Stabbing Information of a Simple Polygon *

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Abstract

The purpose of this paper is to investigate a new combinatorial object describing the structure of a simple polygon and compare it to other well-known objects such as the internal and external visibility graphs, the convex hull and the order type of the vertex set. We call the new object the ***stabbing information***. In fact, we define three variations of the stabbing information, strong, weak and labelled, and explore the relationships among them. The main result of the paper is that strong stabbing information is sufficient to recover the convex hull, the internal and external visibility graphs and to determine which vertices are reflex and it is not sufficient to recover the order type of the vertex set. We give algorithms for computing each of these new structures.

1 Introduction

A simple polygon in the plane is typically represented by means of an ordered list of the real-valued coordinates of its vertices. Although such a representation completely describes the polygon it may not be convenient to store and manipulate. Alternatively then, we may wish to compute and store only that combinatorial information about the polygon required for a particular application. Three well known combinatorial objects defined for simple polygons are the convex hull, the internal visibility graph and the external visibility graph.

Now suppose that we are given a combinatorial object. We ask whether there is a simple polygon which realizes it and we would like to actually construct such a polygon if it exists. This is quite straightforward in the case of the convex hull! However, in the case of visibility graphs these realizability problems are open and appear difficult. A simpler problem might be the following: given one (or several) of these combinatorial objects recover the remaining ones. For example, given the visibility graph of a simple polygon determine which vertices lie on the convex hull. In fact, this problem has been solved: the convex hull cannot be recovered from the visibility graph [2].

In this paper we explore such relationships among several well-known combinatorial objects. We start with the following four objects: the convex hull, the internal visibility graph, the external visibility graph, and a classification of the vertices into convex and reflex. We first note that any three of these taken together is not sufficient to recover the fourth; see Figures 1 to 3. Thus each object gives some new information about the combinatorial structure of the polygon. It is somewhat

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cumbersome to use these four independent objects; it would be nice to have all of this information together in one unified structure.

The order type provides a concise description of the combinatorial structure of a point set. Two sets of labelled points are said to have the same order type if corresponding triples of points are similarly oriented. The order type of a simple polygon is the order type of its vertices. As we shall see later, the order type can serve as the unified structure we are looking for; all of the four objects mentioned earlier, the convex hull, the internal and external visibility graphs, and the reflex/convex characterization of the vertices, can be recovered from the order type. However, as Figure 4 shows, the converse is not true; the order type contains strictly more information than our four objects.

We propose a new object called the *stabbing information* of a simple polygon which contains strictly less information than the order type but all the information in the four basic objects. Consider the oriented line l_{ij} from vertex i to vertex j. Divide the line into three portions: the ray from vertex j forward, the segment ij, and the ray arriving at vertex i. Notice that $\{i, j\}$ is an edge in either the internal or the external visibility graph if no polygon edge intersects the segment ij. The idea of the new structure is to store, for each pair of vertices, i and j, the intersections that each of the three portions of l_{ij} makes with the other edges of the polygon. Thus we think of the line through a pair of vertices as a stabbing line of a set of line segments, in this case, a set of polygon edges. The new structure can be viewed as a generalization of the visibility graphs which only distinguish between zero and more than zero intersections on one of the three portions of the line.

We define three variants of the new object which differ from each other in the information about the intersections that is stored. The weak stabbing information stores, for each pair of vertices i and j, the number of intersections with each of the three portions of the line l_{ij}. For example, for the polygon on the left in Figure 2, for the pair $(11, 9)$ we store the ordered triple $(1, 2, 0)$. In the strong stabbing graph we store the number of intersections from each of the two chains joining the two vertices. In our example, for $(11, 9)$, we store two ordered triples, $(1, 2, 0)$ representing intersections from the chain $(11, 0, \ldots, 9)$ and $(0, 0, 0)$ representing intersections from the chain $(9, 10, 11)$. In the labelled stabbing graph we store a list of the edges that intersect each portion of the line sorted as the occur on the line. So for the pair $(11, 9)$ edge $\{0, 1\}$ intersects the ray arriving at vertex 11, edges $\{4, 5\}$ and $\{5, 6\}$ intersect segment $(11, 9)$ in that order, and no segments intersect the ray leaving vertex 9.

Figures 4 to 6 show that these structures along with the four basic objects and the order type form a strict hierarchy. Figure 4 shows that weak stabbing information is not determined by the internal visibility graph, the external visibility graph, the convex hull and the list of reflex vertices. Figure 5 shows that strong stabbing information is not determined by weak stabbing information. Figure 6 shows that labelled stabbing information is not determined by strong stabbing information.

On the other hand, it turns out (see Theorem 4.3), that the labelled stabbing information is equivalent to the order type. It follows from their definitions that strong stabbing information can be obtained from labelled stabbing information and that weak stabbing information can be obtained from strong. We prove, (see Theorem 4.2), that the four basic objects can be recovered from strong stabbing information. In summary then, the strong stabbing information is our unified structure that contains all the information of the four basic structures but strictly less than that contained in the order type. We have not been able to prove that the four basic objects can be recovered from weak stabbing information and we leave this as an open problem. Finally, efficient algorithms exist for computing each of the new structures, (see Theorem 4.4).

2 Definitions and background

Let P be a simple polygon whose vertices $\{v_0, \ldots, v_{n-1}\}$ are in general position; that is, no three are collinear. When the meaning is clear we will refer to the vertices of a polygon by their indices. The vertices of the polygon are labelled in counter-clockwise order; that is, the interior of the polygon lies to the left as its boundary is traversed. A vertex v_i is called *reflex* if the interior angle $v_{i-1}v_iv_{i+1} > 180$; otherwise it is *convex*. The *convex hull* of P is the boundary of the smallest convex set containing P and is represented as an ordered list of vertices. There are a host of algorithms which compute the convex hull of a simple polygon; see [8] for a linear time algorithm.

The *internal (vertex) visibility graph* of polygon P is a graph whose vertices correspond to the vertices of P and whose edges correspond to internally visible vertices; that is, there is an edge between two vertices if the line segment connecting the corresponding polygon vertices does not intersect the exterior of P. The *external visibility graph* is defined similarly except that graph edges correspond to pairs of polygon vertices that are externally visible, that is, the line segment connecting them does not intersect the interior of P. The boundary edges of P are included in both the interior and exterior visibility graphs and in both cases we are also given the Hamiltonian circuit which describes the polygon boundary. Optimal algorithms exist for computing these visibility graphs, see [7] and [6].

An ordered triple of non-collinear points (p_1, p_2, p_3) has positive orientation if p_3 lies to the left of the oriented line through p_1 and p_2 and it has negative orientation otherwise. Two sets of labelled points are said to have the same order type if corresponding triples of points are similarly oriented [5]. The order type of a polygon is the order type of its vertex set. It is easy to compute the order type of n points in time $O(n^3)$. Goodman and Pollack have shown that the order type of a point set can be encoded in $O(n^2)$ storage [4] and Edelsbrunner, O'Rourke and Seidel have shown how to compute this structure in $O(n^2)$ time [1]. This encoding has the drawback that the orientation of any particular triple cannot be recovered in constant time.

3 The stabbing information

In this section we define precisely our new structures. Recall that the vertices of a polygon are assumed to be labelled $\{v_0, v_1, \ldots, v_{n-1}\}$ in counter-clockwise order and that no three vertices are collinear. We denote by l_{ij} the oriented line from v_i to v_j. For this oriented line h_{ij} will denote the *head* of the line, from v_j forward, b_{ij} the *body* of the line, between v_i and v_j, and t_{ij} the *tail* of the line, arriving at v_i. The set of polygon edges properly intersecting h_{ij} is denoted H_{ij}, the set intersecting b_{ij} is denoted B_{ij} and the set intersecting t_{ij} is denoted T_{ij}.

The *labelled stabbing information* consists of, for each ordered pair of vertices (v_i, v_j), ordered lists of the elements of H_{ij}, B_{ij} and T_{ij} sorted as their intersections occur on l_{ij}. Notice that if xy intersects l_{ij} we don't know if x is to the left or to the right of l_{ij}. The *weak stabbing information* consists of, for each ordered pair of vertices (v_i, v_j), the ordered triple of integers $|H_{ij}|$, $|B_{ij}|$ and $|T_{ij}|$. The boundary of a polygon from v_i to v_j is called the *left chain* from v_i to v_j if the interior of the polygon lies to the right as the boundary is traversed from v_i to v_j; otherwise it is called the *right chain*. Now LH_{ij} is the set of edges from the left chain intersecting h_{ij}; the other sets RH_{ij}, LB_{ij}, RB_{ij}, LT_{ij}, RT_{ij} are defined similarly. The *strong stabbing information* is obtained from the weak by classifying each of $|H_{ij}|$, $|B_{ij}|$ and $|T_{ij}|$ into $|LH_{ij}|$, $|RH_{ij}|$, $|LB_{ij}|$, $|RB_{ij}|$, $|LT_{ij}|$, $|RT_{ij}|$.

4 Main results

In this section we summarize the main results. For the purposes of this abstract most proofs have been omitted. These proofs can be found in the full paper [3].

Lemma 4.1 *The weak stabbing information determines the convex hull and the reflex vertices.*

Proof idea: First notice that vertex i is reflex if and only if $|H_{ij}|$ is odd. If $\{i, i+1\}$ is on the convex hull then $|T_{i,i+1}| = |H_{i,i+1}| = |B_{i,i+1}| = 0$. This can also happen if $\{i, i+1\}$ is an edge of the internal visibility graph; however, it is possible to distinguish between these two cases. □

Theorem 4.2 *The strong stabbing information of a polygon determines its external visibility graph, internal visibility graph, convex hull and reflex vertices.*

Proof idea: That the convex hull and the reflex vertices can be determined follows from the previous lemma. To determine the internal and external visibility graphs first notice that a non-polygon edge $\{i, j\}$ is not in either visibility graph if some polygon edge intersects b_{ij}; that is, $|LB_{ij}| + |RB_{ij}| \neq 0$. The hard part then is to distinguish between non-polygon edges that are in the external visibility graph from those that are in the internal visibility graph. The idea of the proof is to classify each vertex i with respect to l_{ij} depending on whether $i-1$ and $i+1$ lie above or below l_{ij} and whether i is convex or reflex. Similarly we classify vertex v_j. This yields a number of cases which can be resolved. We use the fact that the parity of $|LT_{ij}|+|RT_{ij}|$ and $|LH_{ij}|+|RH_{ij}|$ depends only on the types of i and j. □

Theorem 4.3 *The order type of a polygon determines its labelled stabbing information and vice versa.*

Proof idea: Given the order type it is relatively straightforward to find the labelled stabbing information. The other direction is more difficult; the proof is divided into three cases: 1) T_{ij} is not empty, 2) T_{ij}, B_{ij} and H_{ij} are not empty and 3) the remaining case. □

Theorem 4.4 *The labelled stabbing information can be computed in $O(n^3)$ time, the strong stabbing information in $O(n^2 \log n)$ time and the weak stabbing information in $O(n^2)$ time.*

Proof: Omitted. □

Note that the size of the representation of the labelled stabbing information described here is $O(n^3)$ and that of the weak and strong information is $O(n^2)$. Thus the algorithm for the strong stabbing information is not optimal whereas the other two are.

References

[1] H. Edelsbrunner, J. O'Rourke and R. Seidel, Constructing arrangements of lines and hyperplanes, with applications, *SIAM J. Comput.* 15 (1986) 341-363.

[2] H. Everett, A. Lubiw, and J. O'Rourke, Recovery of convex hulls from external visibility graphs, in *Proc. 5th Canadian Conference on Comp. Geom.* (1993) 309-314.

[3] H. Everett, F. Hurtado, and M. Noy, The stabbing information of a simple polygon, manuscript (1996).

[4] J. Goodman and R. Pollack, Multidimensional sorting, *SIAM J. Comput.* 12 (1983) 484-507.

[5] J. Goodman and R. Pollack, Allowable Sequences and Order Types in Discrete and Computational Geometry, in *New Trends in Discrete and Computational Geometry*, J. Pach, Ed., Springer-Verlag (1993) 103-134.

[6] S.K. Ghosh and D.M. Mount, An output-sensitive algorithm for computing visibility graphs, *SIAM J. Comput.* 20 (1991) 888-910.

[7] J. Hershberger, An optimal visibility graph algorithm for triangulated simple polygons, *Algorithmica* 4 (1989) 141-155.

[8] A. Melkman, On-line construction of the convex hull of a simple polyline, *Inform. Process. Lett.* 25 (1987) 11-12.

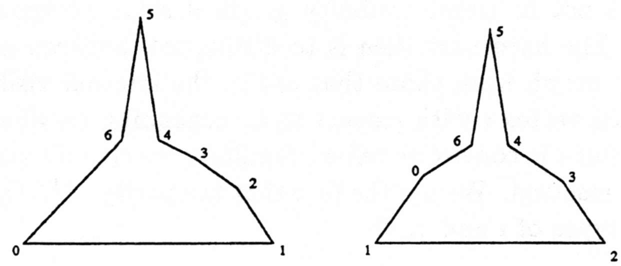

Figure 1: Two polygons with the same reflex vertices, internal and external visibility graphs but different convex hulls [2].

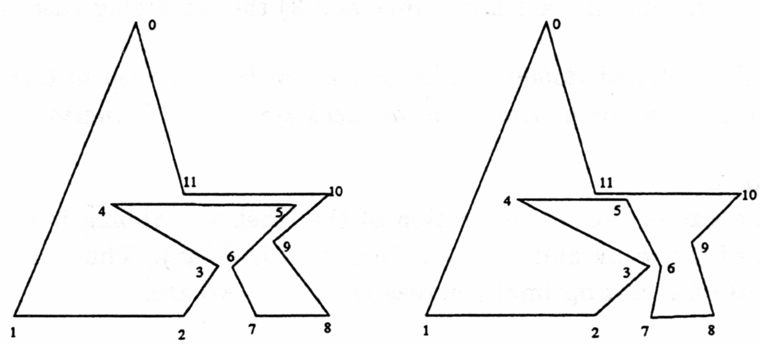

Figure 2: Two polygons with the same internal and external visibility graphs and convex hulls but different reflex vertices

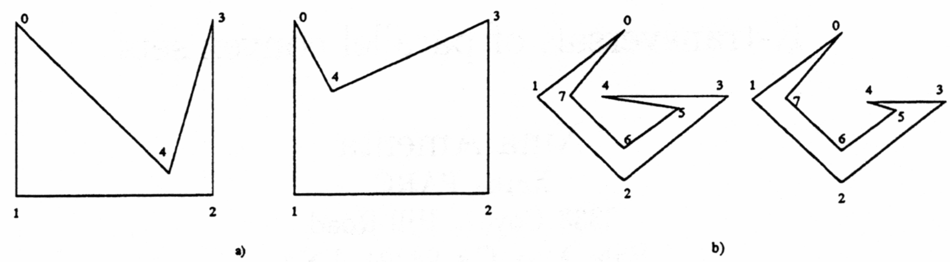

Figure 3: a) The convex hull, external visibility graph and reflex vertices do not determine the internal visibility graph. b) The convex hull, internal visibility graph, and reflex vertices do not determine the external visibility graph.

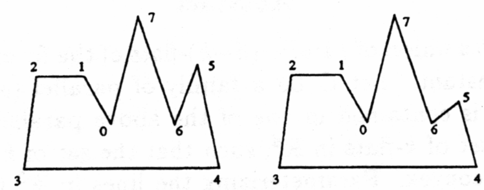

Figure 4: Two polygons with the same internal visibility graphs, external visibility graphs, convex hulls and reflex vertices but different order types and different weak stabbing information; consider $l_{1,2}$.

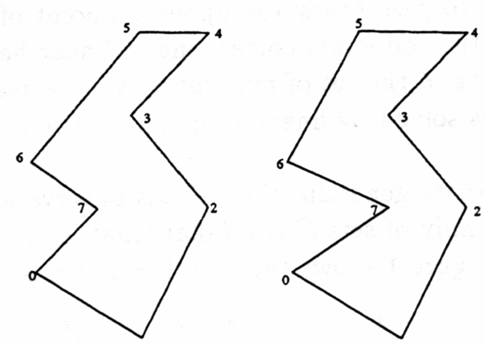

Figure 5: Weak stabbing information does not determine strong stabbing information.

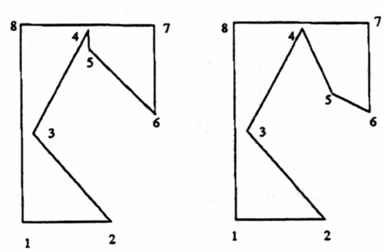

Figure 6: Strong stabbing information does not determine labelled stabbing information.