\to C)\to((B\to C)\to((A\vee B)\to C)) & \text{(Ax8)}
\end{array}
\qquad
\begin{array}{cl}
{\sim}{\sim}A \leftrightarrow A & \text{(Ax9)} \\
{\sim}(A\wedge B) \leftrightarrow ({\sim}A\vee{\sim}B) & \text{(Ax10)} \\
{\sim}(A\vee B) \leftrightarrow ({\sim}A\wedge{\sim}B) & \text{(Ax11)} \\
{\sim}(A\to B) \leftrightarrow (A\to{\sim}B) & \text{(Ax12)} \\
\dfrac{A \quad A\to B}{B} & \text{(MP)}
\end{array}$$

Finally, we write $\Gamma \vdash_{\mathbf{C}} A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of **C**; or (iii) is obtained by (MP) from formulas preceding it in sequence.

Remark 2.4 Note that if we replace (Ax12) by '${\sim}(A\to B) \leftrightarrow (A\wedge{\sim}B)$', then we obtain an axiomatization of Nelson's logic **N4**.

We also note that the deduction theorem is provable.

Proposition 2.5 *For any* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, $\Gamma, A \vdash_{\mathbf{C}} B$ *iff* $\Gamma \vdash_{\mathbf{C}} A\to B$.

Proof. It can be proved in the usual manner in the presence of axioms (Ax1) and (Ax2), given that (MP) is the sole rule of inference. □

2.3 Basic results and an observation

As expected, we have a soundness and completeness result, established in [38].

Theorem 2.6 ([38]) *For any* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, $\Gamma \vdash_{\mathbf{C}} A$ *iff* $\Gamma \vDash_{\mathbf{C}} A$.

A highly unusual feature of **C** is non-trivial but inconsistent.

Proposition 2.7 *For any* $A \in \mathsf{Form}$, $\vdash_{\mathbf{C}} (A\wedge{\sim}A)\to A$ *and* $\vdash_{\mathbf{C}} {\sim}((A\wedge{\sim}A)\to A)$.

Proof. The first item is (Ax3). For the second item, it will suffice to establish $\vdash (A\wedge{\sim}A)\to{\sim}A$ in view of (Ax12) and (MP), but this is just an instance of (Ax4). □

Note that the proof of inconsistency relies on very weak assumptions. Indeed, even with an extremely weak relevant implication (cf. [26, p. 477]) and the very weak conditional considered in [41], the above inconsistency result will hold. This may motivate to address the question if the axiom (Ax12), or the falsity condition for implication from **C**, will *always* give rise to inconsistency of the system. The short answer is: No.

Theorem 2.8 *There is a consistent system with the axiom* (Ax12)*.*

Proof. Take the propositional language only with $\sim$ and $\rightarrow$ (without $\wedge$), and consider an axiomatic proof system with (Ax9), (Ax12), (MP), and any axioms for $\rightarrow$ that only have an even number of occurrences of variables.

Then, we can see that this proof system is consistent. Indeed, we can take the two valued matrix for classical logic, and interpret $\sim$ and $\rightarrow$ as classical negation and classical biconditional, respectively. This clearly shows that the proof system is indeed consistent. □

In the rest of this paper, we will take the propositional language $\mathcal{L}$ that contains both conjunction and disjunction, and thus the inconsistency will be part of the features enjoyed by the system.

3 An extension of C

We now turn to the main extension of **C**. For the semantics, we simply close the gap.

Definition 3.1 A **C3**-model for the language $\mathcal{L}$ is a triple $\langle W, \leq, V \rangle$, where W is a non-empty set (of states); $\leq$ is a partial order on W; and $V : W \times \mathsf{Prop} \longrightarrow \{\{0\}, \{1\}, \{0,1\}\}$ is an assignment of truth values to state-variable pairs with the condition that $i \in V(w_1, p)$ and $w_1 \leq w_2$ only if $i \in V(w_2, p)$ for all $p \in \mathsf{Prop}$, all $w_1, w_2 \in W$ and $i \in \{0, 1\}$. Valuations V are then extended to interpretations I of state-formula pairs by the same conditions as with **C**. The semantic consequence $\models_{\mathbf{C3}}$ is defined in a similar manner.

Remark 3.2 Note that the persistence condition carries over to all formulas.

For the proof system, we add the law of excluded middle with respect to the strong negation.

Definition 3.3 The axiomatic proof system for **C3** is obtained by adding $A \vee {\sim}A$, namely LEM, to the axiomatic proof system for **C**. We then define $\vdash_{\mathbf{C3}}$ in a similar manner.

As usual, the soundness part is rather straightforward.

Proposition 3.4 (Soundness) *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$*, if* $\Gamma \vdash_{\mathbf{C3}} A$ *then* $\Gamma \models_{\mathbf{C3}} A$*.*

Proof. We only note that the elimination of the gappy value in the semantics guarantees that LEM is valid in all **C3**-models. □

For the completeness proof, we first introduce some standard notions.

Definition 3.5 $\Sigma \subseteq \mathsf{Form}$ is *deductively closed* iff if $\Sigma \vdash A$ then $A \in \Sigma$. The set Σ is *prime* iff $A \vee B \in \Sigma$ implies $A \in \Sigma$ or $B \in \Sigma$. Moreover, Σ is *prime deductively closed* (pdc) if it is both. Finally, Σ is *non-trivial* if $A \notin \Sigma$ for some A.

The following two lemmas are well-known, and thus the proofs are omitted.

Lemma 3.6 *If* $\Sigma \nvdash A$*, there is a non-trivial pdc* Δ *such that* $\Sigma \subseteq \Delta$ *and* $\Delta \nvdash A$*.*

Lemma 3.7 *If Σ is pdc and $A \to B \notin \Sigma$, there is a non-trivial pdc Θ such that $\Sigma \subseteq \Theta$, $A \in \Theta$ and $B \notin \Theta$.*

Now, we are ready to prove the completeness.

Theorem 3.8 (Completeness) *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \vDash_{\mathbf{C3}} A$ then $\Gamma \vdash_{\mathbf{C3}} A$.*

Proof. Suppose that $\Gamma \nvdash_{\mathbf{C3}} A$. Then by Lemma 3.6, there is a $\Pi \supseteq \Gamma$ such that Π is a pdc and $A \notin \Pi$. Define the model $\mathfrak{A} = \langle X, \leq, I \rangle$, where $X = \{\Delta : \Delta$ is a non-trivial pdc$\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and I is defined thus. For every state Σ and propositional variable p:

$$1 \in I(\Sigma, p) \text{ iff } p \in \Sigma \text{ and } 0 \in I(\Sigma, p) \text{ iff } {\sim}p \in \Sigma$$

Note that $\mathfrak{A}$ is indeed a **C3**-model since $1 \in I(\Sigma, p)$ or $0 \in I(\Sigma, p)$ holds for all Σ and p in view of LEM.

We now show that the above definition holds for arbitrary formula, B:

$$1 \in I(\Sigma, B) \text{ iff } B \in \Sigma \text{ and } 0 \in I(\Sigma, B) \text{ iff } {\sim}B \in \Sigma$$

This can be proved by a simultaneous induction on the complexity of B with respect to the positive and the negative clause.[7] It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \nvDash_{\mathbf{C3}} A$. □

4 Questions 1 and 2 in view of C3

We now turn to address first two questions, raised in our introduction, in light of the new system **C3**.

4.1 An answer to Question 1: C3 is a generalization of CN

In brief, our first question concerned the system **CN** defended and explored by Cantwell in [6]. More specifically, we asked if we can replace the classical material conditional by a constructive conditional.

Note first that from a semantic perspective, it is easy to see that **C3** expands positive intuitionistic propositional logic conservatively. Moreover, we have the following result that justifies to claim that **C3** is a generalization of **CN**.

Proposition 4.1 *The extension of* **C3** *by Peirce's law is sound and complete with respect to the semantics induced by the following matrix with* **t** *and* **b** *as designated values. In other words, the extension is the system* **CN** *of Cantwell.*

A	${\sim}A$
t	**f**
b	**b**
f	**t**

$A \wedge B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**f**

$A \vee B$	**t**	**b**	**f**
t	**t**	**t**	**t**
b	**t**	**b**	**b**
f	**t**	**b**	**f**

$A \to B$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**t**	**b**	**f**
f	**b**	**b**	**b**

Proof. For soundness, just note that every one-element model validates Peirce's law. For completeness, note first that the presence of Peirce's law makes the partial order on the canonical model trivial. More specifically, for two non-trivial pdcs Σ and Δ, we obtain that $\Sigma \subseteq \Delta$ only if $\Delta \subseteq \Sigma$. Indeed,

[7] The proof is the same as the one for [25, Theorem 2] with an obvious change to be made for the negative clause for the conditional.

suppose for reductio that $\Sigma \subseteq \Delta$ and that for some A_0, $A_0 \in \Delta$ but $A_0 \notin \Sigma$. Then, in view of Peirce's law, we also have $A \vee (A \to B)$ as a derivable formula, and thus we have $A_0 \vee (A_0 \to B) \in \Sigma$ for arbitrary B. In view of $A_0 \notin \Sigma$ and that Σ is prime, we obtain $(A_0 \to B) \in \Sigma$. This together with $\Sigma \subseteq \Delta$ implies $(A_0 \to B) \in \Delta$, and with $A_0 \in \Delta$, we obtain $B \in \Delta$. But since B is arbitrary, Δ will be trivial and this contradicts the assumption that Δ is non-trivial.

We can then consider the submodel of the canonical model with $X = \{\Pi\}$ where $\Pi \supseteq \Gamma$ such that Π is pdc and $A \notin \Pi$, obtained in view of Lemma 3.6. This completes the proof. □

Remark 4.2 Note that Grigory Olkhovikov, in [24], also introduced and discusses the above three-valued truth table for the conditional. Olkhovikov also motivates the truth table by considering conditionals in natural language, but with a different reading of the third value.

Remark 4.3 Compare the above result with **C** in which the addition of Peirce's law results in a four-valued logic, called **MC** (for material connexive logic) in [39], induced by the matrix obtained by adding the following truth table for implication in addition to the truth tables of **FDE**:

$A{\to}B$	**t**	**b**	**n**	**f**
t	**t**	**b**	**n**	**f**
b	**t**	**b**	**n**	**f**
n	**b**	**b**	**b**	**b**
f	**b**	**b**	**b**	**b**

An expansion of **MC** by the Boolean complement is explored in [27].

Remark 4.4 In view of the above result, there will be intermediate logics between **C3** and **CN**, as well as **C** and **MC**. Then, recalling that intermediate extensions of **N3** and **N4** (as well as $\mathbf{N4}^{\perp}$) are explored by Marcus Kracht in [18] and Sergei Odintsov in [23] respectively, describing the intermediate extensions of **C3** and **C** is an interesting open problem.

4.2 An answer to Question 2: C3 gives us all if we are careful

We now turn to the second question of how much connexivity can be captured through the present approach via a slightly different falsity condition. Our considerations build on a list provided by Estrada-González and Ramírez-Cámara in [12] which includes the following four theses that are contra-classical:

Aristotle $\sim(\sim A \to A)$, $\sim(A \to \sim A)$;
Boethius $(A \to B) \to \sim(A \to \sim B)$, $(A \to \sim B) \to \sim(A \to B)$;
Aristotle 2nd $\sim((A \to B) \wedge (\sim A \to B))$;
Abelard $\sim((A \to B) \wedge (A \to \sim B))$.

Then, we first observe that systems **C** and **C3** are not able to capture all of the above four theses.

Proposition 4.5 *Both Aristotle's and Boethius' theses are derivable in* **C** *and* **C3**. *However, (i)* **C** *fails to include Aristotle's second thesis and Abelard's thesis (this was mentioned already in [40,41]); (ii)* **C3** *includes Abelard's thesis* *but not Aristotle's second thesis.*

Proof. The first half, namely that **C** and **C3** include Aristotle's and Boethius' theses as derivable formulas is well-known and immediate in view of (Ax12). For the second half, we can make use of the truth tables.

- For the failures in **C**, it is enough to establish that the concerned theses are not valid in **MC**. Now, take the four-valued truth tables for **MC**. Then, if we assign values **b** and **t** to A and B respectively, Aristotle's second thesis receives the value **f**. Moreover, if we assign values **t** and **n** to A and B respectively, Abelard's second thesis receives the value **n**.
- For the cases of **C3**, we first observe that Aristotle's second thesis is not valid in **CN**. To this end, if we assign values **b** and **t** to A and B respectively, we obtain the desired result. For the derivability of Abelard's thesis, it is enough to establish that $(A{\rightarrow}B)\lor(A{\rightarrow}{\sim}B)$ is derivable in **C3**, and this follows by LEM and (Ax1).

This completes the proof. □

The above results show that the move to **C3** from **C** will allow us to capture one more thesis, namely Abelard's thesis. However, Aristotle's second thesis is not captured, but this is not the end of the story. This is because we may consider another very natural conditional, $\Rightarrow$, in **C** and **C3** defined as follows:

$$A \Rightarrow B := (A \rightarrow B) \land ({\sim}B \rightarrow {\sim}A).$$

One of the obvious differences is that the contraposition rule holds for the "strong implication" $\Rightarrow$, which was not the case with $\rightarrow$. More importantly, we have the following result.

Theorem 4.6 *All four theses listed above, formulated in terms of* $\Rightarrow$*, are derivable in* **C3**. *However, only Aristotle's and Boethius' theses, formulated in terms of* $\Rightarrow$*, are derivable in* **C**.

Proof. We first check that Aristotle's and Boethius' theses, formulated in terms of $\Rightarrow$, are derivable in **C**, and thus also in **C3**. To this end, note that the following equivalences are derivable in view of the definition of $\Rightarrow$:

- $(A \Rightarrow {\sim}B) \leftrightarrow ((A \rightarrow {\sim}B) \land (B \rightarrow {\sim}A))$;
- ${\sim}(A \Rightarrow B) \leftrightarrow ((A \rightarrow {\sim}B) \lor ({\sim}B \rightarrow A))$;
- ${\sim}(A \Rightarrow {\sim}B) \leftrightarrow ((A \rightarrow B) \lor (B \rightarrow A))$.

Then, it is obvious that Aristotle's theses are derivable by the last equivalence. For Boethius' theses, we need to check that the following holds in **C**:

- $(A \Rightarrow B) \rightarrow {\sim}(A \Rightarrow {\sim}B)$;
- $(A \Rightarrow {\sim}B) \rightarrow {\sim}(A \Rightarrow B)$.

But these are obvious in view of the above equivalences, and thus we obtain Boethius' theses in **C** and also in **C3**.

We now turn to check that Abelard's thesis, formulated in terms of $\Rightarrow$, is derivable in **C3**. For this purpose, simply note that the thesis is equivalent to $((A{\rightarrow}{\sim}B)\lor({\sim}B{\rightarrow}A))\lor((A{\rightarrow}B)\lor(B{\rightarrow}A))$. Then, by looking at the first and the third disjuncts, we can see that the above formula is derivable since $(A \to B) \vee (A \to {\sim}B)$ is derivable in **C3**, as we observed in the previous proposition.

Similarly, for the case of Aristotle's second thesis, it suffices to derive $((A \to {\sim}B) \vee ({\sim}B \to A)) \vee (({\sim}A \to {\sim}B) \vee ({\sim}B \to {\sim}A))$. This time, by looking at the second and the fourth disjuncts, we can see that the above formula is derivable for the same reason.

Finally, in order to see that Aristotle's second thesis and Abelard's thesis are not derivable in **C**, it suffices to see that the above two formulas have a counter-model $\langle W, \leq, V \rangle$ defined as follows:

- $W := \{w_0, w_1, w_2\}$, $\leq := \{\langle w_0, w_0\rangle, \langle w_0, w_1\rangle, \langle w_0, w_2\rangle, \langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}$,
- $V(w_0, p)$=$V(w_0, q)$=$V(w_1, p)$=$V(w_2, q)$=$\{\}$ and $V(w_1, q)$=$V(w_2, p)$=$\{1, 0\}$.

This completes the proof. □

Remark 4.7 Note that all four theses, formulated in terms of $\Rightarrow$, hold in **CN** and **MC**. This is due to the fact that ${\sim}(A \Rightarrow B)$ is equivalent to $(A \to {\sim}B) \vee ({\sim}B \to A)$, an instance of the linearity axiom. Thus, for *arbitrary* formulas A and B, we have ${\sim}(A \Rightarrow B)$ as valid since $\to$ in both **CN** and **MC** is classical. This is in sharp contrast with **C3** since we have $\nvDash_{\mathbf{C3}} {\sim}(A \Rightarrow B)$. Indeed, we can consider a counter-model $\langle W, \leq, V \rangle$ defined as follows:

- $W := \{w_0, w_1, w_2\}$, $\leq := \{\langle w_0, w_0\rangle, \langle w_0, w_1\rangle, \langle w_0, w_2\rangle, \langle w_1, w_1\rangle, \langle w_2, w_2\rangle\}$,
- $V(w_0, p)$=$V(w_1, p)$=$\{0\}$, $V(w_0, q)$=$V(w_2, q)$=$\{1\}$, $V(w_1, q)$=$V(w_2, p)$=$\{1, 0\}$.

We summarize our observations from this subsection in a table as follows:

	C		**C3**		**MC**		**CN**	
	$\to$	$\Rightarrow$	$\to$	$\Rightarrow$	$\to$	$\Rightarrow$	$\to$	$\Rightarrow$
Aristotle's theses	✓	✓	✓	✓	✓	✓	✓	✓
Boethius' theses	✓	✓	✓	✓	✓	✓	✓	✓
Aristotle's second theses	×	×	×	✓	×	✓	×	✓
Abelard's theses	×	×	✓	✓	✓	✓	✓	✓
Negated strong conditional	×	×	×	×	×	✓	×	✓

In short, the contraposible strong implication $\Rightarrow$ of **C3** seems to be a very interesting conditional, satisfying all four contra-classical theses, without having the problematic-looking formula, i.e. ${\sim}(A \Rightarrow B)$, as a valid formula.

4.3 Beyond Question 2: Totally connexive logics

The four theses we focused on in the previous subsection are part of a bigger list provided by Estrada-González and Ramírez-Cámara. Indeed, they also considered the following desiderata on top of the four theses.

Positive Paradox of Implication $\nvDash A \to (B \to A)$;
Negative Paradox of Implication $\nvDash A \to (\neg A \to B)$;
Paradox of Necessity $\nvDash A \to (B \to C)$ where A is a contingent truth and $B \to C$ is a logical truth;
Simplification $\vDash (A \wedge B) \to A$, $\vDash (A \wedge B) \to B$;
Idempotence $\vDash (A \wedge A) \to A$, $\vDash A \to (A \wedge A)$;
Kapsner-strong (i) $A \to {\sim}A$ is unsatisfiable and (ii) $A \to B$ and $A \to {\sim}B$ are

not simultaneously satisfiable.

Estrada-González and Ramírez-Cámara then introduced the notion of *totally connexive logics* as logics that satisfy all the desiderata, including the four theses. Moreover, they left as an open problem whether there are totally connexive logics, and if so then which is the minimal one (cf. [12, p. 348]).

In view of our observations in the previous section, there are three candidates for totally connexive logics, namely **C3**, **MC** and **CN**, by looking at the contraposible conditional $\Rightarrow$.[8] Since both **C3** and **MC** are subsystems of **CN**, and since the above desiderata involve some invalidities, let us focus on **CN**. Then, we obtain the following truth table for $\Rightarrow$ in **CN**:

$A \Rightarrow B$	**t**	**b**	**f**
t	**b**	**f**	**f**
b	**b**	**b**	**f**
f	**b**	**b**	**b**

This truth table is not new, but was already introduced and discussed by Chris Mortensen in [21] in a logic, later called **M3V** in [20]. Recall that **M3V** is an expansion of **LP** obtained by adding the above conditional. Before adding further remarks, we observe that **CN** and **M3V** are equivalent systems.

Proposition 4.8 **CN** *and* **M3V** *are equivalent.*

Proof. All connectives in **M3V** are definable in **CN** in view of the above discussion. For the other half, we need to show that the non-contraposible conditional of **CN** is definable in **M3V**. This can be checked by observing that $((A \Rightarrow B) \vee B) \wedge {\sim}(A \Rightarrow ({\sim}(A \Rightarrow B) \wedge (B \wedge {\sim}B)))$ defines $\rightarrow$ of **CN**. □

In fact, **M3V** is examined by Estrada-González and Ramírez-Cámara, and is shown to satisfy all the desiderata except Kapsner-strong. Building on this observation, we conclude that all three candidates enjoy the same status since Simplification and Idempotence are easy to check for the contraposible conditional of both **C** and **C3**.

For the Kapsner-strong condition, we wish to add a remark. We are fully aware of Andreas Kapsner's original motivation, argued in [17], as well as his intended way of spelling out the details in a rather classical manner.[9] However, if we are in the vicinity of **FDE**, then there are always at least two ways to formalize classical notions since truth and non-falsity (or, falsity and non-truth) are not necessarily equivalent.[10] In particular, the notion of satisfiability, which plays the crucial role in the Kapsner-strong condition, can be formalized as follows in **M3V**:

[8] Actually, **CN** is one of the systems examined by Estrada-González and Ramírez-Cámara, but with respect to the non-contraposible conditional $\rightarrow$.

[9] Recall that the original proposal of Kapsner in [17] was to call for *strong connexivity*, and that notion encompasses not only the theses of Aristotle and Boethius, but also the unsatisfiability clauses which we refer to as Kapsner-strong conditions following [12].

[10] For example, think of discussions on $p-$ and $q-$consequence relations that have revived recently through a series of papers by Pablo Cobreros, Paul Egré, Dave Ripley, and Robert van Rooij (e.g. [7,8]).

- A is *positively satisfiable* iff for some **M3V**-valuation V, $1 \in I(A)$.
- A is *negatively satisfiable* iff for some **M3V**-valuation V, $0 \notin I(A)$.

Then, as observed by Estrada-González and Ramírez-Cámara, **M3V** is not Kapsner-strong, if satisfiability is understood as *positive* satisfiability. However, it *is* Kapsner-strong, if satisfiability is understood as *negative* satisfiability since $0 \in I(A \Rightarrow B)$ for all **M3V**-valuations V and for all A and B.

For **C3**, we may formulate two kinds of satisfiability as follows.

- A is *positively satisfiable* iff for some **C3**-model $\mathcal{M} = \langle W, \leq, V \rangle$, $1 \in I(w, A)$ for some $w \in W$.
- A is *negatively satisfiable* iff for some **C3**-model $\mathcal{M} = \langle W, \leq, V \rangle$, $0 \notin I(w, A)$ for some $w \in W$.

Then, **C3** is Kapsner-strong if satisfiability is understood as *negative* satisfiability. Indeed, for all **C**-models $\mathcal{M} = \langle W, \leq, V \rangle$ and for all $w \in W$, we have $0 \in I(w, A \Rightarrow {\sim}A)$ and $0 \in I(w, (A \Rightarrow B) \wedge (A \Rightarrow {\sim}B))$.

Based on these observations, we conclude that there are totally connexive logics, and examples include **C3**, **MC** and **CN** by looking at the contraposible conditional. Note also that **C3** enjoys the additional feature of being totally connexive *without* all negated conditionals being valid (cf. Remark 4.7).

Remark 4.9 The key idea in this section has been to consider the contraposible conditional in **C** and its extensions. Then, in view of results reported by Matthew Spinks and Robert Veroff, such as those in [33] and references therein, establishing clear and neat connections between Nelson logics and relevant logics, it is a natural and interesting question to explore the connections between extensions of **C** and relevant logics as well.

Moreover, a deeper understanding of these connections may also give us some new insights into the problem of finding sound and complete semantics for systems introduced by Everett Nelson. More specifically, Nelson, in his PhD thesis, introduced an axiomatic system of connexive logic, called **NL** by Edwin Mares and Francesco Paoli in [19]. Then, the open problem, noted in [39], is to find a sound and complete semantics for **NL**. Since one of the subsystems, called $\mathbf{NL}^{-}$ in [19], is close to the relevant logic **DK**, we may seek for a suitable semantics via the contraposible conditional of one of the systems related to **C3**.

Before moving further, here is a table indicating the relations between extensions of **C** we discussed so far, with a comparison to extensions of **N4**.

$$\begin{tikzcd}
\mathbf{MC} \arrow[r, "+\mathrm{LEM}"] & \mathbf{CN}(=\mathbf{M3V}) & \mathbf{HBe} \arrow[r, "+\mathrm{LEM}"] & \mathbf{CLuNs}(=\mathbf{RM3}) \\
\mathbf{C} \arrow[u, "+\mathrm{PL}"] \arrow[r, "+\mathrm{LEM}"] & \mathbf{C3} \arrow[u, "+\mathrm{PL}"] & \mathbf{N4} \arrow[u, "+\mathrm{PL}"] \arrow[ur, "+\mathrm{LEM}"] &
\end{tikzcd}$$

Note here that PL stands for Peirce's law. Moreover, **HBe** is an expansion of **FDE** explored by Arnon Avron in [3], and the equivalence of **CLuNs** and **RM3** is shown in [2, (a) of 2.10 Theorem] (Avron refers to **CLuNs** as $\mathbf{RM}_3^{\supset}$).

5 An extension of QC

We will now consider the expansion of **C** and **C3** to their quantified versions **QC** and **QC3** in a language without function symbols and equality.

We extend the propositional language $\mathcal{L}$ to a first-order language by adding denumerably many individual variables, x, y, z, ..., constants a, b, c, ..., and predicate symbols of finite arity. Terms (i.e, individual variables or constants) are denoted by $t, t_1, t_2, \ldots$, atomic formulas by $P, Q, \ldots$, and arbitrary formulas by A, B, etc.

5.1 Axiomatic proof systems

Definition 5.1 ([38, §4]) The schematic axioms and rules of **QC** are those of **C** together with:

$\sim\exists xA \leftrightarrow \forall x{\sim}A$;	$\sim\forall xA \leftrightarrow \exists x{\sim}A$
$A(t) \to \exists xA(x)$ (t is free for x in A);	$\forall xA(x) \to A(t)$ (t is free for x in A)
$\dfrac{A \to B(x)}{A \to \forall xB(x)}$ (x not free in A);	$\dfrac{A(x) \to B}{\exists xA(x) \to B}$ (x not free in B)

Definition 5.2 The axiom schemata and rules of **QC3** and **QC3at** are those of **QC** together with LEM, resp. (LEMat): $P \vee {\sim}P$, for atomic formulas P.

Deducibility in **QC**, **QC3**, and **QC3at** and the consequence relations $\vdash_{\mathbf{QC}}$, $\vdash_{\mathbf{Q3C}}$, and $\vdash_{\mathbf{Q3Cat}}$ are defined in the usual way. In [38, Prop. 11] the axiomatic system **QC** is shown to be complete with respect to a suitable Kripke semantics by means of a faithful embedding into positive first-order intuitionistic logic.

Remark 5.3 Note that the embedding-based method is not available to us here since it is not clear into which system we can embed **QC3**. Thus, we leave this as an open problem.

Since the proof-theoretic aspect was not explored so far even for **QC**, we here focus on sequent calculi for **QC**, **QC3**, and **QC3at**.

5.2 Sequent calculi

In this section, we define cut-free sequent calculi for **QC**, **QC3**, and **QC3at**. The presentation is based on [22, §5.4] by Sara Negri and Jan von Plato, where the sequent calculus **G3i** for intuitionistic predicate logic (without equality) is extended by a rule, *Gem-at*, that captures LEMat. Whereas the addition of *Gem-at* to the sequent calculus **G3ip** for intuitionistic propositional logic results in a sequent system for classical propositional logic, the addition of *Gem-at* to **G3i** results in a proof system for an extension of classical propositional logic by the intuitionistic universal and particular quantifiers. Negri and von Plato [22, p. 121] remark that the proof of admissibility of excluded middle for arbitrary formulas for **G3ip** + *Gem-at* cannot be extended to quantified formulas. We shall therefore add an excluded middle rule for arbitrary formulas, *Gem*, to a sequent calculus **G3C** for **QC**. For the addition of *Gem-at* to **G3C** we prove the Existence Property and a Dual Existence Property.

We first present the sequent calculus **G3C** for **QC**. Uppercase Greek letters now stand for finite, possibly empty multisets of formulas, A, Γ stands for

$\{A\} \uplus \Gamma$, and Δ, Γ stands for $\Delta \uplus \Gamma$, where $\uplus$ is multiset union. Sequents are of the form $\Gamma \Rightarrow A$ ($\Rightarrow$ is used in this way, hereafter, not for strong implication).

Definition 5.4 The rules of the calculus **G3C** are the following:
Logical axioms:
$P, \Gamma \Rightarrow P \qquad {\sim}P, \Gamma \Rightarrow {\sim}P$, for atomic formulas P
Logical rules:

$$\frac{A, B, \Gamma \Rightarrow C}{(A \wedge B), \Gamma \Rightarrow C}\; L\wedge \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \wedge B)}\; R\wedge$$

$$\frac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{(A \vee B), \Gamma \Rightarrow C}\; L\vee \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow (A \vee B)}\; R\vee_1 \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow (A \vee B)}\; R\vee_2$$

$$\frac{(A \to B), \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{(A \to B), \Gamma \Rightarrow C}\; L\to \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow (A \to B)}\; R\to$$

$$\frac{A(t/x), \forall x A, \Gamma \Rightarrow B}{\forall x A, \Gamma \Rightarrow B}\; L\forall \qquad \frac{\Gamma \Rightarrow A(y/x)}{\Gamma \Rightarrow \forall x A}\; R\forall \qquad \frac{A(y/x), \Gamma \Rightarrow B}{\exists x A, \Gamma \Rightarrow B}\; L\exists$$

$$\frac{\Gamma \Rightarrow A(t/x)}{\Gamma \Rightarrow \exists x A}\; R\exists \qquad \frac{A, \Gamma \Rightarrow C}{{\sim\sim}A\Gamma \Rightarrow C}\; L{\sim\sim} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow {\sim\sim}A}\; R{\sim\sim}$$

$$\frac{{\sim}A, {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \vee B), \Gamma \Rightarrow C}\; L{\sim}\vee \qquad \frac{\Gamma \Rightarrow {\sim}A \quad \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \vee B)}\; R{\sim}\vee$$

$$\frac{{\sim}A, \Gamma \Rightarrow C \quad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \wedge B), \Gamma \Rightarrow C}\; L{\sim}\wedge \qquad \frac{\Gamma \Rightarrow {\sim}A}{\Gamma \Rightarrow {\sim}(A \wedge B)}\; R{\sim}\wedge_1 \qquad \frac{\Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \wedge B)}\; R{\sim}\wedge_2$$

$$\frac{{\sim}(A \to B), \Gamma \Rightarrow A \quad {\sim}B, \Gamma \Rightarrow C}{{\sim}(A \to B), \Gamma \Rightarrow C}\; L{\sim}{\to} \qquad \frac{A, \Gamma \Rightarrow {\sim}B}{\Gamma \Rightarrow {\sim}(A \to B)}\; R{\sim}{\to}$$

$$\frac{{\sim}A(y/x), \Gamma \Rightarrow B}{{\sim}\forall x A, \Gamma \Rightarrow B}\; L{\sim}\forall \qquad \frac{\Gamma \Rightarrow {\sim}A(t/x)}{\Gamma \Rightarrow {\sim}\forall x A}\; R{\sim}\forall$$

$$\frac{{\sim}A(t/x), {\sim}\exists x A, \Gamma \Rightarrow B}{{\sim}\exists x A, \Gamma \Rightarrow B}\; L{\sim}\exists \qquad \frac{\Gamma \Rightarrow {\sim}A(y/x)}{\Gamma \Rightarrow {\sim}\exists x A}\; R{\sim}\exists$$

where (i) in $R\forall$ and in $R{\sim}\exists$, y must not occur free in $\Gamma, \forall x A$, resp. in $\Gamma, {\sim}\exists x A$ and (ii) in $L\exists$ and in $L{\sim}\forall$, y must not occur free in $\exists x A, \Gamma, B$, resp. in ${\sim}\forall x A, \Gamma, B$.

Definition 5.5 The rules of the calculus **G3C3**, respectively **G3C3at**, are those of **G3C** plus:

$$\frac{B, \Gamma \Rightarrow A \quad {\sim}B, \Gamma \Rightarrow A}{\Gamma \Rightarrow A}\; \textit{Gem} \quad \text{resp.} \quad \frac{P, \Gamma \Rightarrow A \quad {\sim}P, \Gamma \Rightarrow A}{\Gamma \Rightarrow A}\; \textit{Gem-at}$$

for atomic formulas P.

6 Some basic proof-theoretic results

As in [22, Lemmas 4.1 and 4.1.2], one can prove a formal version of the principle of renaming bound variables and a lemma showing that derivability of sequents is preserved with the same derivation height if a term t is substituted for a free variable x in a sequent $\Gamma \Rightarrow A$, provided that t is free for x in formulas from $\Gamma \Rightarrow A$. Also, one can easily show that for any formula A, sequents of the form $A, \Gamma \Rightarrow A$ are provable in **G3C** (and hence in **G3C3at** and **G3C3**). Moreover, the versions of the sequent rules $L{\rightarrow}$, $L{\sim}{\rightarrow}$, $L\forall$, and $L{\sim}\exists$ without repetitions of principal formulas are admissible in the calculi where they are present.

The proof of height-preserving admissibility of weakening and contraction

$$\frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B}\ Wk \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma, A, A \Rightarrow B}\ Ctr$$

in [22] can be adapted to **G3C**, **G3C3at**, and **G3C3**. In particular, one has to show that the rules $L{\sim}\vee$, $L{\sim}\wedge$, $L{\sim}\forall$ are height-preserving invertible and that the rule $L{\sim}\rightarrow$ is height-preserving invertible for its second (right) premise.

Theorem 6.1 *The Cut rule*

$$\frac{\Gamma \Rightarrow A \quad A, \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow B}\ Cut$$

is an admissible rule of **G3C**, **G3C3at**, *and* **G3C3**.

Proof. The proof follows the standard pattern presented in [22]. In particular, if one of the premises of *Cut* is derived by an LEM rule, *Cut* with the same cut-formula is permuted upwards to applications of *Cut* with a smaller cut-height. The derivation

$$\frac{\Gamma \Rightarrow A \qquad \dfrac{B, A, \Delta \Rightarrow C \quad {\sim}B, A, \Delta \Rightarrow C}{A, \Delta \Rightarrow C}\ Gem}{\Gamma, \Delta \Rightarrow C}\ Cut$$

for example, is replaced by

$$\frac{\dfrac{\Gamma \Rightarrow A \quad B, A, \Delta \Rightarrow C}{B, \Gamma, \Delta \Rightarrow C}\ Cut \qquad \dfrac{\Gamma \Rightarrow A \quad {\sim}B, A, \Delta \Rightarrow C}{{\sim}B, \Gamma, \Delta \Rightarrow C}\ Cut}{\Gamma, \Delta \Rightarrow C}\ Gem$$

This completes the proof (sketch). □

To every finite set of formulas Γ, there corresponds a unique multiset (with no multiplicity of elements). If Γ is such a set, let $\bigwedge \Gamma$ be a conjunction of all formulas from the corresponding multiset. Conversely, to every finite multiset Γ, there corresponds a unique set, which we will also denote by Γ.

Theorem 6.2 (Equivalence of proof systems) *Let* Γ *be a finite set of formulas and let* $\bigwedge \varnothing = (P \rightarrow P)$*, for some fixed atomic formula* P*. Then* $\Gamma \vdash_{\mathbf{QC3}} A$ *iff* $\Rightarrow \bigwedge \Gamma \rightarrow A$ *is derivable in* **G3C3**.

Proof. Left-to-right: It is enough to show that $\Rightarrow A$ is derivable in **G3C3** for every theorem A of **QC3** and that the inference rules preserve derivability. We present two cases. For (LEM), we have

$$\frac{\dfrac{\dfrac{\vdots}{A \Rightarrow A}}{A \Rightarrow (A \vee {\sim}A)} \qquad \dfrac{\dfrac{\vdots}{{\sim}A \Rightarrow {\sim}A}}{{\sim}A \Rightarrow (A \vee {\sim}A)}}{\Rightarrow (A \vee {\sim}A)}$$

where the vertical dots indicate routine derivations. For the $\forall$-rule of **QC3** we have

$$\dfrac{\dfrac{\Rightarrow A \to B(x) \quad \overset{\vdots}{A, (A \to B(x)) \Rightarrow B(x)}}{A \Rightarrow B(x)}}{A \Rightarrow \forall x B(x)}\; R\forall$$

for x not free in A.
Right-to-left: By induction on the height of derivations in **G3C3** because $\Rightarrow \bigwedge\Gamma \to A$ is derivable in **G3C3** iff $\Gamma \Rightarrow A$ is. For axioms $P, \Delta \Rightarrow P$ and ${\sim}P, \Delta \Rightarrow {\sim}P$ we have $P \in \Delta \uplus \{P\}$, respectively ${\sim}P \in \Delta \uplus \{{\sim}P\}$. For the induction steps we consider the sequent rules. In the case of *Gem*, by the induction hypothesis we have $B, \Gamma \vdash_{\mathbf{QC3}} A$ and ${\sim}B, \Gamma \vdash_{\mathbf{QC3}} A$. By (LEM) and reasoning in positive intuitionistic propositional logic, as we can do in **QC3**, we obtain $\Gamma \vdash_{\mathbf{QC3}} A$. We present two more cases involving negation. Consider the rule $L{\sim}\forall$. By the induction hypothesis, we have ${\sim}A(y/x), \Gamma \vdash_{\mathbf{QC3}} B$, where y does not occur free in $\exists x A, \Gamma, B$. We easily obtain ${\sim}A(y/x) \vdash_{\mathbf{QC3}} \bigwedge\Gamma \to B$. By the $\exists$-rule of **QC3**, respecting its side-condition, we obtain $\exists x{\sim}A(x) \vdash_{\mathbf{QC3}} \bigwedge\Gamma \to B$. By axiom ${\sim}\forall x A(x) \leftrightarrow \exists x{\sim}A(x)$, we get ${\sim}\forall A(x) \vdash_{\mathbf{QC3}} \bigwedge\Gamma \to B$ and then ${\sim}\forall A(x), \Gamma \vdash_{\mathbf{QC3}} B$. Consider the rule $L{\sim}{\to}$. Since the version of the rule without repetition of the principal formula is admissible, we may assume by the induction hypothesis that $\Gamma \vdash_{\mathbf{QC3}} A$, and ${\sim}B, \Gamma \vdash_{\mathbf{QC3}} C$. Since $A, (A \to {\sim}B) \vdash_{\mathbf{QC3}} {\sim}B$, we successively obtain $(A \to {\sim}B), \Gamma \vdash_{\mathbf{QC3}} {\sim}B$ and $(A \to {\sim}B), \Gamma \vdash_{\mathbf{QC3}} C$. By (Ax12), we then get ${\sim}(A \to {\sim}B), \Gamma \vdash_{\mathbf{QC3}} C$. This completes the proof. □

Proposition 6.3 *The Disjunction Property and the Constructible Falsity Property fail for* **G3C3at***.*

Proof. Both $\Rightarrow (P \vee {\sim}P)$ and $\Rightarrow {\sim}(P \wedge {\sim}P)$ are derivable in **G3C3at** for atomic formulas P. However, for no atomic formula P, *both* $\Rightarrow P$ and $\Rightarrow {\sim}P$ are derivable with the aid of *Gem-at*. □

Theorem 6.4 *The excluded middle rule Gem is admissible in* **G3C3at** *for arbitrary quantifier-free formulas.*

Proof. As in [22, Theorem 5.4.6], the proof is by induction on the length of a formula D. The rules shown to be admissible may be used, *Inv* indicates invertible rules, and *Ind* indicates applications of the induction hypothesis. For atomic formulas we have *Gem-at*.
D is a disjunction $(A \vee B)$. Apply the induction hypothesis to the following two derivations:

$$\dfrac{(A \vee B), \Gamma \Rightarrow C}{B, \Gamma \Rightarrow C}\; Inv$$

$$\dfrac{\dfrac{\dfrac{(A \vee B), \Gamma \Rightarrow C}{A, \Gamma \Rightarrow C}\; Inv}{A, {\sim}B, \Gamma \Rightarrow C}\; Wk \qquad \dfrac{\dfrac{\overset{\vdots}{({\sim}A \wedge {\sim}B) \Rightarrow {\sim}(A \vee B)} \quad {\sim}(A \vee B), \Gamma \Rightarrow C}{({\sim}A \wedge {\sim}B), \Gamma \Rightarrow C}\; Cut}{{\sim}A, {\sim}B, \Gamma \Rightarrow C}\; Inv}{{\sim}B, \Gamma \Rightarrow C}\; Ind$$

D is a conjunction $(A \wedge B)$. Apply the induction hypothesis to the following two derivations:

$$\dfrac{\dfrac{(A \wedge B), \Gamma \Rightarrow C}{A, B, \Gamma \Rightarrow C}\ \mathit{Inv} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ (\sim A \vee \sim B) \Rightarrow \sim(A \wedge B)\end{matrix} \quad \sim(A \wedge B), \Gamma \Rightarrow C}{(\sim A \vee \sim B) \Rightarrow C}\ \mathit{Cut}}{\sim A, \Gamma \Rightarrow C}\ \mathit{Inv}}{\sim A, B, \Gamma \Rightarrow C}\ \mathit{Wk}}{B, \Gamma \Rightarrow C}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ (\sim A \vee \sim B) \Rightarrow \sim(A \wedge B)\end{matrix} \quad \sim(A \wedge B), \Gamma \Rightarrow C}{(\sim A \vee \sim B), \Gamma \Rightarrow C}\ \mathit{Cut}}{\sim B, \Gamma \Rightarrow C}$$

D is an implication $(A \rightarrow B)$: Apply the induction hypothesis to the following two derivations

$$\dfrac{(A \rightarrow B), \Gamma \Rightarrow C}{B, \Gamma \Rightarrow C}\ \mathit{Inv}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ A, \sim B \Rightarrow \sim(A \rightarrow B)\end{matrix} \quad \sim(A \rightarrow B), \Gamma \Rightarrow C}{A, \sim B, \Gamma \Rightarrow C}\ \mathit{Cut} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ (A \rightarrow \sim B) \Rightarrow \sim(A \rightarrow B)\end{matrix} \quad \sim(A \rightarrow B), \Gamma \Rightarrow C}{(A \rightarrow \sim B), \Gamma \Rightarrow C}\ \mathit{Cut}}{\sim B, \Gamma \Rightarrow C}\ \mathit{Inv}}{\sim A, \sim B, \Gamma \Rightarrow C}\ \mathit{Wk}}{\sim B, \Gamma \Rightarrow C}\ \mathit{Ind}$$

D is of the form $\sim(A \sharp B)$, $\sharp \in \{\vee, \wedge, \rightarrow\}$: Similar to the previous cases because $(A \sharp B), \Gamma \Rightarrow C$ is derivable from $\sim\sim(A \sharp B), \Gamma \Rightarrow C$.
D is a double negation $\sim\sim A$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots \\ A \Rightarrow A\end{matrix}}{A \Rightarrow \sim\sim A} \quad \sim\sim A, \Gamma \Rightarrow B}{A, \Gamma \Rightarrow B}\ \mathit{Cut} \qquad \dfrac{\dfrac{\begin{matrix}\vdots \\ \sim A \Rightarrow \sim A\end{matrix}}{\sim A \Rightarrow \sim\sim\sim A} \quad \sim\sim\sim A, \Gamma \Rightarrow A}{\sim A, \Gamma \Rightarrow B}\ \mathit{Cut}}{\Gamma \Rightarrow B}\ \mathit{Ind}$$

This completes the proof. □

Theorem 6.5 (Existence Properties) *If* $\Rightarrow \exists x A$ *is derivable in* **G3C3at***, then so is* $\Rightarrow A(t/x)$ *for some term* t*. If* $\Rightarrow \sim\forall x A$ *is derivable in* **G3C3at***, then so is* $\Rightarrow \sim A(t/x)$ *for some term* t*.*

Proof. We consider the first claim; the proof of the second claim is analogous. Suppose $\Rightarrow \exists x A$ is derivable in **G3C3at**. Then the last step in the derivation is either an application of $R\exists$, and we are done, or it is an application of *Gem-at*:

$$\dfrac{P \Rightarrow \exists x A \quad \sim P \Rightarrow \exists x A}{\Rightarrow \exists x A}$$

The same kind of case distinction applies to the derivations of $P \Rightarrow \exists A$ and $\sim P \Rightarrow \exists A$. Since every derivation is a finite tree, the derivation of $\Rightarrow \exists x A$ has the shape

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Delta_1 \Rightarrow A(t_1/x)}}{\Delta_1 \Rightarrow \exists x A}\,R\exists \quad \cdots \quad \dfrac{\dfrac{\vdots}{\Delta_n \Rightarrow A(t_n/x)}}{\Delta_n \Rightarrow \exists x A}\,R\exists}{\dfrac{P \Rightarrow \exists x A \quad \sim P \Rightarrow \exists x A}{\Rightarrow \exists x A}}
$$

for some $n \in \mathbb{N}$ with $2 \leq n$, where every Δ_i $(1 \leq i \leq n)$ is a multiset of atomic formulas or negated atomic formulas that disappear in the derivation of $\Rightarrow \exists A$ by applications of *Gem-at* only. Suppose now for reductio that for every term t, the sequent $\Rightarrow A(t/x)$ is not derivable in **G3C3at**. Then for any term t, $(P \vee \sim P) \Rightarrow A(t/x)$ is not derivable because $\Rightarrow (P \vee \sim P)$ is derivable and *Cut* is admissible. Hence either $P \Rightarrow A(t/x)$ or $\sim P \Rightarrow A(t/x)$ is not derivable because otherwise $(P \vee \sim P) \Rightarrow A(t/x)$ is derivable by applying $L\vee$. Assume without loss of generality that $P \Rightarrow A(t/x)$ is not derivable. Then $P, (Q \vee \sim Q) \Rightarrow A(t/x)$ is not derivable for any atomic formula Q because $\Rightarrow (Q \vee \sim Q)$ is derivable and *Cut* is admissible. Therefore either $P, Q \Rightarrow A(t/x)$ or $P, \sim Q \Rightarrow A(t/x)$ is not derivable. Iterating this reasoning, we may conclude that for some i with $1 \leq i \leq n$, $\Delta_i \Rightarrow A(t_i/x)$ is not derivable, contrary to the assumption that we are considering a derivation of $\Rightarrow \exists x A$. □

Remark 6.6 It is known from work by Nobu-Yuki Suzuki [35] that the Disjunction Property and the Existence Property can come apart in intermediate predicate logics, in particular, that in general the Existence Property does not imply the Disjunction Property. The logic **QC3at** is an example of a naturally arising and independently motivated logic for which the Disjunction Property fails, whereas the Existence Property holds.

Remark 6.7 The proof of Theorem 6.5 uses classical logic in the meta-language. In [36, p. 206 f.], Dirk van Dalen presents a constructive proof of the Existence Property for intuitionistic predicate logic (without identity). One may wonder whether a constructive proof of Theorem 6.5 is possible.

7 Concluding remarks

In this article, we introduced an extension of the connexive logic **C** from [38], with the following three questions as our motivations.

Q1 Can we improve John Cantwell's **CN**?
Q2 How much of the desiderata, listed by Estrada-González and Ramírez-Cámara, can be met by the approach to connexivity à la **C**?
Q3 What is the relation between constructivity and LEM?

Our answers to these questions, in view of the new extension **C3**, are as follows.

A1 Cantwell's classical conditional can be replaced by a constructive one.
A2 **C3** is a totally connexive logic with respect to the strong implication.

A3 LEM does not necessarily exclude properties that are usually regarded as indicating constructivity.

These answers give rise to additional questions such as:

Q1' Can we take other conditionals than the constructive conditional?
Q2' Which system is minimal among the family of totally connexive logics?
Q3' Are there any interesting variants of the Disjunction Property and the Existence Property, discussed in [34], that hold in **QC3** or related systems?

These questions, together with the open problems noted in Remarks 4.4, 4.9, 5.3 and 6.7 seem to show that there is a lot of room for further investigations. We hope some readers will be motivated to join the authors to continue with the development of connexive logics.

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