Bi-approximation Semantics for Substructural Logic at Work

Tomoyuki Suzuki

Department of Computer Science, University of Leicester
Leicester, LE1 7RH, United Kingdom
tomoyuki.suzuki@mcs.le.ac.uk

Abstract

In this paper, we introduce bi-approximation semantics, a two-sorted relational semantics, via the canonical extension of lattice expansions. To characterise Ghilardi and Meloni's parallel computation, we introduce doppelgänger valuations which allow us to evaluate sequents and not only formulae. Moreover, by introducing the bi-directional approximation and bases, we track down a connection to Kripke-type semantics for distributive substructural logics through a relationship between basis and the existential quantifier. Based on the framework, we give a possible interpretation of the two sorts, and prove soundness via bi-approximation and completeness via an algebraic representation theorem plus invariance of validity along a back-and-force correspondences.

Keywords: substructural logic, relational semantics, canonicity

1 Introduction

What is a natural relational semantics for substructural logic or resource sensitive logics? Unlike Kripke semantics for modal logic, we can find several types of relational semantics for substructural logic based on their philosophy or on their mathematical frameworks. For example, the study of relational semantics for distributive substructural logics has led to an operational semantics for relevant implication [20]. In [17], a ternary relational semantics, a.k.a. Routley-Meyer semantics, has been introduced by a different interpretation of relevant implication. For distributive substructural logics, we can also find other relational semantics, see e.g. [16]. Reasoning about relational-type semantics for non-distributive substructural logics, one encounters the interpretation problem of disjunction, namely how to avoid the distributivity of *conjunction* and *disjunction*. For orthologic, one can solve the problem on Goldblatt frames [12], by introducing a non-standard interpretation of disjunction. With Dedekind-MacNeille frames and the closure operator interpretation in [13] and [14], one can also solve the problem by using a closure operator. Generalized Kripke frames [7], which are introduced by characterising the intermediate level of canonical extensions of lattice expansions (see e.g. [5] or [8]), provide another semantics in which one can avoid the disjunction problem by a Galois connection.

The aim of the current paper is to propose another possible relational-type semantics for substructural logic. To achieve our goal, we introduce a two-sorted relational-type semantics, called *bi-approximation semantics*, and describe Ghilardi and Meloni's parallel computation [11], see also [19]. Our framework is closely related to the works [13], [14] and [7]. On the other hand, bi-approximation semantics has novel aspects: *bi-directional approximation*, *bases*, and *doppelgänger valuations* which allow us to evaluate *sequents* (Section 3). Based on our setting, we will come across one possible interpretation of the two sorts, *premises* and *conclusions*, and discover a relationship to Kripke-type semantics for distributive substructural logics through *bases* and *the existential quantifier* (Section 4). Furthermore, the connection between *bases* and *the existential quantifier* provides an *effective evaluation of sequents* in bi-approximation semantics, which is useful to prove the soundness theorem (Theorem 6.1). In Section 5, we prove the representation theorem of FL-algebras via p-frames, which is used to show the completeness theorem in Section 6.

2 Substructural logic

In this paper, we denote propositional variables by $p, q, r, p_1, \ldots$, the set of all propositional variables by Φ, and $\mathbf{t}$ and $\mathbf{f}$ are logical constants representing *true* and *false*, respectively. As logical connectives, we use *disjunction* $\vee$, *conjunction* $\wedge$, *fusion (multiplication)* $\circ$, *implications (residuals)* $\rightarrow$ and $\leftarrow$. Formulae of substructural logic are denoted by ϕ, ψ, $\phi_1, \ldots$ and $\psi_1, \ldots$, and the set of all formulae is denoted by Λ. The following BNF generates formulae of substructural logic.

$$\phi ::= p \mid \mathbf{t} \mid \mathbf{f} \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \circ \phi \mid \phi \rightarrow \phi \mid \phi \leftarrow \phi$$

$\Gamma, \Delta, \Sigma, \Pi$ are (possibly empty) finite lists of formulae, and φ is a list of at most one formula. Then, we call $\Gamma \Rightarrow \varphi$ a *sequent.*

Gentzen's sequent system for substructural logic Let ϕ, ψ be arbitrary formulae, $\Gamma, \Delta, \Sigma, \Pi$ arbitrary (possibly empty) finite lists of formulae, φ a list of at most one formula: see e.g. [15]. The sequent system FL is the following.

Initial sequents :

$$\phi \Rightarrow \phi \qquad\qquad \Rightarrow \mathbf{t} \qquad\qquad \mathbf{f} \Rightarrow$$

Cut rule :

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \phi, \Pi \Rightarrow \varphi}{\Sigma, \Gamma, \Pi \Rightarrow \varphi}\,(\text{cut})$$

Rules for constants :

$$\frac{\Gamma, \Delta \Rightarrow \varphi}{\Gamma, \mathbf{t}, \Delta \Rightarrow \varphi} \ (\mathbf{t}\mathrm{w}) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{f}} \ (\mathbf{f}\mathrm{w})$$

Rules for logical connectives :

$$\frac{\Gamma, \phi, \Delta \Rightarrow \varphi \qquad \Gamma, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \vee \psi, \Delta \Rightarrow \varphi} \ (\vee \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \psi} \ (\Rightarrow \vee_1) \qquad \frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \vee \psi} \ (\Rightarrow \vee_2)$$

$$\frac{\Gamma, \phi, \Delta \Rightarrow \varphi}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi} \ (\wedge_1 \Rightarrow) \qquad \frac{\Gamma, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \wedge \psi, \Delta \Rightarrow \varphi} \ (\wedge_2 \Rightarrow)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \wedge \psi} \ (\Rightarrow \wedge)$$

$$\frac{\Gamma, \phi, \psi, \Delta \Rightarrow \varphi}{\Gamma, \phi \circ \psi, \Delta \Rightarrow \varphi} \ (\circ \Rightarrow) \qquad \frac{\Gamma \Rightarrow \phi \qquad \Sigma \Rightarrow \psi}{\Gamma, \Sigma \Rightarrow \phi \circ \psi} \ (\Rightarrow \circ)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \psi, \Pi \Rightarrow \varphi}{\Sigma, \Gamma, \phi \to \psi, \Pi \Rightarrow \varphi} \ (\to\Rightarrow) \qquad \frac{\phi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \phi \to \psi} \ (\Rightarrow\to)$$

$$\frac{\Gamma \Rightarrow \phi \qquad \Sigma, \psi, \Pi \Rightarrow \varphi}{\Sigma, \psi \leftarrow \phi, \Gamma, \Pi \Rightarrow \varphi} \ (\leftarrow\Rightarrow) \qquad \frac{\Gamma, \phi \Rightarrow \psi}{\Gamma \Rightarrow \psi \leftarrow \phi} \ (\Rightarrow\leftarrow)$$

In the sequent system FL, a formula ϕ is *provable in FL* if the sequent $\Rightarrow \phi$ is derivable in FL. The substructural logic **FL** is the set of all provable formulae in FL.

Proposition 2.1 *For all formulae ϕ and ψ, we have*

(i) *ϕ is provable if and only if $\mathbf{t} \Rightarrow \phi$ is derivable,*

(ii) *$\phi \Rightarrow \psi$ is derivable if and only if $\phi \to \psi$ is provable in FL if and only if $\psi \leftarrow \phi$ is provable in FL,*

(iii) *$\phi_1, \ldots, \phi_n \Rightarrow \varphi$ is derivable in FL if and only if $\phi_1 \circ \cdots \circ \phi_n \Rightarrow \varphi$ is derivable in FL.*

The algebraic counterparts of substructural logic **FL** are known as FL-algebras [6].

Definition 2.2 [FL-algebra] An 8-tuple $\mathbb{A} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0\rangle$ is a *FL-algebra*, if $\langle A, \vee, \wedge\rangle$ is a lattice, $\langle A, *, 1\rangle$ is a monoid, 0 is a constant in A, and for all $a, b, c \in A$,

$$a * b \leq c \iff b \leq a\backslash c \iff a \leq c/b.$$

By Proposition 2.1, we sometimes state that **FL** is the set of all sequents derivable in FL. On FL-algebras, each sequent $\phi_1, \ldots, \phi_n \Rightarrow \varphi$ is interpreted as an inequality $\phi_1 * \cdots * \phi_n \leq \varphi$.

3 Bi-approximation semantics

In this section, we firstly introduce a *polarity*, see [1] or [21], which is the foundation of *bi-approximation semantics*.

Polarity and bi-directional approximation

Definition 3.1 [Polarity] A triple $\langle X, Y, B\rangle$ is a *polarity*, if X and Y are non-empty sets, and B a binary relation on $X \times Y$, i.e. $B \subseteq X \times Y$.

Given a polarity $\langle X, Y, B\rangle$, we induce a preorder $\leq_B$ on $X \cup Y$ as follows, see [7]: for all $x_1, x_2 \in X$ and all $y_1, y_2 \in Y$, we let

(i) $x_1 \leq_B x_2 \iff$ for each $y \in Y$, $x_2 B y$ implies $x_1 B y$,

(ii) $y_1 \leq_B y_2 \iff$ for each $x \in X$, $x B y_1$ implies $x B y_2$,

(iii) $x_1 \leq_B y_1 \iff x_1 B y_1$,

(iv) $y_1 \leq_B x_1 \iff$ for each $x' \in X$ and each $y' \in Y$, $x' B y'$ if $x' B y_1$ and $x_1 B y'$.

Hereinafter, we sometimes omit the subscript $_B$ from the induced preorder $\leq_B$, and refer to the triple $\langle X, Y, \leq\rangle$ as the polarity. That is, a polarity $\langle X, Y, \leq\rangle$ is a preordered set $\langle X \cup Y, \leq\rangle$.

Next, we introduce two approximation functions for polarities. Let $\langle X, Y, \leq\rangle$ be a polarity, $\wp(X)$ the poset of all subsets of X ordered by inclusion $\subseteq$, and $\wp(Y)^\partial$ the poset of all subsets of Y ordered by reverse-inclusion $\supseteq$. Two functions $\lambda : \wp(X) \to \wp(Y)^\partial$ and $\upsilon : \wp(Y)^\partial \to \wp(X)$ are defined as follows: for each $\mathfrak{X} \in \wp(X)$ and each $\mathfrak{Y} \in \wp(Y)^\partial$,

(i) $\lambda(\mathfrak{X}) := \{y \in Y \mid \forall x \in \mathfrak{X}.\ x \leq y\}$,

(ii) $\upsilon(\mathfrak{Y}) := \{x \in X \mid \forall y \in \mathfrak{Y}.\ x \leq y\}$.

The functions λ and υ form a *Galois connection*, i.e. $\lambda \dashv \upsilon$. Hence, the images $\lambda[\wp(X)]$ and $\upsilon[\wp(Y)^\partial]$ are isomorphic. Hereafter, we denote the image $\lambda[\wp(X)]$ by $\mathbb{U}$ and the image $\upsilon[\wp(Y)^\partial]$ by $\mathbb{D}$. We mention that the images are *the Dedekind-MacNeille completion* of the quotient poset of $\langle X, Y, \leq\rangle$ with respect to the equivalence relation associated with $\leq$, see [1] or [4]. We call each element in $\mathbb{D}$ a *Galois stable X-set* and refer to each Galois stable X-set by adding the superscript $\_^\downarrow$, e.g. $\alpha^\downarrow$. We call each element in $\mathbb{U}$ a *Galois stable Y-set* and refer to each Galois stable Y-set by adding the subscript $\_{}_\uparrow$, e.g. $\alpha_\uparrow$. Since every Galois stable X-set is an image of some (not necessarily unique) subset of Y, and every Galois stable Y-set is an image of some (not necessarily unique) subset of X, we introduce the following terminology.

Definition 3.2 [Approximation and basis] Let $\mathfrak{X} \in \wp(X)$, $\mathfrak{Y} \in \wp(Y)^\partial$, $\alpha^\downarrow \in \mathbb{D}$ and $\beta_\uparrow \in \mathbb{U}$. An element $\alpha^\downarrow$ is *approximated from above by* $\mathfrak{Y}$ and $\mathfrak{Y}$ is a *(Y-)basis of* α, if $\alpha^\downarrow = \upsilon(\mathfrak{Y})$. An element $\beta_\uparrow$ is *approximated from below by* $\mathfrak{X}$ and $\mathfrak{X}$ is a *(X-)basis of* β, if $\beta_\uparrow = \lambda(\mathfrak{X})$.

Later, we will construct two isomorphic FL-algebras on $\mathbb{D}$ and $\mathbb{U}$: see Section 5. Namely, we will take the abstract algebra whose underlying poset is isomorphic to both $\mathbb{D}$ and $\mathbb{U}$. Then, we can see every point α as $\alpha^\downarrow$ and as $\alpha_\uparrow$. In other words, every point in an abstract algebra is approximated from both *above* and *below.* The main concept of *bi-approximation semantics* is to keep the two directions of approximation: see e.g. Proposition 4.7.

Bi-approximation model Based on a polarity, we introduce *bi-approximation semantics* for substructural logic.

Definition 3.3 [P-frame for substructural logic] A *p-frame for substructural logic*, *p-frame* for short, is a 8-tuple $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$, where $\langle X, Y, \leq \rangle$ is a polarity, $R \subseteq X \times X \times Y$ a ternary relation, O_X a non-empty Galois stable X-set, N_X a Galois stable X-set, O_Y and N_Y Galois stable Y-sets, and $\mathbb{F}$ satisfies

R-order: for all $x, x' \in X$, $x' \leq x$ if and only if
$$\exists o \in O_X.[\forall y \in Y.[R(x, o, y) \Rightarrow x' \leq y] \text{ or } \forall y \in Y.[R(o, x, y) \Rightarrow x' \leq y]],$$

R-identity: for each $x \in X$, $[\exists o_2 \in O_X, \forall y \in Y.[R(x, o_2, y) \Rightarrow x \leq y]$
$$\text{and } \exists o_1 \in O_X, \forall y \in Y.[R(o_1, x, y) \Rightarrow x \leq y]],$$

R-transitivity: for all $x_1, x_1', x_2, x_2' \in X$ and $y, y' \in Y$,
$$x_1' \leq x_1, x_2' \leq x_2, y \leq y' \text{ and } R(x_1, x_2, y) \Rightarrow R(x_1', x_2', y'),$$

R-associativity: for all $x_1, x_2, x_3, x \in X$,
$$\exists x' \in X.[\forall y \in Y.(R(x_1, x', y) \Rightarrow x \leq y) \text{ and } \forall y' \in Y.(R(x_2, x_3, y') \Rightarrow x' \leq y')]$$
$$\text{if and only if}$$
$$\exists x'' \in X.[\forall y \in Y.(R(x'', x_3, y) \Rightarrow x \leq y) \text{ and } \forall y'' \in Y.(R(x_1, x_2, y'') \Rightarrow x'' \leq y'')],$$

O-isom: $O_X = \upsilon(O_Y)$ and $O_Y = \lambda(O_X)$,

N-isom: $N_X = \upsilon(N_Y)$ and $N_Y = \lambda(N_X)$,

$\circ$-tightness: for all $x_1, x_2 \in X$, there exists $x \in X$ such that
$$\forall y \in Y.[R(x_1, x_2, y) \text{ if and only if } x \leq y],$$

$\rightarrow$-tightness: for each $x_1 \in X$ and each $y \in Y$, there exists $y_2 \in Y$ such that
$$\forall x_2 \in X.[R(x_1, x_2, y) \text{ if and only if } x_2 \leq y_2],$$

$\leftarrow$-tightness: for each $x_2 \in X$ and each $y \in Y$, there exists $y_1 \in Y$ such that
$$\forall x_1 \in X.[R(x_1, x_2, y) \text{ if and only if } x_1 \leq y_1].$$

A p-frame $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$ is intuitively explained as follows: the Galois stable sets O_X, O_Y, N_X and N_Y define the worlds where we assume $\mathbf{t}$, conclude $\mathbf{t}$, assume $\mathbf{f}$ and conclude $\mathbf{f}$. The conditions O-isom and N-isom guarantee that every $x \in X$ where we assume the formula $\mathbf{t}$ ($\mathbf{f}$), if and only if every $y \in Y$ where we conclude the formula $\mathbf{t}$ ($\mathbf{f}$) have the consequence relation $x \leq y$. The ternary relation R is another consequence relation which allows us to reason about logical consequences between two premises and one conclusion. The R-order condition says that the induced relation on X, $x' \leq x$ is also obtained by the ternary consequence relation R. The tightness conditions guarantee that the ternary consequence relation R respects $\leq$.

Remark 3.4 In Definition 3.3 one may feel that the conditions R-order, R-identity and R-associativity look too complicated. However, we reformulate them in Remark 4.3.

Our framework is similar to generalized Kripke frames in [7]. However, we do not assume neither Separation axioms nor Reduced axioms, hence p-frames may not be RS-frames. Our current purpose is to characterise Ghilardi and Meloni's parallel computation [11], see also [19]. The most distinct points are how to evaluate formulae on bi-approximation semantics, i.e. the valuation on two-sorted frames by introducing *doppelgänger valuation*, and how to interpret the satisfaction relation $\Vdash$ on each sort, X and Y.

Definition 3.5 [Doppelgänger valuation] Let $\mathbb{F}$ be a p-frame. A pair $V = \langle V^{\downarrow}, V_{\uparrow}\rangle$ of two functions $V^{\downarrow} : \Phi \to \mathbb{D}$ and $V_{\uparrow} : \Phi \to \mathbb{U}$ is a *doppelgänger valuation*, if $V^{\downarrow}(p)$ and $V_{\uparrow}(p)$ coincide for every propositional variable $p \in \Phi$. That is, $V^{\downarrow}(p) = \upsilon(V_{\uparrow}(p))$ and $V_{\uparrow}(p) = \lambda(V^{\downarrow}(p))$ for each propositional variable $p \in \Phi$.

Given a p-frame $\mathbb{F}$ and a doppelgänger valuation V, we call the pair $\mathbb{M} = \langle \mathbb{F}, V\rangle$ a *bi-approximation model*. On a bi-approximation model $\mathbb{M} = \langle \mathbb{F}, V\rangle$, we inductively define a *satisfaction relation* $\Vdash$ as follows: for each $x \in X$,

X-1: $\mathbb{M}, x \Vdash p \iff x \in V^{\downarrow}(p)$ for each $p \in \Phi$,

X-2: $\mathbb{M}, x \Vdash \mathbf{t} \iff x \in O_X$,

X-3: $\mathbb{M}, x \Vdash \mathbf{f} \iff x \in N_X$,

X-4: $\mathbb{M}, x \Vdash \phi \vee \psi \iff \forall y \in Y.\ [\mathbb{M}, y \Vdash \phi \vee \psi \Rightarrow x \leq y]$,

X-5: $\mathbb{M}, x \Vdash \phi \wedge \psi \iff \mathbb{M}, x \Vdash \phi$ and $\mathbb{M}, x \Vdash \psi$,

X-6: $\mathbb{M}, x \Vdash \phi \circ \psi \iff \forall y \in Y.\ [\mathbb{M}, y \Vdash \phi \circ \psi \Rightarrow x \leq y]$,

X-7: $\mathbb{M}, x \Vdash \phi \to \psi \iff \forall x' \in X, y \in Y.\ [\mathbb{M}, x' \Vdash \phi$ and $\mathbb{M}, y \Vdash \psi \Rightarrow R(x', x, y)]$,

X-8: $\mathbb{M}, x \Vdash \psi \leftarrow \phi \iff \forall x' \in X, y \in Y.\ [\mathbb{M}, x' \Vdash \phi$ and $\mathbb{M}, y \Vdash \psi \Rightarrow R(x, x', y)]$.

For each $y \in Y$,

Y-1: $\mathbb{M}, y \Vdash p \iff y \in V_{\uparrow}(p)$ for each $p \in \Phi$,

Y-2: $\mathbb{M}, y \Vdash \mathbf{t} \iff y \in O_Y$,

Y-3: $\mathbb{M}, y \Vdash \mathbf{f} \iff y \in N_Y$,

Y-4: $\mathbb{M}, y \Vdash \phi \vee \psi \iff \mathbb{M}, y \Vdash \phi$ and $\mathbb{M}, y \Vdash \psi$,

Y-5: $\mathbb{M}, y \Vdash \phi \wedge \psi \iff \forall x \in X.\ [\mathbb{M}, x \Vdash \phi \wedge \psi \Rightarrow x \leq y]$,

Y-6: $\mathbb{M}, y \Vdash \phi \circ \psi \iff \forall x_1, x_2 \in X.\ [\mathbb{M}, x_1 \Vdash \phi$ and $\mathbb{M}, x_2 \Vdash \psi \Rightarrow R(x_1, x_2, y)]$,

Y-7: $\mathbb{M}, y \Vdash \phi \to \psi \iff \forall x \in X.\ [\mathbb{M}, x \Vdash \phi \to \psi \Rightarrow x \leq y]$,

Y-8: $\mathbb{M}, y \Vdash \psi \leftarrow \phi \iff, \forall x \in X.\ [\mathbb{M}, x \Vdash \psi \leftarrow \phi \Rightarrow x \leq y]$.

In bi-approximation models, the satisfaction relation $\Vdash$ has two distinct interpretations depending on the domains X and Y. On X, we comprehend $\mathbb{M}, x \Vdash \phi$ as the formula ϕ is *assumed at* x, and on Y, $\mathbb{M}, y \Vdash \phi$ as the formula ϕ is *concluded at* y. Moreover, we also define $\mathbb{F}, x \Vdash \phi$ and $\mathbb{F}, y \Vdash \phi$ as usual: for every doppelgänger valuation V, we have $\mathbb{F}, V, x \Vdash \phi$ and $\mathbb{F}, V, y \Vdash \phi$, respectively.

An interpretation of the two-sorted semantics To reason about resource sensitive logics, we make a clear distinction between premises and conclusions, and evaluate logical consequences as relations between premises and conclusions. On p-frames, we think about X as a set of *premise worlds* where we evaluate only premises, and about Y as a set of *conclusion worlds* where we evaluate just conclusions. One may feel that the satisfaction relation $\mathbb{M}, y \Vdash \phi$, which says "the formula ϕ is *concluded* at the conclusion world y", is the same with "the formula ϕ is *true* at y." However, these two concepts are not the same. This is because, even if we conclude a formula ϕ at y, we cannot logically judge whether the formula is true or not. For example, if we conclude a formula ϕ meaning "tomorrow is Sunday" at a conclusion world y, we do not have any clue to justify that the formula is a fact. In other words, we may explain $\mathbb{M}, y \Vdash \phi$ as someone is just claiming "ϕ should be concluded" without any reason. Of course, we cannot consider it as logical reasoning. Only when we also have a reasonable premise like "today is Saturday" or "tomorrow is Sunday," we can justify that the logical consequence is *true*. More precisely, only when we have a pair of a premise and a conclusion, we can justify the logical consequence.

Formally the concept of *truth* of logical consequences on bi-approximation models is defined as follows. To reason about truth on bi-approximation models, it is necessary to extend the satisfaction relation $\Vdash \subseteq (X\times\Lambda)\cup(Y\times\Lambda)$ to a relation between $X \times Y$ and pairs of two formulae $\Lambda \times \Lambda$, or *sequents*. For our purpose, we fix the interpretation between sequents and pairs of two formulae. Given a sequent $\phi_1, \dots, \phi_n \mapsto \varphi$, we translate it to $(\phi_1 \circ \cdots \circ \phi_n, \varphi)$. If $n = 0$, the left-hand side is empty and we write $(\mathbf{t}, \varphi)$. If the right-hand side is empty, we write $(\phi_1 \circ \ldots \circ \phi_n, \mathbf{f})$. But, whenever it is not confusing, we do not make any distinction between sequents and pairs of two formulae. So, both are called just *sequents* and are denoted by $\Gamma \mapsto \varphi$.

Definition 3.6 [Truth] Let $\mathbb{M} = \langle \mathbb{F}, V \rangle$ be a bi-approximation model and $\Gamma \mapsto \varphi$ a sequent. We let

(i) $\mathbb{M}, (x, y) \Vdash \Gamma \mapsto \varphi \iff x \leq y$ whenever $\mathbb{M}, x \Vdash \Gamma$ and $\mathbb{M}, y \Vdash \varphi$,

(ii) $\mathbb{F}, (x, y) \Vdash \Gamma \mapsto \varphi \iff \langle \mathbb{F}, V \rangle, (x, y) \Vdash \Gamma \mapsto \varphi$ for each doppelgänger valuation V,

(iii) $\mathbb{M} \Vdash \Gamma \mapsto \varphi \iff \mathbb{M}, (x, y) \Vdash \Gamma \mapsto \varphi$ for all $x \in X$ and $y \in Y$,

(iv) $\mathbb{F} \Vdash \Gamma \mapsto \varphi \iff \langle \mathbb{F}, V \rangle, (x, y) \Vdash \Gamma \mapsto \varphi$ for all $x \in X$ and $y \in Y$, and every doppelgänger valuation V.

We interpret $\mathbb{M}, (x, y) \Vdash \Gamma \mapsto \varphi$ as the sequent $\Gamma \mapsto \varphi$ is *true at the pair* (x, y), and $\mathbb{F} \Vdash \Gamma \mapsto \varphi$ as the sequent $\Gamma \mapsto \varphi$ is *valid on* $\mathbb{F}$.

Remark 3.7 Unlike what happens in the setting of the normal Kripke semantics, in bi-approximation models we reason about *sequents* but not *formulae*, in general. But, thanks to Proposition 2.1, this distinction is not critical when we consider substructural logic.

Hereinafter, we sometimes write $(x, y) \Vdash \phi \mapsto \psi$ instead of $\mathbb{M}, (x, y) \Vdash \phi \mapsto \psi$.

External reasoning and internal reasoning on p-frames Before we show preliminary results for bi-approximation semantics, we explain how to evaluate premises, conclusions and logical consequences on p-frames.

Recall the satisfaction relation $\Vdash$ in (X-1) - (X-8) and (Y-1) - (Y-8). We notice that there are two types of reasoning: *internal* and *external*. Namely, there is the reasoning on X, e.g. (X-4), or on Y, e.g. (Y-5), and there is the reasoning given by the relation $\leq$ or R between X and Y, e.g. (X-4) or (Y-6). Intuitively speaking, the internal reasoning derives a premise from premises, or a conclusion from conclusions, e.g. we assume $\phi \wedge \psi$ at x if and only if we assume ϕ and ψ at x (X-5). On the other hand, the external reasoning evaluates logical consequences. That is, we describe a premise world by conclusion worlds, and vise versa. For example, a conclusion world y where we conclude $\phi \wedge \psi$ is described by all premise worlds where we assume ϕ and ψ (Y-5). We also say that the conclusion world y is *approximated* by the corresponding premise worlds. Analogously, e.g. (X-4), a premise world is *approximated* by the corresponding conclusion worlds. See also Proposition 3.9. This is what we call *bi-approximation* in our framework.

Whereas the external reasoning is fundamental in bi-approximation models, we also have the internal reasoning as well. One may feel that the internal reasoning (Y-4) is far from our intuition. However, we can also explain it as follows. Recall the sequent calculus LK. In LK, we consider a sequent as a pair of a finite list of premises and a finite list of conclusions, $\phi_1, \ldots, \phi_m \mapsto \psi_1, \ldots, \psi_n$. The intuitive interpretation of this sequent is "if we assume *all* premises $\phi_1, \ldots, \phi_m$ then we conclude *one of these* conclusions $\psi_1, \ldots, \psi_n$." In other words, premises are *compulsory* and conclusions are *elective*. Therefore, it is natural to consider (Y-4) as "ϕ and ψ are possible conclusions at y if and only if $\phi \vee \psi$ is a possible conclusion at y."

Preliminary results for bi-approximation semantics In this paragraph, we show basic properties on bi-approximation semantics. The following proposition corresponds to Hereditary property in Kripke semantics for intuitionistic logic, e.g. [3]. But, it is two-sorted in our case.

Proposition 3.8 (Hereditary) *Let $\mathbb{M}$ be a bi-approximation model and ϕ a formula. For all elements $x, x' \in X$ and $y, y' \in Y$, we have*

(i) *if $x' \leq x$ and ϕ is assumed at x, $x \Vdash \phi$, then it is also assumed at x', $x' \Vdash \phi$,*

(ii) *if $y \leq y'$ and ϕ is concluded at y, $y \Vdash \phi$, then it is also concluded at y', $y' \Vdash \phi$.*

Proposition 3.9 *For each bi-approximation model $\mathbb{M}$, each $x \in X$, each $y \in Y$, and every formula ϕ, if $\mathbb{M}, x \Vdash \phi$ and $\mathbb{M}, y \Vdash \phi$, then $x \leq y$. Furthermore, we have*

(i) $\mathbb{M}, x \Vdash \phi \iff$ *for every $y \in Y$. if $\mathbb{M}, y \Vdash \phi$ then $x \leq y$,*

(ii) $\mathbb{M}, y \Vdash \phi \iff$ *for every $x \in X$. if $\mathbb{M}, x \Vdash \phi$ then $x \leq y$.*

Remark 3.10 Proposition 3.9 tells us initial sequents $\phi \mapsto \phi$ is valid on every p-frame $\mathbb{F}$. Intuitively, if ϕ is assumed at x, then it should be concluded everywhere in Y above x. Conversely, if ϕ is concluded at y, then it should be assumed everywhere in X below y.

As a corollary of Proposition 3.9, we obtain the following.

Corollary 3.11 *For every p-frame $\mathbb{F}$, each doppelgänger valuation V is naturally extended from the set of all propositional variables Φ to the set of all formulae Λ, i.e. for each formula ϕ, we let*

(i) $\tilde{V}^{\downarrow}(\phi) := \{x \in X \mid \mathbb{F}, V, x \Vdash \phi\}$,

(ii) $\tilde{V}_{\uparrow}(\phi) := \{y \in Y \mid \mathbb{F}, V, y \Vdash \phi\}$.

$$\begin{tikzcd}
 & \Lambda \arrow[ddl, bend right, "\tilde{V}^{\downarrow}"'] \arrow[ddr, bend left, "\tilde{V}_{\uparrow}"] \arrow[d, dash, dashed] & \\
 & \Phi \arrow[dl, "V^{\downarrow}"'] \arrow[dr, "V_{\uparrow}"] & \\
\mathbb{D} \arrow[rr, bend left, "\lambda"] \arrow[rr, phantom, "\cong"] & & \mathbb{U} \arrow[ll, bend left, "v"]
\end{tikzcd}$$

4 Bi-approximation, bases and the existential quantifier

In Kripke semantics, we have a simple interpretation of modal operators $\Diamond$ and $\Box$ as follows: for each Kripke model $\mathbb{M}$ and each possible world w, we let

(i) $\mathbb{M}, w \Vdash \Diamond\phi \iff \exists v \in W$ such that $R(w, v)$ and $\mathbb{M}, v \Vdash \phi$,

(ii) $\mathbb{M}, w \Vdash \Box\phi \iff \forall v \in W$. if $R(w, v)$ then $\mathbb{M}, v \Vdash \phi$,

whereas, in bi-approximation models, all logical connectives are interpreted uniformly with *conjunction*, *implication* and *universal quantifier* $\forall$. For example, if we introduce $\Diamond$ on bi-approximation semantics, it is interpreted as follows:

(iii) $\mathbb{M}, x \Vdash \Diamond\phi \iff \forall y \in Y$. if $\mathbb{M}, y \Vdash \Diamond\phi$ then $x \leq y$,

(iv) $\mathbb{M}, y \Vdash \Diamond\phi \iff \forall x \in X$. if $\mathbb{M}, x \Vdash \phi$ then $R(x, y)$,

where R is a binary relation on $X \times Y$. This is because it is essential to set up our interpretation to return Galois stable sets. Note that item (iv) gives the definition of $\Diamond$ on $\mathbb{U}$, and item (iii) copies the same value to $\mathbb{D}$: see also Section 5. As we saw in Corollary 3.11, this setting allows us to assign the corresponding Galois stable X-set and Y-set for every formula between $\mathbb{D}$ and $\mathbb{U}$. On the other hand, to evaluate any formula on bi-approximation models, we encounter the universal quantifier $\forall$ and an implication in each step, which generates considerable complexity.

However, in this section, we will show that we can reduce the complexity in specific cases by introducing *auxiliary relations* for R. In other words, some logical connectives are translated into other simpler conditions with *the existential quantifier*, which may not be equivalent to the original conditions anymore. Through these simpler conditions, we will find the relationship between relational semantics and bi-approximation semantics. Furthermore, we will also unearth a connection among bi-approximation, bases and the existential quantifier.

Definition 4.1 [Auxiliary relations] For every bi-approximation model $\mathbb{M}$ and the ternary relation $R \subseteq X \times X \times Y$, we let the following three ternary relations $R^{\circ} \subseteq X \times X \times X$, $R^{\rightarrow} \subseteq X \times Y \times Y$ and $R^{\leftarrow} \subseteq Y \times X \times Y$:

(i) $R^{\circ}(x_1, x_2, x) \iff$ for every $y \in Y$. if $R(x_1, x_2, y)$ then $x \leq y$,

(ii) $R^{\rightarrow}(x_1, y_2, y) \iff$ for every $x_2 \in X$. if $R(x_1, x_2, y)$ then $x_2 \leq y_2$,

(iii) $R^{\leftarrow}(y_1, x_2, y) \iff$ for every $x_1 \in X$. if $R(x_1, x_2, y)$ then $x_1 \leq y_1$.

Note that R° is related to $R^{\downarrow}$ in [7], but we also introduce $R^{\rightarrow}$ and $R^{\leftarrow}$ to show Theorem 6.1. Thanks to the tightness conditions in p-frames, see Definition 3.3, we obtain the following.

Lemma 4.2 *For every bi-approximation model* $\mathbb{M}$ *and the ternary relation* $R \subseteq X \times X \times Y$*,*

(i) $R(x_1, x_2, y) \iff$ *for every* $x \in X$*. if* $R^{\circ}(x_1, x_2, x)$ *then* $x \leq y$*,*

(ii) $R(x_1, x_2, y) \iff$ *for every* $y_2 \in Y$*. if* $R^{\rightarrow}(x_1, y_2, y)$ *then* $x_2 \leq y_2$*,*

(iii) $R(x_1, x_2, y) \iff$ *for every* $y_1 \in Y$*. if* $R^{\leftarrow}(y_1, x_2, y)$ *then* $x_1 \leq y_1$*.*

Remark 4.3 By Definition 4.1, we can reformulate R-order, R-identity and R-associativity in Definition 3.3 as follows:

R-order: for all $x, x' \in X$, $x' \leq x \iff \exists o \in O_X.\ [R^{\circ}(x, o, x')$ or $R^{\circ}(o, x, x')]$,

R-identity: for every $x \in X$. $[\exists o_2 \in O_X.\ R^{\circ}(x, o_2, x)$ and $\exists o_1 \in O_X.\ R^{\circ}(o_1, x, x)]$,

R-associativity: for all $x_1, x_2, x_3, x \in X$.

$$\exists x' \in X.\ [R^{\circ}(x_1, x', x) \text{ and } R^{\circ}(x_2, x_3, x')] \iff \exists x'' \in X.\ [R^{\circ}(x'', x_3, x) \text{ and } R^{\circ}(x_1, x_2, x'')].$$

We note that similar conditions for R-order, R-identity and R-associativity can be found in a relational semantics for distributive substructural logics, e.g. [18, Definition 6].[1] Thanks to the auxiliary relations R°, $R^{\rightarrow}$ and $R^{\leftarrow}$, we obtain other interpretations of formulae on bi-approximation semantics.

Theorem 4.4 *For every bi-approximation model* $\mathbb{M}$ *and all formulae* ϕ*,* ψ*, we have*

(i) $y \Vdash \phi \circ \psi \iff \forall x_1 \in X, y_2 \in Y.$ *if* $x_1 \Vdash \phi$ *and* $R^{\rightarrow}(x_1, y_2, y)$ *then* $y_2 \Vdash \psi$*,*

(ii) $y \Vdash \phi \circ \psi \iff \forall x_2 \in X, y_1 \in Y.$ *if* $x_2 \Vdash \psi$ *and* $R^{\leftarrow}(y_1, x_2, y)$ *then* $y_1 \Vdash \phi$*,*

(iii) $x_2 \Vdash \phi \to \psi \iff \forall x_1, x \in X.$ *if* $x_1 \Vdash \phi$ *and* $R^{\circ}(x_1, x_2, x)$ *then* $x \Vdash \psi$*,*

(iv) $x_2 \Vdash \phi \to \psi \iff \forall y_1, y \in Y.$ *if* $y \Vdash \psi$ *and* $R^{\leftarrow}(y_1, x_2, y)$ *then* $y_1 \Vdash \phi$*,*

(v) $x_1 \Vdash \psi \leftarrow \phi \iff \forall x_2, x \in X.$ *if* $x_2 \Vdash \phi$ *and* $R^{\circ}(x_1, x_2, x)$ *then* $x \Vdash \psi$*,*

(vi) $x_1 \Vdash \psi \leftarrow \phi \iff \forall y_2, y \in Y.$ *if* $y \Vdash \psi$ *and* $R^{\rightarrow}(x_1, y_2, y)$ *then* $y_2 \Vdash \phi$*,*

(vii) $x \Vdash \phi \circ \psi \Longleftarrow \exists x_1, x_2 \in X$ *such that* $x_1 \Vdash \phi$*,* $x_2 \Vdash \psi$ *and* $R^{\circ}(x_1, x_2, x)$*,*

(viii) $y_2 \Vdash \phi \to \psi \Longleftarrow \exists x_1 \in X$*,* $\exists y \in Y$ *such that* $x_1 \Vdash \phi$*,* $y \Vdash \psi$ *and* $R^{\rightarrow}(x_1, y_2, y)$*,*

[1] The order of the ternary relation is different. That is, $R^{\circ}(x_1, x_2, x)$ in this paper is the same with $R_{\circ}(x, x_1, x_2)$ in [18].

(ix) $y_1 \Vdash \psi \leftarrow \phi \Longleftarrow \exists x_2 \in X$, $\exists y \in Y$ *such that* $x_2 \Vdash \phi$, $y \Vdash \psi$ *and* $R^{\leftarrow}(y_1, x_2, y)$.

In Theorem 4.4, item (iii) and item (v) correspond to the normal interpretations in Kripke semantics. The same results for item (iii) and item (v) are obtained by generalized Kripke frames [7]. Moreover, item (vii) looks similar to the interpretation on ternary-relational semantics of distributive substructural logics. Item (vii) must be closely related to the discussion in [7, p.264]. However, unlike what happens in the setting of generalized Kripke frames, the conditions of item (vii), item (viii) and item (ix) are *more beneficial* to evaluate formulae in our framework. More precisely, the auxiliary relations R°, $R^{\rightarrow}$ and $R^{\leftarrow}$ provide bases of $V(\phi \circ \psi)$, $V(\phi \rightarrow \psi)$ and $V(\psi \leftarrow \phi)$: see Theorem 4.6 and Proposition 4.7.

Related to Theorem 4.4, we also obtain the following results for $\vee$ and $\wedge$.

Theorem 4.5 *Let* $\mathbb{M}$ *be an arbitrary bi-approximation model,* ϕ, ψ *be all formulae. For each* $x \in X$ *and each* $y \in Y$*,*

(i) $\mathbb{M}, x \Vdash \phi \vee \psi \Longleftarrow \mathbb{M}, x \Vdash \phi$ *or* $\mathbb{M}, x \Vdash \psi$,

(ii) $\mathbb{M}, y \Vdash \phi \wedge \psi \Longleftarrow \mathbb{M}, y \Vdash \phi$ *or* $\mathbb{M}, y \Vdash \psi$.

Items (vii)-(ix) in Theorem 4.4 and Theorem 4.5 indicate that, when we reason about formulae with the existential quantifier and disjunction, we may not accumulate all worlds in X (in Y) where the formulae are assumed (concluded). However, as we will see below, we can still collect *essential worlds* in X (in Y) to gather *all worlds* in Y (in X) where the formulae are concluded (assumed): see Theorem 4.6. Hereinafter, to discuss the connection between the existential quantifier and the bi-approximation clearly, we introduce an auxiliary relation $\Vdash_{\mathfrak{bs}}$ of $\Vdash$ as follows (the subscript $_{-\mathfrak{bs}}$ refers to *bases*, see Theorem 4.6):

(i) $x \Vdash_{\mathfrak{bs}} \phi \vee \psi \iff x \Vdash_{\mathfrak{bs}} \phi$ or $x \Vdash_{\mathfrak{bs}} \psi$,

(ii) $y \Vdash_{\mathfrak{bs}} \phi \wedge \psi \iff y \Vdash_{\mathfrak{bs}} \phi$ or $y \Vdash_{\mathfrak{bs}} \psi$,

(iii) $x \Vdash_{\mathfrak{bs}} \phi \circ \psi \iff \exists x_1, x_2 \in X$ s.t. $x_1 \Vdash_{\mathfrak{bs}} \phi$, $x_2 \Vdash_{\mathfrak{bs}} \psi$ and $R^{\circ}(x_1, x_2, x)$,

(iv) $y_2 \Vdash_{\mathfrak{bs}} \phi \rightarrow \psi \iff \exists x_1 \in X, \exists y \in Y$ s.t. $x_1 \Vdash_{\mathfrak{bs}} \phi$, $y \Vdash_{\mathfrak{bs}} \psi$ and $R^{\rightarrow}(x_1, y_2, y)$,

(v) $y_1 \Vdash_{\mathfrak{bs}} \psi \leftarrow \phi \iff \exists x_2 \in X, \exists y \in Y$ s.t. $x_2 \Vdash_{\mathfrak{bs}} \phi$, $y \Vdash_{\mathfrak{bs}} \psi$ and $R^{\leftarrow}(y_1, x_2, y)$.

(vi) $x \Vdash_{\mathfrak{bs}} \phi \iff x \Vdash \phi$, whenever ϕ is a propositional variable or a constant, or the outermost connective of ϕ is either $\wedge$, $\rightarrow$ or $\leftarrow$,

(vii) $y \Vdash_{\mathfrak{bs}} \psi \iff y \Vdash \psi$, whenever ψ is a propositional variable or a constant, or the outermost connective of ψ is either $\vee$ or $\circ$.

By parallel induction, we obtain the following straightforwardly. For every formula ϕ, each $x \in X$ and each $y \in Y$, we have

(i) if $x \Vdash_{\mathfrak{bs}} \phi$, then $x \Vdash \phi$,

(ii) if $y \Vdash_{\mathfrak{bs}} \phi$, then $y \Vdash \phi$.

Furthermore, we also obtain the following.

Theorem 4.6 *Let $\mathbb{M}$ be an arbitrary bi-approximation model and ϕ each formula. Then, we have the following (recall $\tilde{V}$ in Corollary 3.11 and basis in Definition 3.2):*

(i) *the set $\{x \in X \mid \mathbb{M}, x \Vdash_{\mathfrak{bs}} \phi\}$ is a basis of $\tilde{V}_{\uparrow}(\phi)$,*

(ii) *the set $\{y \in Y \mid \mathbb{M}, y \Vdash_{\mathfrak{bs}} \phi\}$ is a basis of $\tilde{V}^{\downarrow}(\phi)$.*

Proof. Parallel induction. Base cases are trivial.

(i) $\upsilon(\{y \in Y \mid y \Vdash_{\mathfrak{bs}} \phi \wedge \psi\}) = \tilde{V}^{\downarrow}(\phi \wedge \psi)$. ($\subseteq$). For each x, suppose that $x \leq y$, if $y \Vdash_{\mathfrak{bs}} \phi$ or $y \Vdash_{\mathfrak{bs}} \psi$ for every y. It is equivalent to both $x \leq y$ if $y \Vdash_{\mathfrak{bs}} \phi$ and $x \leq y$ if $y \Vdash_{\mathfrak{bs}} \psi$. By induction hypothesis, we have $x \Vdash \phi$ and $x \Vdash \psi$, hence $x \Vdash \phi \wedge \psi$. ($\supseteq$). trivial.

(ii) $\lambda(\{x \in X \mid x \Vdash_{\mathfrak{bs}} \phi \circ \psi\}) = \tilde{V}_{\uparrow}(\phi \circ \psi)$. For each $y \in Y$, by Theorem 4.4,

$$\begin{aligned} y \Vdash \phi \circ \psi &\iff \forall x_1, x_2.[x_1 \Vdash_{\mathfrak{bs}} \phi, x_2 \Vdash_{\mathfrak{bs}} \psi \Rightarrow R(x_1, x_2, y)] \\ &\iff \forall x_1, x_2, y_2.[x_1 \Vdash_{\mathfrak{bs}} \phi, R^{\rightarrow}(x_1, y_2, y) \Rightarrow (x_2 \Vdash_{\mathfrak{bs}} \psi \Rightarrow x_2 \leq y_2)] \\ &\iff \forall x_1, x_2' \in X.[x_1 \Vdash_{\mathfrak{bs}} \phi, x_2' \Vdash \psi \Rightarrow R(x_1, x_2', y)]. \end{aligned}$$

Note that $x_2 \Vdash_{\mathfrak{bs}} \psi$ changes to $x_2' \Vdash \psi$. Repeat the same replacement for x_1.

The other cases are analogous. □

Theorem 4.6 tells us that, in bi-approximation semantics, bases are (partly) inductively characterised by the existential quantifier and disjunction: see also $\cup$-terms, $\cap$-terms, pseudo-$\cup$-terms and pseudo-$\cap$-terms in [19]. Moreover, this property works beneficially together with the following proposition: see Remark 6.2.

Proposition 4.7 *Let $\mathbb{M}$ be an arbitrary bi-approximation model and ϕ, ψ all formulae. Then, we have*

$$\mathbb{M} \Vdash \phi \mapsto \psi \iff \forall x \in X, \forall y \in Y. \text{ if } \mathbb{M}, x \Vdash_{\mathfrak{bs}} \phi \text{ and } \mathbb{M}, y \Vdash_{\mathfrak{bs}} \psi, \text{ then } x \leq y.$$

5 The Representation theorem

In this section, to prove Theorem 6.3, we show that FL-algebras can be represented by p-frames. By analogy to the situation in modal logic (e.g. [2]), we will show that the dual frames of FL-algebras are p-frames and the dual algebras of p-frames are FL-algebras. Moreover, the validity relations between p-frames and FL-algebras are also proved as in the case of modal logic: see Theorem 5.3 and Theorem 5.5.

Dual algebra of p-frame For each p-frame $\mathbb{F}$, we construct two isomorphic FL-algebras *in parallel* based on the isomorphic posets $\mathbb{D}$ and $\mathbb{U}$. Namely, we define the operations $\vee$, $\wedge$, $*$, $\backslash$ and $/$, and the constants 1 and 0 on both $\mathbb{D}$ and $\mathbb{U}$, as they are isomorphic FL-algebras, i.e. $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, 1, 0\rangle \cong \langle \mathbb{U}, \vee, \wedge, *, \backslash, /, 1, 0\rangle$.

Since $\mathbb{D}$ and $\mathbb{U}$ are isomorphic through the Galois connection $\lambda \dashv \upsilon$, we have two natural ways to define each operation, in general. That is, an operation on $\mathbb{U}$, *approximated from below*, and take the copy to the other side via $\upsilon : \mathbb{U} \to \mathbb{D}$. Or, an operation on $\mathbb{D}$, *approximated from above*, and take the copy to the other side via $\lambda : \mathbb{D} \to \mathbb{U}$. We define additive operations $\vee$ and $*$ are defined on $\mathbb{U}$, approximated from below, and multiplicative operations $\wedge$, $\backslash$ and $/$ are defined on $\mathbb{D}$, approximated from above. Otherwise, we cannot prove the residuality (see [9] and [10]).

For each p-frame $\mathbb{F} = \langle X, Y, \leq, R, O_X, O_Y, N_X, N_Y \rangle$, we define $\vee$, $\wedge$, $*$, $\backslash$ and $/$ are defined as follows: on $\mathbb{D}$, for all $\alpha^{\downarrow}, \beta^{\downarrow} \in \mathbb{D}$,

$\mathbb{D}$-1: $\alpha^{\downarrow} \vee \beta^{\downarrow} := \upsilon(\alpha_{\uparrow} \vee \beta_{\uparrow})$,

$\mathbb{D}$-2: $\alpha^{\downarrow} \wedge \beta^{\downarrow} := \alpha^{\downarrow} \cap \beta^{\downarrow}$,

$\mathbb{D}$-3: $\alpha^{\downarrow} * \beta^{\downarrow} := \upsilon(\alpha_{\uparrow} * \beta_{\uparrow})$,

$\mathbb{D}$-4: $\alpha^{\downarrow} \backslash \beta^{\downarrow} := \{x_2 \in X \mid \forall x_1 \in \alpha^{\downarrow}, \forall y \in \beta_{\uparrow}.\ R(x_1, x_2, y)\}$,

$\mathbb{D}$-5: $\beta^{\downarrow} / \alpha^{\downarrow} := \{x_1 \in X \mid \forall x_2 \in \alpha^{\downarrow}, \forall y \in \beta_{\uparrow}.\ R(x_1, x_2, y)\}$.

On $\mathbb{U}$, for all $\alpha_{\uparrow}, \beta_{\uparrow} \in \mathbb{U}$,

$\mathbb{U}$-1: $\alpha_{\uparrow} \vee \beta_{\uparrow} := \alpha_{\uparrow} \cap \beta_{\uparrow}$,

$\mathbb{U}$-2: $\alpha_{\uparrow} \wedge \beta_{\uparrow} := \lambda(\alpha^{\downarrow} \wedge \beta^{\downarrow})$,

$\mathbb{U}$-3: $\alpha_{\uparrow} * \beta_{\uparrow} := \{y \in Y \mid \forall x_1 \in \alpha^{\downarrow}, \forall x_2 \in \beta^{\downarrow}.\ R(x_1, x_2, y)\}$,

$\mathbb{U}$-4: $\alpha_{\uparrow} \backslash \beta_{\uparrow} := \lambda(\alpha^{\downarrow} \backslash \beta^{\downarrow})$,

$\mathbb{U}$-5: $\beta_{\uparrow} / \alpha_{\uparrow} := \lambda(\beta^{\downarrow} / \alpha^{\downarrow})$.

Based on these operations, we can show the following.

Theorem 5.1 *Both* $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, O_X, N_X \rangle$ *and* $\langle \mathbb{U}, \vee, \wedge, *, \backslash, /, O_Y, N_Y \rangle$ *are FL-algebras, and they are isomorphic.*

By Theorem 5.1, we naturally define the dual FL-algebras of p-frames.

Definition 5.2 [Dual algebra] Let $\mathbb{F}$ be a p-frame. *The dual algebra of* $\mathbb{F}$ is an algebra $\mathbb{F}^{+} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ which is isomorphic to $\langle \mathbb{D}, \vee, \wedge, *, \backslash, /, O_X, N_X \rangle$ and $\langle \mathbb{U}, \vee, \wedge, *, \backslash, /, O_Y, N_Y \rangle$.

Along with the definition of dual algebras, we obtain the equivalence of validity, as usual.

Theorem 5.3 *For every p-frame* $\mathbb{F}$ *and each sequent* $\Gamma \Rightarrow \varphi$*, the sequent* $\Gamma \Rightarrow \varphi$ *is valid on* $\mathbb{F}$ *if and only if it is valid on the dual algebra* $\mathbb{F}^{+}$.

$$\mathbb{F} \Vdash \Gamma \Rightarrow \varphi \iff \mathbb{F}^{+} \models \Gamma \leq \varphi$$

Dual frame of FL-algebras Here we construct the dual frames of FL-algebras. We mention that the dual frame corresponds to *the intermediate level* introduced in [11] but see also [5] and [19].

Let $\mathbb{A} = \langle A, \vee, \wedge, *, \backslash, /, 1, 0 \rangle$ be a FL-algebra. The set of all filters and the set of all ideals are denoted by $\mathcal{F}$ and $\mathcal{I}$. On $\mathcal{F} \cup \mathcal{I}$, we define a binary relation $\sqsubseteq$ as follows: for all filters $F, F_1, F_2 \in \mathcal{F}$ and all ideals $I, I_1, I_2 \in \mathcal{I}$,

(i) $F_1 \sqsubseteq F_2 \iff F_1 \supseteq F_2$,

(ii) $F \sqsubseteq I \iff F \cap I \neq \emptyset$,

(iii) $I \sqsubseteq F \iff \forall a \in I, \forall b \in F.\ a \leq b$,

(iv) $I_1 \sqsubseteq I_2 \iff I_1 \subseteq I_2$.

Next, on the triple $\langle \mathcal{F}, \mathcal{I}, \sqsubseteq \rangle$, we build a ternary relation R, and subsets $O_{\mathcal{F}}$, $O_{\mathcal{I}}$, $N_{\mathcal{F}}$ and $N_{\mathcal{I}}$ as follows: for all $F_1, F_2 \in \mathcal{F}$ and each $I \in \mathcal{I}$,

(i) $R(F_1, F_2, I) \iff F_1 * F_2 \sqsubseteq I$,
where $F_1 * F_2 := \{a \in A \mid \exists f_1 \in F_1, \exists f_2 \in F_2.\ f_1 * f_2 \leq a\}$,

(ii) $O_{\mathcal{F}}$ is the set of all filters containing 1,

(iii) $O_{\mathcal{I}}$ is the set of all ideals containing 1,

(iv) $N_{\mathcal{F}}$ is the set of all filters containing 0,

(v) $N_{\mathcal{I}}$ is the set of all ideals containing 0.

Then, the 8-tuple $\mathbb{A}_+ = \langle \mathcal{F}, \mathcal{I}, \sqsubseteq, R, O_{\mathcal{F}}, O_{\mathcal{I}}, N_{\mathcal{F}}, N_{\mathcal{I}} \rangle$ is *the dual frame of* $\mathbb{A}$. To prove the following theorems, we here mention that, for all $F, F_1, F_2 \in \mathcal{F}$ and each $I, I_1, I_2 \in \mathcal{I}$,

(i) $F_1 * F_2$ is a filter,

(ii) $F \backslash I := \{a \in A \mid \exists f \in F, \exists i \in I.\ a \leq f \backslash i\}$ is an ideal,

(iii) $I/F := \{a \in A \mid \exists i \in I, \exists f \in F.\ a \leq i/f\}$ is an ideal.

(iv) $R^{\circ}(F_1, F_2, F) \iff F \sqsubseteq F_1 * F_2$,

(v) $R^{\rightarrow}(F_1, I_2, I) \iff F_1 \backslash I \sqsubseteq I_2$,

(vi) $R^{\leftarrow}(I_1, F_2, I) \iff I/F_2 \sqsubseteq I_1$.

Then, we can prove the following.

Theorem 5.4 *For any FL-algebra $\mathbb{A}$, the dual frame $\mathbb{A}_+$ is a p-frame.*

We prove the validity relationship between FL-algebras and the dual p-frames.

Theorem 5.5 *Let $\mathbb{A}$ be every FL-algebra and $\Gamma \mapsto \varphi$ each sequent. If the sequent is valid on the dual frame $\mathbb{A}_+$, it is also valid on the original FL-algebra $\mathbb{A}$.*

$$\mathbb{A} \models \Gamma \leq \varphi \Longleftarrow \mathbb{A}_+ \Vdash \Gamma \mapsto \varphi$$

6 Soundness and Completeness

In this section, we will show that p-frames are a sound and complete semantics for the substructural logic **FL**. Unlike what happens in the setting of relational semantics for distributive substructural logics, soundness is not straightforward. This is because bi-approximation models evaluate formulae through *the Galois connection* $\lambda \dashv \upsilon$. To avoid this complex argument, we can use the relationship between the *bi-approximation* and the *bases*: recall Proposition 4.7.

Theorem 6.1 (Soundness) *Let $\Gamma \mapsto \varphi$ be an arbitrary sequent. If the sequent $\Gamma \mapsto \varphi$ is derivable in FL, it is valid on every p-frame $\mathbb{F}$.*

Proof. Let $\mathbb{F}$ be an arbitrary p-frame and V an arbitrary doppelgänger valuation on $\mathbb{F}$. On the bi-approximation model $\mathbb{M} = \langle \mathbb{F}, V \rangle$, all initial sequents are true, by Proposition

3.9. Note that we use Proposition 4.7 to prove the inductive steps. We mention that (**f**w) and ($\circ \Rrightarrow$) are trivial, and ($\Rrightarrow \vee_2$), ($\wedge_2 \Rrightarrow$), ($\Rrightarrow\rightarrow$) and ($\leftarrow\Rrightarrow$) are analogous to ($\Rrightarrow \vee_1$), ($\wedge_1 \Rrightarrow$), ($\Rrightarrow\leftarrow$) and ($\rightarrow\Rrightarrow$), respectively.

(cut): For arbitrary $x \in X$ and $y \in Y$, let $\mathbb{M}, x \Vdash_{\mathfrak{bs}} \Sigma \circ \Gamma \circ \Pi$ and $\mathbb{M}, y \Vdash_{\mathfrak{bs}} \varphi$. Then, there exist $x_1, x_2, x_3, x' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Sigma$, $x_2 \Vdash_{\mathfrak{bs}} \Gamma$, $x_3 \Vdash_{\mathfrak{bs}} \Pi$, $R^\circ(x_1, x', x)$ and $R^\circ(x_2, x_3, x')$. By induction hypothesis, $\Gamma \Rrightarrow \phi$ is true on $\mathbb{M}$. By Proposition 3.9, we obtain that $x_2 \Vdash \phi$, hence $x \Vdash \Sigma \circ \phi \circ \Pi$. Again, by induction hypothesis, $\mathbb{M} \Vdash \Sigma \circ \phi \circ \Pi \Rrightarrow \varphi$, which concludes $x \leq y$.

(tw): For arbitrary $x \in X$ and $y \in Y$, let $\mathbb{M}, x \Vdash_{\mathfrak{bs}} \Gamma \circ \mathbf{t} \circ \Delta$ and $y \Vdash_{\mathfrak{bs}} \varphi$. Then, there exist $x_1, x_2, x_3, x' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Gamma$, $x_2 \Vdash_{\mathfrak{bs}} \mathbf{t}$, $x_3 \Vdash_{\mathfrak{bs}} \Delta$, $R^\circ(x_1, x', x)$ and $R^\circ(x_2, x_3, x')$. Because $x_2 \in O_X$ and $R^\circ(x_2, x_3, x')$, we obtain $x' \leq x_3$ by R-order. By Hereditary (Proposition 3.8), we also have $x' \Vdash \Delta$, hence $x \Vdash \Gamma \circ \Delta$ holds. Finally, by induction hypothesis, $\mathbb{M} \Vdash \Gamma \circ \Delta \Rrightarrow \varphi$. Therefore, $x \leq y$.

($\Rrightarrow \circ$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma \circ \Sigma$ and $y \Vdash \phi \circ \psi$. Then, there exist $x_1, x_2 \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Gamma$, $x_2 \Vdash_{\mathfrak{bs}} \Sigma$ and $R^\circ(x_1, x_2, x)$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Gamma \Rrightarrow \phi$ and $\mathbb{M} \Vdash \Sigma \Rrightarrow \psi$. We obtain $x_1 \Vdash \phi$ and $x_2 \Vdash \psi$. By definition, since $y \Vdash \phi \circ \psi$, $R(x_1, x_2, y)$ holds. Because of Definition 4.1, we conclude $x \leq y$.

($\rightarrow\Rrightarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Sigma \circ \Gamma \circ (\phi \rightarrow \psi) \circ \Pi$ and $y \Vdash_{\mathfrak{bs}} \varphi$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Sigma \circ \psi \circ \Pi \Rrightarrow \varphi$, hence $y \Vdash \Sigma \circ \psi \circ \Pi$. Moreover, there exist $x_1, x_2, x_3, x_4, x', x'' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Sigma$, $x_2 \Vdash_{\mathfrak{bs}} \Gamma$, $x_3 \Vdash_{\mathfrak{bs}} \phi \rightarrow \psi$, $x_4 \Vdash_{\mathfrak{bs}} \Pi$, $R^\circ(x_2, x_3, x')$, $R^\circ(x', x_4, x'')$ and $R^\circ(x_1, x'', x)$. By inductive hypothesis, $\mathbb{M} \Vdash \Gamma \Rrightarrow \phi$, hence $x_2 \Vdash \phi$. Furthermore, because $x_2 \Vdash \phi$ and $x_3 \Vdash \phi \rightarrow \psi$, we have that, for each $x''' \in X$, if $R^\circ(x_2, x_3, x''')$ holds, then $x''' \Vdash \psi$ (Theorem 4.4). Because of $R^\circ(x_2, x_3, x')$, we obtain $x' \Vdash \psi$. Hence, we derive $x \Vdash \Sigma \circ \psi \circ \Pi$. Therefore, $x \leq y$. □

Remark 6.2 We mention that, in the proof of Theorem 6.1, we effectively use the bi-approximation, bases and the existential quantifier, i.e. Theorem 4.4, Theorem 4.6 and Proposition 4.7, to *stay away from taking the Galois connection.*

Theorem 6.3 (Completeness) *Let $\Gamma \Rrightarrow \varphi$ be an arbitrary sequent. If the sequent $\Gamma \Rrightarrow \varphi$ is valid on every p-frame $\mathbb{F}$, then it is derivable in FL.*

Proof. Let $\mathbb{L}$ be Lindenbaum-Tarski algebra of substructural logic **FL**. If $\Gamma \Rrightarrow \varphi$ is not derivable in FL, then $\Gamma \Rrightarrow \varphi$ is not valid on $\mathbb{L}$. By Theorem 5.4, the dual frame $\mathbb{L}_+$ of $\mathbb{L}$ is a p-frame. Furthermore, by theorem 5.5, the sequent $\Gamma \Rrightarrow \varphi$ is not valid on $\mathbb{L}_+$. □

Therefore, combined with the canonicity results in [19], we obtain the following.

Theorem 6.4 *Let Ω be a set of sequents which have consistent variable occurrence (see [19]). A substructural logic extended by Ω is complete with respect to a class of p-frames.*

7 Conclusion

We introduced bi-approximation semantics to describe Ghilardi and Meloni's parallel computation of the canonical extension of lattice expansions. Unlike what happens in the setting of standard relational semantics, like Kripke semantics or Routley-Meyer semantics, bi-approximation semantics is two-sorted. However, we claim that this is a natural framework for the study of logic, because logic is *a priori* two-sorted: premises and conclusions. In other words, logic is the study of a consequence relation.

From this point of view, bi-approximation semantics is a reasonable relational-type semantics for lattice-based logics. This framework could be valuable when we think about resource sensitive logics, since we explicitly distinguish premises from conclusions. Even over distributive lattice-based logics like intuitionistic logic, our two-sorted semantics may be worthwhile. For example, the first-order definability for intuitionistic modal logic on Kripke semantics is still open (see the footnote in [11, p.2]), whereas the first-order definability on bi-approximation semantics is effectively solved (in preparation).

References

[1] Birkhoff, G., "Lattice theory," American Mathematical Society Colloquium Publications **XXV**, American Mathematical Society, New York, 1948, revised edition.

[2] Blackburn, P., M. de Rijke and Y. Venema, "Modal logic," Cambridge Tracts in Theoretical Computer Science **53**, Cambridge University Press, Cambridge, 2002.

[3] Chagrov, A. and M. Zakharyaschev, "Modal logic," Oxford Logic Guides **35**, Oxford Science Publications, New York, 1997.

[4] Davey, B. and H. Priestley, "Introduction to Lattices and Order," Cambridge University Press, Cambridge, 2002, 2nd edition.

[5] Dunn, M., M. Gehrke and A. Palmigiano, *Canonical extensions and relational completeness of some substructural logics*, The Journal of Symbolic Logic **70** (2005), pp. 713–740.

[6] Galatos, N., P. Jipsen, T. Kowalski and H. Ono, "Residuated lattices: an algebraic glimpse at substructural logics," Studies in Logics and the Foundation of Mathematics **151**, Elsevier, Amsterdam, 2007.

[7] Gehrke, M., *Generalized Kripke frames*, Studia Logica **84** (2006), pp. 241–275.

[8] Gehrke, M. and J. Harding, *Bounded lattice expansions*, Journal of Algebra **239** (2001), pp. 345–371.

[9] Gehrke, M. and H. Priestley, *Non-canonicity of MV-algebras*, Houston Journal of Mathematics **28** (2002), p. 449.

[10] Gehrke, M. and H. Priestley, *Canonical extensions of double quasioperator algebras: an algebraic perspective on duality for certain algebras with binary operators*, Journal of Pure and Applied Algebra **209** (2007), pp. 269–290.

[11] Ghilardi, S. and G. Meloni, *Constructive canonicity in non-classical logics*, Annals of Pure and Applied Logic **86** (1997), pp. 1–32.

[12] Goldblatt, R., *Semantic analysis of orthologic*, Journal of Philosophical Logic **3** (1974), pp. 19–35.

[13] Hartonas, C., *Duality for lattice-ordered algebras and normal algebraizable logics*, Studia Logica **58** (1997), pp. 403–450.

[14] Hartonas, C. and J. M. Dunn, *Stone duality for lattices*, Algebra Universalis **37** (1997), pp. 391–401.

[15] Ono, H., *Substructural logics and residuated lattices - an introduction*, in: V. F. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica: Trends in Logic*, Kluwer Academic Publishers, Dordrecht, 2003 pp. 193–228.

[16] Restall, G., “An Introduction to Substructural Logics,” Routledge, London, 2000.

[17] Routley, R., V. Plumwood, R. K. Meyer and R. T. Brady, “Relevant logics and their rivals. Part 1. The basic philosophical and semantical theory,” Ridgeview Publishing Company, Atascadero, 1982.

[18] Suzuki, T., *A relational semantics for distributive substructural logics and the topological characterization of the descriptive frames*, CALCO-jnr 2007 Report No.367, Department of Informatics, University of Bergen (2008).
URL `http://www.ii.uib.no/publikasjoner/texrap/pdf/2008-367.pdf`

[19] Suzuki, T., *Canonicity results of substructural and lattice-based logics*, The Review of Symbolic Logic **3** (2010).

[20] Urquhart, A., “The semantics of entailment,” Ph.D. thesis, University of Pittsburgh (1972).

[21] Wright, F. B., *Polarity and duality*, Pacific Journal of Mathematics **10** (1960), pp. 723–730.

A Appendix of proofs

Proof. [Proposition 3.8] Parallel induction. Base cases are straightforward, since every Galois stable X-set is a downset and every Galois stable Y-set is an upset.

Inductive steps: $\vee$**:** Assume $y \Vdash \phi \vee \psi$. By definition, $y \Vdash \phi$ and $y \Vdash \psi$. By induction hypothesis, we obtain $y' \Vdash \phi$ and $y' \Vdash \psi$, hence $y' \Vdash \phi \vee \psi$. Suppose $x \Vdash \phi \vee \psi$. For each $y \Vdash \phi \vee \psi$, we have $x \leq y$. Because of $x' \leq x$, we obtain $x' \leq y$, hence $x' \Vdash \phi \vee \psi$.

$\wedge$**:** Assume $x \Vdash \phi \wedge \psi$. By definition, $x \Vdash \phi$ and $x \Vdash \psi$. By induction hypothesis, we obtain $x' \Vdash \phi$ and $x' \Vdash \psi$, hence $x' \Vdash \phi \wedge \psi$. Suppose $y \Vdash \phi \wedge \psi$. For each $x \Vdash \phi \wedge \psi$, we have $x \leq y$. Because of $y \leq y'$, we obtain $x \leq y'$, hence $y' \Vdash \phi \wedge \psi$.

$\circ$**:** Assume $y \Vdash \phi \circ \psi$. If $x_1 \Vdash \phi$ and $x_2 \Vdash \psi$, then we have $R(x_1, x_2, y)$. Since $y \leq y'$, by R-transitivity, we obtain $R(x_1, x_2, y')$, hence $y' \Vdash \phi \circ \psi$. Suppose $x \Vdash \phi \circ \psi$. For each $y \Vdash \phi \circ \psi$, we have $x \leq y$. Because of $x' \leq x$, we obtain $x' \leq y$, hence $x' \Vdash \phi \circ \psi$.

$\rightarrow$**:** Assume $x \Vdash \phi \rightarrow \psi$. For each $x_1 \Vdash \phi$ and each $y \Vdash \psi$, we have $R(x_1, x, y)$. By R-transitivity, we have $R(x_1, x', y)$, hence $x' \Vdash \phi \rightarrow \psi$. Suppose $y \Vdash \phi \rightarrow \psi$. For each $x \Vdash \phi \rightarrow \psi$, we have $x \leq y$. Since $y \leq y'$, we obtain $x \leq y'$, hence $y' \Vdash \phi \rightarrow \psi$.

$\leftarrow$**:** Assume $x \Vdash \psi \leftarrow \phi$. For each $x_2 \Vdash \phi$ and each $y \Vdash \psi$, we have $R(x, x_2, y)$. By R-transitivity, we have $R(x', x_2, y)$, hence $x' \Vdash \psi \leftarrow \phi$. Suppose $y \Vdash \psi \leftarrow \phi$. For each $x \Vdash \psi \leftarrow \phi$, we have $x \leq y$. Because of $y \leq y'$, we obtain $x \leq y'$, hence $y' \Vdash \psi \leftarrow \phi$.

□

Proof. [Lemma 4.2] Item (i). ($\Rightarrow$). Suppose $R^\circ(x_1, x_2, x)$, i.e. if $R(x_1, x_2, y')$ then $x \leq y'$ for every $y' \in Y$. By assumption, we obtain $R(x_1, x_2, y)$, which derives $x \leq y$.

($\Leftarrow$). Contraposition. Namely, we claim that there exists $x \in X$ such that $R^\circ(x_1, x_2, x)$ and $x \not\leq y$, under the assumption that $R(x_1, x_2, y)$ does not hold. Suppose that $R(x_1, x_2, y)$ does not hold. By $\circ$-tightness, there exists $x \in X$ such that, $R^\circ(x_1, x_2, x)$, and, for each $y' \in Y$, if $x \leq y'$, then $R(x_1, x_2, y')$. Since $R(x_1, x_2, y)$ does not hold, we have $x \not\leq y$. Item (ii) and item (iii) are analogous to item (i). □

Proof. [Theorem 4.4] Items (i) - (v) are analogous to item (vi). And, item (viii) and item (ix) are analogous to item (vii).

(vi) By Proposition 3.9, Definition 4.1 and Lemma 4.2, we can prove as follows.

$$\begin{aligned} x_1 \Vdash \psi \leftarrow \phi &\iff \forall x_2 \in X, \forall y \in Y.[x_2 \Vdash \phi, y \Vdash \psi \Rightarrow R(x_1, x_2, y)] \\ &\iff \forall x_2 \in X, \forall y_2, y \in Y.[x_2 \Vdash \phi, y \Vdash \psi, R^{\rightarrow}(x_1, y_2, y) \Rightarrow x_2 \leq y_2] \\ &\iff \forall y_2, y \in Y.[y \Vdash \psi, R^{\rightarrow}(x_1, y_2, y) \Rightarrow y_2 \Vdash \phi] \end{aligned}$$

(vii) Suppose that there exist $x_1, x_2 \in X$ such that $x_1 \Vdash \phi$, $x_2 \Vdash \psi$ and $R^\circ(x_1, x_2, x)$. We claim that every element $y \in Y$ at which $\phi \circ \psi$ is concluded is above x. If $y \Vdash \phi \circ \psi$ holds, then, by definition, $R(x_1, x_2, y)$ holds. By Definition 4.1, we also obtain that $x \leq y'$, whenever $R(x_1, x_2, y')$ holds for every $y' \in Y$. Hence, $x \leq y$ holds, which derives $x \Vdash \phi \circ \psi$.

□

Proof. [Theorem 4.5] (i). Suppose that $x \Vdash \phi$ or $x \Vdash \psi$. For an arbitrary $y \in Y$, if $y \Vdash \phi \vee \psi$, by definition, $y \Vdash \phi$ and $y \Vdash \psi$. By Proposition 3.9, $x \Vdash \phi$ or $x \Vdash \psi$, either way, $x \leq y$ holds. Therefore, $x \Vdash \phi \vee \psi$. Item (ii) is analogous to item (i). □

Proof. [Proposition 4.7] ($\Rightarrow$). Since $x \Vdash \phi$ ($y \Vdash \psi$) whenever $x \Vdash_{\mathfrak{bs}} \phi$ ($y \Vdash_{\mathfrak{bs}} \psi$), this is trivial. ($\Leftarrow$). Let x be an arbitrary element where ϕ is premised, y an arbitrary element where ψ is concluded. By our assumption, for an arbitrary $x_B \Vdash_{\mathfrak{bs}} \phi$, we have $x_B \leq y_B$ for every $y_B \Vdash_{\mathfrak{bs}} \psi$. By Theorem 4.6, we obtain $x_B \Vdash \psi$, hence $x_B \leq y$ (Proposition 3.9). As x_B is arbitrary, by Theorem 4.6, $y \Vdash \phi$ also holds. Therefore, $x \leq y$ (Proposition 3.9). □

Proof. [Theorem 5.1] Firstly, we need to check well-definedness of each operation. Namely, it is necessary to show that every value returns a Galois stable set. The copying parts are trivial, hence we need to check the following definition parts.

$\vee$**:** We claim that $\alpha_\uparrow \cap \beta_\uparrow = \lambda(\alpha^\downarrow \cup \beta^\downarrow)$. ($\subseteq$). For each $y \in \alpha_\uparrow \cap \beta_\uparrow$, since $y \in \alpha_\uparrow$ and $y \in \beta_\uparrow$, $x \leq y$ for each $x \in \alpha^\downarrow \cup \beta^\downarrow$. ($\supseteq$). If $y \in \lambda(\alpha^\downarrow \cup \beta^\downarrow)$, for arbitrary $x_a \in \alpha^\downarrow$ and $x_b \in \beta^\downarrow$, we have $x_a \leq y$ and $x_b \leq y$, hence $y \in \alpha_\uparrow$ and $y \in \beta_\uparrow$.

$\wedge$**:** We claim that $\alpha^\downarrow \cap \beta^\downarrow = \upsilon(\alpha_\uparrow \cup \beta_\uparrow)$. ($\subseteq$). For each $x \in \alpha^\downarrow \cap \beta^\downarrow$, since $x \in \alpha^\downarrow$ and $x \in \beta^\downarrow$, $x \leq y$ for each $y \in \alpha_\uparrow \cup \beta_\uparrow$. ($\supseteq$). If $x \in \upsilon(\alpha_\uparrow \cup \beta_\uparrow)$, for arbitrary $y_a \in \alpha_\uparrow$ and $y_b \in \beta_\uparrow$, we have $x \leq y_a$ and $x \leq y_b$, hence $x \in \alpha^\downarrow$ and $x \in \beta^\downarrow$.

$*$**:** We claim that $\alpha_\uparrow * \beta_\uparrow = \lambda(\{x \in X \mid x_1 \in \alpha^\downarrow, x_2 \in \beta^\downarrow, R^\circ(x_1, x_2, x)\})$.

$$\begin{aligned}\alpha_\uparrow * \beta_\uparrow &= \{y \in Y \mid \forall x_1 \in \alpha^\downarrow, \forall x_2 \in \beta^\downarrow, R(x_1, x_2, y)\}\\ &= \{y \in Y \mid \forall x \in X, \forall x_1 \in \alpha^\downarrow, \forall x_2 \in \beta^\downarrow, R^\circ(x_1, x_2, x) \Rightarrow x \leq y\}\\ &= \lambda(\{x \in X \mid x_1 \in \alpha^\downarrow, x_2 \in \beta^\downarrow, R^\circ(x_1, x_2, x)\})\end{aligned}$$

: We claim that $\alpha^\downarrow \backslash \beta^\downarrow = \upsilon(\{y_2 \in Y \mid x_1 \in \alpha^\downarrow, y \in \beta_\uparrow, R^\rightarrow(x_1, y_2, y)\})$.

$$\begin{aligned}\alpha^\downarrow \backslash \beta^\downarrow &= \{x_2 \in X \mid \forall x_1 \in \alpha^\downarrow, \forall y \in \beta_\uparrow, R(x_1, x_2, y)\}\\ &= \{x_2 \in X \mid \forall y_2 \in Y, \forall x_1 \in \alpha^\downarrow, \forall y \in \beta_\uparrow, R^\rightarrow(x_1, y_2, y) \Rightarrow x_2 \leq y_2\}\\ &= \upsilon(\{y_2 \in Y \mid x_1 \in \alpha^\downarrow, y \in \beta_\uparrow, R^\rightarrow(x_1, y_2, y)\})\end{aligned}$$

/ is analogous to \.

Therefore, all operations are well-defined. Furthermore, these two algebras are isomorphic by definition. Next, we prove they are FL-algebras.

$\langle \mathbb{D}, \vee, \wedge \rangle$ and $\langle \mathbb{U}, \vee, \wedge \rangle$ are lattices. For all α, β, γ, we claim that [2]

$$\alpha \leq \gamma \text{ and } \beta \leq \gamma \iff \alpha \vee \beta \leq \gamma, \tag{A.1}$$

$$\gamma \leq \alpha \text{ and } \gamma \leq \beta \iff \gamma \leq \alpha \wedge \beta. \tag{A.2}$$

($\Rightarrow$) of the condition (A.1). For each $y \in \gamma_\uparrow$, since $\alpha_\uparrow \supseteq \gamma_\uparrow$ and $\beta_\uparrow \supseteq \gamma_\uparrow$, we have $y \in \alpha_\uparrow$ and $y \in \beta_\uparrow$, hence $y \in \alpha_\uparrow \cap \beta_\uparrow$. ($\Leftarrow$) of the condition (A.1). For each $y \in \gamma_\uparrow$, since $\alpha_\uparrow \cap \beta_\uparrow \supseteq \gamma_\uparrow$, we obtain $y \in \alpha_\uparrow$ and $y \in \beta_\uparrow$. The condition (A.2) is analogous.

[2] Recall that the order $\leq$ is $\subseteq$ on $\mathbb{D}$ and $\supseteq$ on $\mathbb{U}$.

$\langle \mathbb{D}, *, O_X \rangle$ and $\langle \mathbb{U}, *, O_Y \rangle$ are monoids. For all α, β, γ, we claim that

$$\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma, \tag{A.3}$$

$$\alpha * O = \alpha = O * \alpha, \tag{A.4}$$

where O is either O_X or O_Y depending on the domain. The condition (A.3). Let y be an arbitrary element in $\alpha_\uparrow * (\beta_\uparrow * \gamma_\uparrow)$. By Theorem 4.6, for all $x_1, x_2, x_3, x', x \in X$, if $x_1 \in \alpha^\downarrow$, $x_2 \in \beta^\downarrow$, $x_3 \in \gamma^\downarrow$, $R^\circ(x_1, x', x)$ and $R^\circ(x_2, x_3, x')$, then $x \leq y$ holds. By R-associativity (see Remark 4.3), the condition is equivalent to that, for each element x, for all $x'' \in X$, if $x_1 \in \alpha^\downarrow$, $x_2 \in \beta^\downarrow$, $x_3 \in \gamma^\downarrow$, $R^\circ(x'', x_3, x)$ and $R^\circ(x_1, x_2, x'')$, then $x \leq y$, which concludes $y \in (\alpha_\uparrow * \beta_\uparrow) * \gamma_\uparrow$.

The left equality of the condition (A.4). ($\subseteq$). For each $x_1 \in \alpha^\downarrow$, by R-identity, there exists $o_2 \in O_X$ such that $R^\circ(x_1, o_2, x_1)$. By definition, for every $y' \in Y$, if $R(x_1, o_2, y')$, then $x_1 \leq y'$ holds. Now, for every $y \in \alpha_\uparrow * O_Y$, by definition, $R(x_1, o_2, y)$ holds, hence $x_1 \leq y$. Since x_1 is arbitrary in $\alpha^\downarrow$, which derives $y \in \alpha_\uparrow$. ($\supseteq$). For arbitrary $x \in \alpha^\downarrow$ and $o \in O_X$, by $\circ$-tightness, there exists $x' \in X$ such that $R^\circ(x, o, x')$ and $x' \leq y' \Rightarrow R(x, o, y')$ for each $y' \in Y$. For every $y \in \alpha_\uparrow$, we have $x \leq y$, because of $x \in \alpha^\downarrow$. Furthermore, by R-order, $x' \leq x$ holds. Since $\leq$ is transitive, we obtain $x' \leq y$, hence $R(x, o, y)$. The right equality of the condition (A.4) is analogous.

Finally, we will show the residuality: for all α, β, γ,

$$\alpha * \beta \leq \gamma \iff \beta \leq \alpha \backslash \gamma \iff \alpha \leq \gamma / \beta. \tag{A.5}$$

($\Rightarrow$) of the first equivalence in the condition (A.5). Let x_2 be an arbitrary element in $\beta^\downarrow$. For arbitrary $x_1 \in \alpha^\downarrow$ and $y \in \gamma_\uparrow$, since $\alpha_\uparrow * \beta_\uparrow \supseteq \gamma_\uparrow$, we have $R(x_1, x_2, y)$. Hence, $x_2 \in \alpha^\downarrow \backslash \gamma^\downarrow$. ($\Leftarrow$) of the first equivalence in the condition (A.5). Let y be an arbitrary element in $\gamma_\uparrow$. For arbitrary $x_1 \in \alpha^\downarrow$ and $x_2 \in \beta^\downarrow$, since $\beta^\downarrow \subseteq \alpha^\downarrow \backslash, \gamma^\downarrow$, we obtain $R(x_1, x_2, y)$. Hence, $y \in \alpha_\uparrow * \beta_\uparrow$. The other equivalence is analogous. □

Proof. [Theorem 5.4] By definition, $\langle \mathcal{F}, \mathcal{I}, \sqsubseteq \rangle$ is a polarity.

R-order: Let F, F' be arbitrary filters. Suppose $F' \sqsubseteq F$. Since $F = F * {\uparrow}1$, we obtain $F' \sqsubseteq F * {\uparrow}1$. Conversely, if $F' \sqsubseteq F * O$ or $F' \sqsubseteq O * F$ for some $O \in O_{\mathcal{F}}$, because $1 \in O$, we obtain $F * O \sqsubseteq F$ or $O * F \sqsubseteq F$, hence $F' \sqsubseteq F$.

R-identity: Let ${\uparrow}1$ be the principal filter generated by 1. For each filter F, we have $F * {\uparrow}1 = {\uparrow}1 * F = F$, hence $R^\circ(F, {\uparrow}1, F)$ and $R^\circ({\uparrow}1, F, F)$.

R-transitivity: For all $F_1, F_1', F_2, F_2' \in \mathcal{F}$ and all $I, I' \in \mathcal{I}$, if $F_1' \sqsubseteq F_1$, $F_2' \sqsubseteq F_2$, $I \sqsubseteq I'$ and $F_1 * F_2 \sqsubseteq I$, then there exist $f_1 \in F_1$, $f_2 \in F_2$ and $i \in I$ such that $f_1 * f_2 \leq i$. Since $f_1 \in F_1'$, $f_2 \in F_2'$ and $i \in I'$, we also have $F_1' * F_2' \sqsubseteq I'$.

R-associativity: For all $F_1, F_2, F_3 \in \mathcal{F}$, we have $F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3$, by the associativity of $*$ on $\mathbb{A}$. If $F \sqsubseteq F_1 * F'$ and $F' \sqsubseteq F_2 * F_3$, we obtain $F \sqsubseteq F_1 * (F_2 * F_3) = (F_1 * F_2) * F_3$. Let $F'' = F_1 * F_2$. Then, $F \sqsubseteq F'' * F_3$ and $F'' \sqsubseteq F_1 * F_2$ hold.

O-isom (N-isom): For each $F \in O_{\mathcal{F}}$ ($N_{\mathcal{F}}$) and each $I \in O_{\mathcal{I}}$ ($N_{\mathcal{I}}$), they have 1 (0) in common.

$\circ$-tightness: For all $F_1, F_2 \in \mathcal{F}$, it is trivially true that $R(F_1, F_2, I)$ if and only if $F_1 * F_2 \sqsubseteq I$ for every $I \in \mathcal{I}$. The other is analogous.

$\rightarrow$-tightness: For each $F_1 \in \mathcal{F}$ and each $I \in \mathcal{I}$, by definition, for each $F_2 \in \mathcal{F}$, $R(F_1, F_2, I)$ if and only if $F_2 \sqsubseteq F_1 \backslash I$.

$\leftarrow$-tightness: For each $F_2 \in \mathcal{F}$ and each $I \in \mathcal{I}$, by definition, for each $F_1 \in \mathcal{F}$, $R(F_1, F_2, I)$ if and only if $F_1 \sqsubseteq I/F_2$.

□

Proof. [Theorem 5.5] Let $f : \Phi \to \mathbb{A}$ be an arbitrary assignment. We also denote the normally extended assignment $f : \Lambda \to \mathbb{A}$ by f. Then, we define a doppelgänger valuation V based on f as follows: for each proposition $p \in \Phi$,

(i) $V^{\downarrow}(p) := \{F \in \mathcal{F} \mid f(p) \in F\} = \upsilon(\{\downarrow f(p)\})$,

(ii) $V_{\uparrow}(p) := \{I \in \mathcal{I} \mid f(p) \in I\} = \lambda(\{\uparrow f(p)\})$.

We claim that, for each filter F, each ideal I and each formula ϕ, $f(\phi) \in F \iff \mathbb{A}_+, V, F \Vdash \phi$ and $f(\phi) \in I \iff \mathbb{A}_+, V, I \Vdash \phi$. Base cases are trivial. Inductive steps. For each filter $F \in \mathcal{F}$ and each ideal $I \in \mathcal{I}$,

$\vee$**:** Suppose that $f(\phi) \vee f(\psi) = f(\phi \vee \psi) \in I$. It is equivalent to $f(\phi) \in I$ and $f(\psi) \in I$. By induction hypothesis, it is also equivalent to $I \Vdash \phi$ and $I \Vdash \psi$, which, by definition, $I \Vdash \phi \vee \psi$.

If $f(\phi \vee \psi) \in F$, then F has non-empty intersection with all ideals containing $f(\phi \vee \psi)$. We obtain $F \Vdash \phi \vee \psi$, because every ideal I satisfying $I \Vdash \phi \vee \psi$ contains $f(\phi \vee \psi)$. Conversely, if $F \Vdash \phi \vee \psi$, then it must have non-empty intersection with $\downarrow f(\phi \vee \psi)$ as well. Therefore, $f(\phi \vee \psi) \in F$.

$\wedge$**:** Suppose that $f(\phi) \wedge f(\psi) = f(\phi \wedge \psi) \in F$. It is equivalent to $f(\phi) \in F$ and $f(\psi) \in F$. By induction hypothesis, it is also equivalent to $F \Vdash \phi$ and $F \Vdash \psi$, which $F \Vdash \phi \wedge \psi$ by definition.

If $f(\phi \wedge \psi) \in I$, then I has non-empty intersection with all filters containing $f(\phi \wedge \psi)$. We obtain $I \Vdash \phi \wedge \psi$, because every filter F satisfying $F \Vdash \phi \wedge \psi$ contains $f(\phi \wedge \psi)$. Conversely, if $I \Vdash \phi \wedge \psi$, then it must have non-empty intersection with $\uparrow f(\phi \wedge \psi)$ as well. Therefore, $f(\phi \wedge \psi) \in I$.

$\circ$**:** Suppose that $f(\phi) * f(\psi) = f(\phi \circ \psi) \in I$. For arbitrary $F_1, F_2 \in \mathcal{F}$, if $F_1 \Vdash \phi$ and $F_2 \Vdash \psi$, by induction hypothesis, $f(\phi) \in F_1$ and $f(\psi) \in F_2$, hence $f(\phi) * f(\psi) \in F_1 * F_2$, which derives $F_1 * F_2 \sqsubseteq I$, i.e. $R(F_1, F_2, I)$. Conversely, assume that $I \Vdash \phi \circ \psi$. By definition, for arbitrary $F_1 \Vdash \phi$ and $F_2 \Vdash \psi$, $F_1 * F_2 \sqsubseteq I$ holds. Then, $\uparrow f(\phi) * \uparrow f(\psi) \sqsubseteq I$ must hold, hence $f(\phi \circ \psi) \in I$.

If $f(\phi \circ \psi) \in F$, then F has non-empty intersection with all ideals containing $f(\phi \circ \psi)$. Since every ideal I satisfying $I \Vdash \phi \circ \psi$ contains $f(\phi \circ \psi)$, we have $F \Vdash \phi \circ \psi$. Conversely, if $F \Vdash \phi \circ \psi$, then it must have non-empty intersection with $\downarrow f(\phi \circ \psi)$ as well. Therefore, $f(\phi \circ \psi) \in F$.

$\rightarrow$**:** Suppose that $f(\phi) \backslash f(\psi) = f(\phi \to \psi) \in F$. For arbitrary $F' \in \mathcal{F}$ and $I \in \mathcal{I}$, if $F' \Vdash \phi$ and $I \Vdash \psi$, by induction hypothesis, $f(\phi) \in F'$ and $f(\psi) \in I$, hence $f(\phi) \backslash f(\psi) \in F' \backslash I$. By the residuality on $\mathbb{A}$, we obtain $F' * F \sqsubseteq I$, i.e. $R(F', F, I)$

holds. Conversely, assume that $F \Vdash \phi \rightarrow \psi$. By definition, for arbitrary $F' \Vdash \phi$ and $I \Vdash \psi$, we have $F' * F \sqsubseteq I$. Then, $\uparrow f(\phi) * F \sqsubseteq \downarrow f(\psi)$ must hold as well. Therefore, there exists $x \in F$ such that $x \leq f(\phi)\backslash f(\psi) = f(\phi \rightarrow \psi)$, hence $f(\phi \rightarrow \psi) \in F$.

If $f(\phi \rightarrow \psi) \in I$, then I has non-empty intersection with all filters containing $f(\phi \rightarrow \psi)$. Since every filter F satisfying $F \Vdash \phi \rightarrow \psi$ contains $f(\phi \rightarrow \psi)$, we have $I \Vdash \phi \rightarrow \psi$. Conversely, if $I \Vdash \phi \rightarrow \psi$, then it must have non-empty intersection with $\uparrow f(\phi \rightarrow \psi)$ as well. Therefore, $f(\phi \rightarrow \psi) \in I$.

$\leftarrow$: Suppose that $f(\psi)/f(\phi) = f(\psi \leftarrow \phi) \in F$. For arbitrary $F' \in \mathcal{F}$ and $I \in \mathcal{I}$, if $F' \Vdash \phi$ and $I \Vdash \psi$, by induction hypothesis, $f(\phi) \in F'$ and $f(\psi) \in I$, hence $f(\psi)/f(\phi) \in I/F'$. By the residuality on $\mathbb{A}$, we obtain $F * F' \sqsubseteq I$, i.e. $R(F, F', I)$ holds. Conversely, assume that $F \Vdash \psi \leftarrow \phi$. By definition, for arbitrary $F' \Vdash \phi$ and $I \Vdash \psi$, we have $F * F' \sqsubseteq I$. Then, $F * \uparrow f(\phi) \sqsubseteq \downarrow f(\psi)$ must hold as well. Therefore, there exists $x \in F$ such that $x \leq f(\psi)/f(\phi) = f(\psi \leftarrow \phi)$, hence $f(\psi \leftarrow \phi) \in F$.

If $f(\psi \leftarrow \phi) \in I$, then I has non-empty intersection with all filters containing $f(\psi \leftarrow \phi)$. Since every filter F satisfying $F \Vdash \psi \leftarrow \phi$ contains $f(\psi \leftarrow \phi)$, we have $I \Vdash \psi \leftarrow \phi$. Conversely, if $I \Vdash \psi \leftarrow \phi$, then it must have non-empty intersection with $\uparrow f(\psi \leftarrow \phi)$ as well. Therefore, $f(\psi \leftarrow \phi) \in F$.

Finally, we finish up the proof. Assume $\Gamma \Rightarrow \varphi$ is not valid on $\mathbb{A}$. Then, there exists an assignment $f : \Phi \rightarrow \mathbb{A}$ such that $f(\Gamma) \not\leq f(\varphi)$. We have that $\uparrow f(\Gamma) \in \mathcal{F}$ and $\downarrow f(\varphi) \in \mathcal{I}$. Moreover, we also have $\mathbb{A}_+, V, \uparrow f(\Gamma) \Vdash \Gamma$ and $\mathbb{A}_+, V, \downarrow f(\varphi) \Vdash \varphi$. However, since $f(\Gamma) \not\leq f(\varphi)$, $\uparrow f(\Gamma) \not\sqsubseteq \downarrow f(\varphi)$. Therefore, $\mathbb{A}_+ \not\Vdash \Gamma \Rightarrow \varphi$. □

Proof. [The other cases of Theorem 6.1]

($\vee \Rightarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma \circ (\phi \vee \psi) \circ \Delta$ and $y \Vdash_{\mathfrak{bs}} \varphi$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Gamma \circ \phi \circ \Delta \Rightarrow \varphi$ and $\mathbb{M} \Vdash \Gamma \circ \psi \circ \Delta \Rightarrow \varphi$. So, we obtain $y \Vdash \Gamma \circ \phi \circ \Delta$ and $y \Vdash \Gamma \circ \psi \circ \Delta$. With repeating Definition 4.1 and Lemma 4.2, we obtain the following:

$$\begin{aligned}
y \Vdash \Gamma \circ \phi \circ \Delta \iff & \forall y', y_2 \in Y, \forall x_1, x_3 \in X. \\
& x_1 \Vdash \Gamma, x_3 \Vdash \Delta, R^{\leftarrow}(y_2, x_3, y'), R^{\rightarrow}(x_1, y', y) \Rightarrow y_2 \Vdash \phi, \\
y \Vdash \Gamma \circ \psi \circ \Delta \iff & \forall y', y_2 \in Y, \forall x_1, x_3 \in X. \\
& x_1 \Vdash \Gamma, x_3 \Vdash \Delta, R^{\leftarrow}(y_2, x_3, y'), R^{\rightarrow}(x_1, y', y) \Rightarrow y_2 \Vdash \psi, \\
y \Vdash \Gamma \circ (\phi \vee \psi) \circ \Delta \iff & \forall y', y_2 \in Y, \forall x_1, x_3 \in X. \\
& x_1 \Vdash \Gamma, x_3 \Vdash \Delta, R^{\leftarrow}(y_2, x_3, y'), R^{\rightarrow}(x_1, y', y) \Rightarrow y_2 \Vdash \phi \vee \psi.
\end{aligned}$$

Therefore, we obtain $y \Vdash \Gamma \circ (\phi \vee \psi) \circ \Delta$, hence $x \leq y$.

($\Rightarrow \vee_1$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma$ and $y \Vdash \phi \vee \psi$. By definition, $y \Vdash \phi$. By induction hypothesis, we have $\mathbb{M} \Vdash \Gamma \Rightarrow \phi$, hence $x \leq y$.

($\wedge_1 \Rightarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma \circ (\phi \wedge \psi) \circ \Delta$ and $y \Vdash_{\mathfrak{bs}} \varphi$. Then, there exist $x_1, x_2, x_3, x' \in X$ such that $x_1 \Vdash_{\mathfrak{bs}} \Gamma$, $x_2 \Vdash \phi \wedge \psi$, $x_3 \Vdash_{\mathfrak{bs}} \Delta$, $R^{\circ}(x_1, x', x)$ and $R^{\circ}(x_2, x_3, x')$. By definition, we also have $x_2 \Vdash \phi$, hence $x \Vdash \Gamma \circ \phi \circ \Delta$. By induction hypothesis, $\mathbb{M} \Vdash \Gamma \circ \phi \circ \Delta \Rightarrow \varphi$, hence $x \leq y$.

($\Rightarrow \wedge$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma$ and $y \Vdash_{\mathfrak{bs}} \phi \wedge \psi$. By inductive hypothesis, we have $\mathbb{M} \Vdash \Gamma \Rightarrow \phi$ and $\mathbb{M} \Vdash \Gamma \Rightarrow \psi$. Therefore, we obtain that $x \Vdash \phi$ and $x \Vdash \psi$, which derives $x \Vdash \phi \wedge \psi$. Then, $x \leq y$.

($\Rightarrow\leftarrow$): For arbitrary $x \in X$ and $y \in Y$, let $x \Vdash_{\mathfrak{bs}} \Gamma$ and $y \Vdash_{\mathfrak{bs}} \psi \leftarrow \phi$. Then, there exist $x_2 \in X$ and $y' \in Y$ such that $x_2 \Vdash_{\mathfrak{bs}} \phi$, $y' \Vdash_{\mathfrak{bs}} \psi$ and $R^{\leftarrow}(y, x_2, y')$. By induction hypothesis, we have $\mathbb{M} \Vdash \Gamma \circ \phi \Rightarrow \psi$, hence $y' \Vdash \Gamma \circ \phi$. By Theorem 4.4, for every $y'' \in Y$, if $R^{\leftarrow}(y'', x_2, y')$, then $y'' \Vdash \Gamma$. Finally, since $x \Vdash \Gamma$, we conclude $x \leq y$.

$\square$