Efficient Algorithms for the Smallest Enclosing Cylinder Problem

(Extended Abstract)

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Abstract

This paper addresses the complexity of computing the smallest-radius infinite cylinder that encloses an input set of n points in 3-space. We show that the problem can be solved in time $O(n^4 \log^{O(1)} n)$ in an algebraic complexity model. We also achieve a time of $O(n^4 L \cdot \mu(L))$ in a bit complexity model.

These and several other results highlight a general *linearization technique* which transforms non-linear problems into some higher dimensional but linear problems. The technique is reminiscent of the use of Plücker coordinates, and is used here in conjunction with Megiddo's parametric searching.

1 Introduction

1.1 Motivation and Problem Statement

A major topic of geometric optimization is to approximate point sets by simple geometric figures. This includes extensively studied planar problems such as smallest enclosing circles, the minimum width annulus, and the minimum width slab. In higher dimensions, there are few non-trivial complexity results for geometric figures beyond hyperplanes or spheres. In this paper, we consider the following:

> **Smallest Cylinder Problem (P1):** Let I be a given set of n points in 3-space. Find a line ℓ which minimizes $\max\{\ d(\ell, c)\ :\ c \in I\ \}$.

Here, $d(\ell, c)$ denotes the minimum Euclidean distance between c and a point of ℓ.

Since cylinders constitute an important primitive shape in computer-aided design and manufacturing, this problem has many applications. We give one example from an area of importance to modern high precision engineering, *dimensional tolerancing and metrology* (see [SV, Ya]). Here the task is, given a physical object, to verify its conformance to tolerance specifications by taking probes of its surface. In industry, highly specialized, expensive equipment (called Coordinate Measurement Machines) is used to perform these probes automatically. Hence, high numerical accuracy is important, and any exact solution is preferable to the frequently used heuristic approaches.

To illustrate the intrinsic complexity of (P1) and the optimization technique used, we shall also consider the following subproblem:

> **Smallest Anchored Cylinder Problem (P2):** Let I be a given set of n points in 3-space. Find a line ℓ through the origin which minimizes $\max\{\ d(\ell, c)\ :\ c \in I\ \}$.

1.2 Outline of Results

We summarize two areas of contribution of this paper. (See [SSTY] for the full version.)

(I) We design efficient algorithms for the smallest cylinder problem in both an algebraic and a bit model of computing.

Most geometric algorithms are developed within one of two distinct computational frameworks. In the *algebraic framework*, the complexity of an algorithm is measured by the number of algebraic operations on real-valued variables, assuming exact computations. The input size corresponds to the number n of input values. In the *bit framework*, the complexity is measured by the number of bitwise boolean operations on binary strings. The input generally consists of integers, and the parameter n is supplemented by an additional parameter L that bounds the maximal bit-size of any input value (in our application, the coordinates of the points in I).

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While the size of the input is measured differently in the algebraic and in the bit model, the output can often be treated in a uniform way by asking for a *combinatorial solution* to the problem. In the case of (P1), we may assume the required output to be a list of those input points that determine the optimal cylinder(s).

Theorem 1
Problem (P1) can be solved in time
(i) $O(n^4 \log^{O(1)} n)$ *in an algebraic model; and*
(ii) $O(L\mu(L)n^4)$ *in a bit model.*

Here, $\mu(L) = O(L \log L \log\log L)$ denotes the complexity of multiplying two L-bit integers.

Result (i) is obtained by using Megiddo's parametric search [Me]. Result (ii) is based on "exact approximation" and a bit complexity analysis that uses multivariate root bounds.

While parametric search provides a clean dependency of running time on the number n of input points, the bit complexity approach is more suitable if accuracy is the main goal to achieve. This gets increasingly important as the algebraic source of complexity comes into play.

(II) We highlight a *linearization technique* for geometric optimization problems.

The heart of both approximation and parametric search algorithms is a *decision scheme* for a *fixed optimization parameter*. To obtain efficient decision algorithms that lend themselves to parametric search, it is often possible to exploit geometric duality transformations. Examples in the recent literature include inversion (as in [FSS]) and Plücker coordinates (as in [ST]).

In this paper, we extend these principles to a more general framework known as linearization.

We give this an abstract formulation. Let $P(\mathbf{x}, \mathbf{y})$ be a polynomial in the real variables $\mathbf{x} = (x_1, \ldots, x_\ell)$ and $\mathbf{y} = (y_1, \ldots, y_m)$.

> **Abstract Decision Problem (D):** Given a set $I \subseteq \mathbf{R}^m$ of n points, decide if there exists a point $c \in \mathbf{R}^\ell$ such that for all $p \in I$, $P(c, p) \leq 0$.

We say $P(\mathbf{x}, \mathbf{y})$ has an *order k linearization* if there exists $2k+1$ polynomials, $X_i = X_i(\mathbf{x})$ $(i = 1, \ldots, k)$ and $Y_i = Y_i(\mathbf{y})$ (for $i = 0, \ldots, k$), such that

$$P(\mathbf{x}, \mathbf{y}) = Y_0 + \sum_{i=1}^{k} X_i Y_i.$$

Theorem 2
(i) If $P(\mathbf{x}, \mathbf{y})$ *has an order k linearization, the decision problem (D) can be solved in* $O(n^{\lfloor k/2 \rfloor})$ *in the algebraic model.*
(ii) In the bit model, if each input coordinate has L bits, the problem (D) can be solved in $O(\mu(L) n^{\lfloor k/2 \rfloor})$.

1.3 Related Work

Problem (P1) belongs to a class of problems that have been considered from a complexity–theoretic viewpoint in [KG]. Although problem (P1) is routinely solved in engineering applications using numerical optimization techniques, few complexity theoretic results have been published. Concrete geometrical properties have first been investigated in [Pa], with focus on the *decision problem* to determine if there exists a cylinder with radius $r = 1$ (a *unit cylinder*) which encloses the input points.

Proposition 1 ([Pa])
(a) If there exists a unit cylinder that encloses all input points, then there also exists a unit enclosing cylinder which touches 4 of the input points, or whose axis is parallel to an edge of the convex hull of I.
(b) There is only a finite number of unit cylinders that touch 4 non-collinear points in 3-space.

With these (geometrically non-trivial) results, the decision problem for fixed radius can be solved by enumerating all cylinders through choices of 4 points, and by checking if one of these encloses the input points. This algorithm has complexity $O(n^5)$. With this, it is not hard to see that the optimization problem can be solved in time $O(n^5 \log n)$ by a straightforward application of parametric search. (The same time bound can also be achieved by computing all locally smallest enclosing cylinders for up to 5 input points, sorting them by radius, and using binary search to find the smallest enclosing among them.) We shall improve this bound by using linearization.

The linearization technique appears to have been used first by Yao [YY]. More recently, Agarwal and Matoušek [AM] use linearization in the context of range searching with semialgebraic sets. They also give a simple procedure for finding the optimal linearization for a given polynomial.

In this paper, linearization is used directly and is combined with the new parallel convex hull algorithm in [AGR] to allow the application of parametric search.

1.4 Notes on Complexity

Problem (P2) can serve to illustrate the complexity of the smallest cylinder problem: already this problem can be shown to be neither convex nor LP-type. In the following paragraph, we give a quadratic lower bound on the number of possible local minima of (P2) (the bound can easily be extended to (P1)):

Consider an even number n of points that are arranged on the unit sphere S^2, $n/2$ on the circle C_1 (C_2) of intersection with the plane $z = 0$ ($y = 0$). We assume that the points on each circle are uniformly stepped and diametrically opposed. Further, let each line through the origin be parameterized by its intersection with the sphere S^2. Now let us ask for the set of cylinders with distance $\geq 1 - \varepsilon$ to one input point c. This set corresponds to a thin stripe on S^2, and describes the forbidden cylinders with respect to c. The set of enclosing cylinders with radius $\leq 1 - \varepsilon$ is the complement of the union of the stripes for all $c \in I$. For ε sufficiently small, this set has quadratic complexity.

Our optimization technique yields a running time of $O(n^2)$ for the decision problem for (P2), assuming the algebraic model. Due to the possible number of minima, this result may be optimal. It is an open question whether (P2) belongs to the class of n^2-hard problems introduced in [GO]. Finally, it is noteworthy that a variant of (P2) where we ask for an enclosing silo instead of a cylinder can be solved in time $O(n \log^3 n \log\log n)$ [Fo].

2 Preliminaries

2.1 Algebraic Formulation

A cylinder C in 3-space is specified by 5 real parameters: its axis line ℓ and its radius r. We follow the approach suggested by Proposition 1, and first specify the set $C(c_1, \ldots, c_4)$ of cylinders that touch 4 given points $c_1, \ldots, c_4 \in I$.

By translation of the coordinate system, we can assume $c_1 = (0,0,0)$. Let $u \in \mathbf{R}^3$ be any direction vector of ℓ. Let E be the plane passing through the origin and orthogonal to u, and let $c_1^*, \ldots, c_4^*$ be the orthogonal projection of the input points $c_1, \ldots, c_4$ onto E. Then the cylinder C passes through $c_1, \ldots, c_4$ if and only if $c_1^*, \ldots, c_4^*$ are cocircular.

The first problem that we face in the algebraic computation of solutions is to find a suitable parametrization for the direction vector u. We shall treat the case when u is not parallel to the plane containing c_2, c_3, c_4. (Otherwise, we have a simpler subproblem.) Let

$$u = xc_2 + yc_3 + zc_4,$$

with $z = 1 - x - y$. The parameters x, y, z are also called the *barycentric coordinates* of u with respect to c_2, c_3, c_4.

Now, let $R_1(x,y,z)$ be the squared radius of the circumcircle of c_1^*, c_2^*, c_3^* in E, and $R_2(x,y,z)$ the squared radius of the circumcircle of c_1^*, c_3^*, c_4^*. Then the set $C(c_1, \ldots, c_4)$ can be interpreted as a 2-dimensional surface in 3-space, defined by $R_1(x,y,z) = R_2(x,y,z)$. This condition is equivalent to $P(x,y,z) = 0$, with

$$\begin{aligned} P(x,y,z) = {} & \Delta_{1,2,4}(xz^2 + x^2z) \\ & + \Delta_{1,3,4}(yz^2 + y^2z) \\ & + \Delta_{1,2,3}(xy^2 + x^2y) \\ & + (\Delta_{1,2,4} + \Delta_{1,3,4} + \Delta_{1,2,3} - \Delta_{2,3,4})(xyz), \end{aligned}$$

where $\Delta_{i,j,k} = c_i(c_j \times c_k)$.

With $z = 1 - x - y$, P can also be interpreted as a polynomial in the 2 variables x and y, or as 1-dimensional curve in the x-y-plane. We note that the total degree of P is 3, and the degree in each variable is 2.

In order to compute the cylinders with fixed radius r in the set $C(c_1, \ldots, c_4)$, the additional condition $R_1(x,y,z) = r$ has to be satisfied. Unfortunately, this leads to a significantly more complicated polynomial equation $Q(x,y) = 0$, with total degree 6.

The set $C_f(c_1, \ldots, c_4, r)$ of all cylinders with radius r that pass through $c_1, \ldots, c_4$ is given by the set of solutions of the system $\{\ Q(x,y) = 0, P(x,y) = 0\ \}$, and can be obtained algebraically by computing the roots of the resultants $F_x = Res(P,Q,y)$ and $F_y = Res(P,Q,x)$. These resultants have degree 12.

Lemma 1 *If $c_1, \ldots, c_4$ are not collinear, the set $C_f(c_1, \ldots, c_4, r)$ contains at most 12 cylinders. Assuming c_i are rational points, each cylinder is specified uniquely by algebraic numbers of degree at most 12.*

2.2 Bit Complexity

Lemma 1 implicates a simple decision algorithm for the fixed-radius problem (P1), and hence an approximation algorithm for the optimization problem. By exploiting ideas from the theory of exact computation, such approximation algorithms can be made "exact" in the sense of determining the combinatorial solution (see 1.2). What is needed is a gap theorem that relates qualitative changes of the solution to quantitative changes in the optimization parameter (here, r).

In the following, it is useful to consider the optimization function (the radius r of a smallest enclosing cylinder) as a function of the axis direction, and thus as a surface in $\mathbf{R}^3$. (Note that for any fixed axis direction there exists a unique smallest enclosing cylinder !) This surface is given by 2-dimensional surface patches (corresponding to cylinders that touch 3 points), 1-dimensional ridges (corresponding to cylinders that touch 4 points), and vertices (defined by tuples of 5 points).

Now assume that r_i°, $i = 1, 2$, denotes a local minimum of a surface patch, a local minimum of a ridge, or the "height" of a vertex. Further, let δ be a *separation gap* between any two values r_1° and r_2° that are not equal, i.e., $|r_1^\circ - r_2^\circ| \geq \delta$ for all $r_1^\circ \neq r_2^\circ$. Then the combinatorial solution of (P1) can easily be derived from a δ–approximate solution of (P1).

The computation of the gap δ is non-trivial, requiring an algebraic characterization of the local minima above, and the application of multivariate root bounds. In the sequel, we shall focus on the computation of δ for the most complicated case, when r_1° and r_2° are the local minima of ridges.

Let $c_1, \ldots, c_4$ be an arbitrary choice of input points. Our goal is to compute a discrete set of values which contains r_1°, a local minimum value with respect to $c_1, \ldots, c_4$. Following subsection 2.1, let $R_1(x, y)$ be the squared radius of the circumcircle of c_1^*, c_2^*, c_3^*, and $P_1(x, y)$ the polynomial which defines the cylinder with direction parameters (x, y) passing through $c_1, \ldots, c_4$. Then the candidates for r_1° are the local minimum values of $\sqrt{R_1(x, y)}$ under the side condition $P_1(x, y) = 0$. By the rule of Lagrange, there exists a parameter λ such that the following 2 conditions hold at the minima:

$$(1) \quad \frac{\partial R_1}{\partial x} + \lambda \frac{\partial P_1}{\partial x} = 0, \quad (2) \quad \frac{\partial R_1}{\partial y} + \lambda \frac{\partial P_1}{\partial y} = 0.$$

Eliminating λ in these equations, let $Q_1(x, y)$ be the numerator of the expression

$$\frac{\partial R_1}{\partial x} - \frac{\partial R_1}{\partial y} \frac{\partial P_1}{\partial x} \left(\frac{\partial P_1}{\partial y} \right)^{-1}.$$

Then $r_1^\circ = \sqrt{R_1(x_1^\circ, y_1^\circ)}$, where (x_1°, y_1°) is a solution of the system $\{ P_1(x, y) = 0, Q_1(x, y) = 0 \}$.

Analogously, let r_2° be a minimum candidate for a different choice of input points, and P_2, Q_2, R_2 the corresponding defining formulas. Then the needed separation gap can be obtained as a lower bound for $|\delta|$ in the system of equations

$$\begin{aligned}
&(1) \quad P_1(x_1, y_1) = 0, \\
&(2) \quad Q_1(x_1, y_1) = 0, \\
&(3) \quad P_2(x_2, y_2) = 0, \\
&(4) \quad Q_2(x_2, y_2) = 0, \\
&(5) \quad \sqrt{R_1(x_1, y_1)} - \sqrt{R_2(x_2, y_2)} = \delta.
\end{aligned}$$

By repeated squaring, formula (5) can be transformed into a polynomial equation $R(x_1, y_1, x_2, y_2, \delta) = 0$ such that the set of solutions is only increased by a finite number of new candidates. Now, a bound for δ can be obtained from the gap-theorem of Canny [Ca].

Proposition 2 ([Ca]) *Let $f_1, \ldots, f_n$ be n polynomials in n variables, with degree $\leq d$ and coefficient magnitude $\leq c$. Assume that the system $\{f_1 = 0, \ldots, f_n = 0\}$ has only a finite number of solutions when homogenized. If $(\alpha_1, \ldots, \alpha_n)$ is a solution with $\alpha_i \neq 0$, then $|\alpha_i| > (3dc)^{-nd^n}$.*

With $c = 2^L$, $d = const$ and $n = 5$, we get $|\delta| = 2^{-O(L)}$. This gives us:

Lemma 2 *Let C be a smallest enclosing cylinder for input set I, with radius r^*. Then any cylinder $C' \neq C$ that touches a different set of points than C has radius $r = r^*$ or $r \geq r^* + 2^{-cL}$, for a suitable constant c.*

Remark 1 *The use of the general gap-theorem (proposition 2) gives constants that are far beyond from being practical. It would be desirable to derive sharper bounds for special cases of this theorem.*

3 Optimization Algorithms

3.1 Linearization

In order to illustrate the basic idea of the linearization technique, we first consider the anchored problem (P2). Our focus is the fixed-parameter problem to decide whether there exists an anchored cylinder of given radius r that encloses all input points.

Let ℓ_{ab} be the line through the points $a, b \in \mathbf{R}^3$. We fix a at the origin and w.l.o.g. require b to lie on the plane $z = 1$:

$$a = (0, 0, 0), \quad b = (b_x, b_y, 1).$$

Further, let $c = (c_x, c_y, c_z)$ be an arbitrary input point. We call ℓ_{ab} *admissible* with respect to c if

$$d(\ell_{ab}, c)^2 \leq r^2, \tag{1}$$

with

$$\begin{aligned}
d(\ell_{ab}, c)^2 = ((c_y^2 + c_z^2)b_x^2 + (c_x^2 + c_z^2)b_y^2 \\
-2c_xc_yb_xb_y - 2c_xc_zb_x - 2c_yc_zb_y \\
+(c_x^2 + c_y^2)) \ / \ (b_x^2 + b_y^2 + 1).
\end{aligned}$$

We *embed* our problem into a higher-dimensional space by setting

$$X_1 = b_x, X_2 = b_y, X_3 = b_x^2, X_4 = b_y^2, X_5 = b_xb_y. \tag{2}$$

Now, equation (1) is true if and only if

$$P_c(X_1, \ldots, X_5) \leq 0, \quad (3)$$

where P_c is the linear equation

$$\begin{aligned} &P_c(X_1, \ldots, X_5) \\ &= (-2c_x c_z)X_1 + (-2c_y c_z)X_2 \\ &\quad +(c_y^2 + c_z^2 - r^2)X_3 + (c_x^2 + c_z^2 - r^2)X_4 \\ &\quad +(-2c_x c_y)X_5 + (c_x^2 + c_y^2 - r^2). \end{aligned}$$

According to this equation, P_c defines a hyperplane in $\mathbf{R}^5$, and inequality (3) a halfspace H_c. The set of equations (2) defines a 2-dimensional manifold which can be written as

$$M = \{ (X_1, \ldots, X_5) : Q(X_1, \ldots, X_5) = 0 \}$$

with

$$\begin{aligned} &Q(X_1, \ldots, X_5) \\ &= (X_1^2 - X_3)^2 + (X_2^2 - X_4)^2 + (X_5 - X_1 X_2)^2. \end{aligned}$$

For the set I of input points, the fixed-parameter problem has a solution if and only if there exists a line ℓ_{ab} which is admissible with respect to each $c \in I$. This is equivalent to the existence of a common intersection of the halfspaces H_c and the manifold M. The intersection

$$H = \bigcap_{c \in I} H_c$$

is a convex polytope of complexity $O(n^2)$, and can be constructed in the same time bound by Chazelle's result [Ch]. In order to intersect H with M, we triangulate H into $O(n^2)$ simplices. Each of these simplices can be tested for intersection with M separately in constant time if we assume an algebraic model of computing, and in time $O(\mu(L))$ if we assume a bit model [Re]. (Note here that M is a semi-algebraic set and the above test corresponds to deciding the satisfiability for a system of polynomial equations and inequalities.)

Concluding, we get a decision algorithm that runs in time $O(n^2)$ (respectively $O(\mu(L)n^2)$) in an algebraic (respectively bit) model. This argument generalizes in a straightforward way to proving the general theorem 2.

To apply the linearization technique to the problem (P1), we consider – w.l.o.g. – axis lines that are not parallel to the plane $z = 0$. Let ℓ_{ab} be the line through the points $a, b \in \mathbf{R}^3$, with

$$a = (a_x, a_y, 0), \quad b = (a_x + b_x, a_y + b_y, 1).$$

Then ℓ_{ab} is admissible with respect to $c = (c_x, c_y, c_z)$ and given radius r iff

$$P_c(a_x, a_y, b_x, b_y) \leq 0,$$

with

$$\begin{aligned} &P_c(a_x, a_y, b_x, b_y) \\ &= c_x^2(b_y^2 + 1) + c_y^2(b_x^2 + 1) + c_z^2(b_x^2 + b_y^2) \\ &\quad +c_x c_y(-2b_x b_y) + c_x c_z(-2b_x) + c_y c_z(-2b_y) \\ &\quad +c_z(2b_y a_y + 2b_x a_x) \\ &\quad +c_x(-2a_x - 2a_x b_y^2 + 2b_x b_y a_y) \\ &\quad +c_y(-2a_y - 2a_y b_x^2 + 2b_x a_x b_y) \\ &\quad +(a_x^2 b_y^2 + a_y^2 b_x^2 + a_x^2 + a_y^2 \\ &\quad\quad -r^2(b_x^2 + b_y^2) - 2b_x a_x b_y a_y) \\ &\quad -r^2. \end{aligned}$$

At first glance, P_c has an order 10 linearization. However, we can save one variable by grouping the terms with factors c_x^2, c_y^2 and c_z^2 differently:

$$\begin{aligned} &c_x^2(b_y^2 + 1) + c_y^2(b_x^2 + 1) + c_z^2(b_x^2 + b_y^2) \\ &= (c_y^2 + c_z^2)b_x^2 + (c_x^2 + c_z^2)b_y^2 + (c_x^2 + c_y^2). \end{aligned}$$

Now, the linearization is given by

$$\begin{aligned} &X_1 = b_x, X_2 = b_y, X_3 = b_x^2, X_4 = b_y^2, X_5 = b_x b_y, \\ &X_6 = b_y a_y + b_x a_x, \\ &X_7 = -a_x - a_x b_y^2 + b_x b_y a_y, \\ &X_8 = -a_y - a_y b_x^2 + b_x a_x b_y, \\ &X_9 = a_x^2(b_y^2 + 1) + a_y^2(b_x^2 + 1) - r^2(b_x^2 + b_y^2) \\ &\quad -2b_x a_x b_y a_y. \end{aligned}$$

Applying theorem 2, we conclude that the fixed-parameter problem for (P1) can be decided in time $O(n^4)$ in an algebraic model, and in time $O(\mu(L)n^4)$ in a bit model.

Remark 2 *If P_c has an order 8 linearization, this fact would not improve the asymptotic complexity of the problem. But it means we could use some of the $O(n^{\lceil k/2 \rceil})$ convex hull algorithms to achieve the same complexity bounds.*

3.2 Parametric Search vs. Exact Approximation

In this subsection we shall apply parametric search and "exact approximation" to problem (P1), based on the decision algorithm from the previous subsection.

Note that the presented techniques apply as well to the restricted setting (P2).

We shall use the parametric search paradigm in its general form (see eg. [AST] for a detailed description). Let T_s denote the running time of a sequential decision algorithm for the fixed-parameter problem, and T_p (resp., P) the time (resp., number of processors) of a parallel decision algorithm, then the optimal value (here, r^*) can be computed in sequential time $O(PT_p + T_sT_p \log P)$. It remains to give a parallel version of the decision algorithm. Here we exploit the new parallel algorithm for convex hulls in [AGR]: For dimension $d \geq 4$, there is an algorithm with time $O(\log n)$ and work $O(n^{\lfloor d/2 \rfloor} \log^{c(\lceil d/2 \rceil - \lfloor d/2 \rfloor)} n)$, for some constant $c > 0$. Further, with $O(n^{\lfloor d/2 \rfloor})$ processors, the test for intersection of H with M can be done in constant time in an algebraic model (resp., a real RAM, see [Re]). Plugging this into the parametric search paradigm, and observing that – in an algebraic model – the combinatorial solution of (P1) can easily be constructed from the computed optimum value r^*, we obtain:

Lemma 3 *A combinatorial solution of (P1) can be computed via parametric search in time $O(n^4 \log^k n)$, for a fixed constant $k > 0$.*

Turning our attention to the bit model, as shown in subsection 2.2, the combinatorial solution of (P1) can be obtained from an ε-approximate solution for r^* if $\varepsilon = 2^{-O(L)}$. To compute this approximate solution, it suffices to run the decision algorithm for the fixed-parameter problem $O(L)$ times, with radii of bit-size $O(L)$ as input. This yields:

Lemma 4 *A combinatorial solution of (P1) can be computed in the bit model in time $O(L\mu(L)n^4)$.*

4 Final Remarks

As the field of geometric optimization matures, it treats problems of increasingly non-trivial algebraic complexity. The traditional neglect of bit complexity is no longer justified. The smallest cylinder problem is one of these problems. By combining the general linearization technique with parametric search and multivariate root bounds, we developed efficient algorithms in both an algebraic and a bit model.

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