The Complexity of Illuminating Polygons by α-Flood-Lights

Jay Bagga [1], Laxmi Gewali [2], and David Glasser [2]

Abstract: The Flood-light Illumination Problem (FIP) is the determination of the minimum number of vertex α-lights to illuminate a given polygon. For simple polygons, we show that this problem is NP-hard for any fixed α in the range $0 < \alpha \leq 360^0$. Furthermore, the problem remains NP-hard even when α-lights are required to be flush with the edges of the polygon. Our technique is based on the construction of beam-machines.

1. Introduction

In most illumination problems, visibility from a point is allowed in all directions (360^0 angular aperture). Recently, several researchers have considered illumination problems that restrict visibility to within a certain angular aperture [3,4,5]. A light source whose illumination angle is restricted to α-degrees is called an **α-flood-light** (or simply **α-light**). A simple polygon may remain unilluminated even if we place a 90^0-flood-light at each vertex; and this holds true even if the polygon is restricted to be monotone [4]. Estivill-Castro et. al. [5] presented a surprise on polygon illumination: there are simple polygons (called **logarithmic spirals**) that can not be illuminated by placing α-lights on all of its vertices for any fixed $\alpha < 180^0$. An open problem posed in [3] is the determination of the minimum number of α-lights required to illuminate a simple polygon. This problem is known to be NP-hard when visibility is allowed in all directions (i.e., $\alpha = 360^0$) [1,2]; the problem is also known to be NP-hard when $\alpha = 90^0$ [6].

In this paper, an approach based on beam-machines is used to show that the problem of illuminating a simple polygon by the minimum number of α-lights is NP-hard for any fixed α in range $0 < \alpha \leq 360^0$. We further show that the problem remains NP-hard (in fact, it is NP-Complete) even if α-lights are required to be flush with the edges of the polygon.

2. Preliminaries

The notation $< v_1, v_2, ..., v_n >$ is used to denote the **polygonal chain** connecting vertices in the order $v_1, v_2, v_3, ..., v_n$. A simple polygon is specified by the polygonal chain describing its boundary. In a polygonal chain, all vertices are distinct and hence the first vertex and the last vertex in the chain are not connected. However, when a polygonal chain $< v_1, v_2, ..., v_n >$ is used for representing a simple polygon then it is considered to be closed, i.e., the first vertex and the last vertex are understood to be connected. We require that α-lights can be placed only on the vertices and that at most one α-light can be placed on a vertex.

The Flood-light Illumination Problem (FIP)
Instance: A simple polygon P of n sides, a positive integer m, and angular aperture α.
Question: Can P be illuminated by at most m α-lights?

The Satisfiability Problem (SAT)
Instance: A collection $W = \{C_1, C_2, C_3, ..., C_k\}$ of clauses of a finite set of boolean variables $X = \{x_1, x_2, x_3, ..., x_r\}$; variable x_i occurs k_{i_1} times as x_i and k_{i_2} times as $\overline{x_i}$.
Question: Is there a truth assignment for variables in X that satisfies all the clauses?

Given an instance I_1 of SAT, we convert it in polynomial time to an instance $I_2 = (P, m)$ of FIP such that I_1 is satisfiable if and only if the interior of P can be illuminated by m or fewer α-lights, where m can be expressed in terms of the number of clauses k and the number of literals r in I_1. Our reduction is similar in spirit to the construction developed by Culberson and Reckhow [7], where an approach based on beam-machines is used to reduce the satisfiability problem to the problem of covering a simple polygon by the minimum number of convex polygons.

---

[1] Department of Computer Science, Ball State University, Muncie, Indiana. email: jay@bsu-cs.bsu.edu

[2] Department of Computer Science, University of Nevada, Las Vegas, Nevada. e-mail: laxmi@cs.unlv.edu. For this research, Laxmi P Gewali was supported in part by a grant from US department of energy (NVOO).

3. Illumination by 45^0-lights

For the purpose of clarity of exposition we first examine the complexity of FIP when the angular aperture of flood-lights is restricted to 45^0. One of the polygonal structures used in the reduction is the beam-machine.

Beam-Machine: A beam-machine is a polygonal chain used for generating a thin illumination beam by the placement of the minimum number of 45^0-lights in its interior. Corresponding to each literal in the given SAT expression we will construct a beam-machine. Specifically, it is a nineteen sided polygonal chain $< a, b, c, ..., j, j', i', ...a' >$ (Figure 1a) which can be viewed as a simple polygon with a tiny opening $< a, a' >$ on its boundary. The interior of a beam-machine is considered to be the interior of the polygon formed by closing its opening. The interior of a beam-machine consist of four parts: (i) the **left-wing** $< d, e, f, g, h, i, j >$, (ii) the **right-wing** $< d', e', f', g', h', i', j' >$, (iii) the **body** (quadrilateral with vertices d, j, j', and d'), and (iv) the **mouth** $< d, c, b, a, a', b', c', d' >$. A beam-machine satisfies the visibility properties listed in Table 1.

Table 1

Pr.#	*Property*
1	Interior angles at e and e' are at most 45^0 each.
2	Vertex e (respectively, e') is the only vertex from which the entire left-wing (respectively, right-wing) can be illuminated by a 45^0-light.
3	(a) Edges $\overline{(a,b)}$, $\overline{(c,d)}$, and $\overline{(c',b')}$ are parallel to each other and their extensions meet $\overline{(i,j)}$ near its middle. (b) Analogous properties hold for the edges $\overline{(a',b')}$, $\overline{(c',d')}$, and $\overline{(c,b)}$.
4	The mouth is visible only from vertices j or j'.
5	The angles subtended by segment $\overline{(d,d')}$ at vertices j and j' are less than 45^0 each.

Due to space limitation proofs of lemmas/theorems and other details are omitted.

Lemma 1: Three 45^0-lights are necessary and sufficient to illuminate a beam-machine.

A beam-machine can be illuminated by three 45^0-lights by placing them either at vertices (1) e, e', and j, or at vertices (2) e, e', and j'; we refer to these placements as **placement-j** and **placement-j'**, respectively.

Observation 1 (*Beam Formation*): Consider the illumination of a beam-machine by placement-j. In addition to the illumination of the interior of the beam-machine, it produces a thin beam of light in the exterior extending from the opening of the mouth and inclined to the right side (Figure 1a). Similarly, **placement-j'** produces a thin beam extending to the left.

Beam Adjustment: It is desirable to be able to adjust the width and orientation of a beam without changing the visibility properties of the corresponding beam-machine. The width of the beam can be adjusted by narrowing or widening the opening in the mouth of the beam-machine, which can be done by adjusting the lengths of the edges $\overline{(a,b)}$ and $\overline{(a',b')}$. It is easily seen that the visibility properties listed in Table 1 do not change by adjusting the lengths of these edges. To rotate the beam to the right, we slightly shift vertex j to the left along the edge $\overline{(j',j)}$ and adjust the chain $< f, g, h, i, j >$ so that the left-wing is still visible from vertex e. The adjustment is shown by dashed chain in Figure 1b. Since the angle subtended by the edge $\overline{(d,d')}$ at vertex j is slightly less than 45^0 (property 5 in Table 1), the mouth of the beam-machine remains illuminated from the 45^0-light appropriately placed at the new position of j. To rotate the beam to the left we slightly shift j to the right along the edge $\overline{(i,j)}$.

The Background of Variable Generators (BVG): BVG is a twenty two sided polygonal chain $< v_1, v_2, ..., v_{23} >$ (Figure 2a) whose interior can be partitioned into five polygonal regions: (i) **left-wing** $< v_2, v_3, ..., v_7 >$, (ii) **right-wing** $< v_{22}, v_{21}, ..., v_{17} >$, (iii) **left-arm** $< v_7, v_8, ..., v_{12} >$, (iv) **right-arm** $< v_{12}, v_{13}, ...v_{17} >$, and (v) **body** $< v_1, v_2, v_7, v_{12}, v_{17}, v_{22}, v_{23} >$. BVG satisfies the visibility properties enumerated in Table 2. VBG can be systematically constructed by first constructing its **skeleton**. The skeleton is then modified to obtain BVG (Figure 2b). (details omitted)

Table 2

Pr.#	*Property*
1	Interior angles at v_4, v_{11}, v_{13}, and v_{20} are at most 45^0 each.
2	Vertex v_{11} (respectively v_{13}) is the only vertex from which the entire left-arm (respectively, right-arm) can be illuminated by a 45^0-light.
3	(a) Both wings can be illuminated by one 45^0-light placed at v_4 or v_{20}. (b) No other placement of a single 45^0-light can illuminate both wings.
4	(a) Vertices v_4, v_6, v_7 are collinear and the line through them meets the edge $\overline{(v_{16}, v_{17})}$ in its interior. (b) Similar properties hold for vertices v_{17}, v_{18}, and v_{20} and edge $\overline{(v_7, v_8)}$
5	(a) v_{19} is not visible from v_5, v_6, and v_7. (b) v_5 is not visible from v_{17}, v_{18}, and v_{19}.
6	(a) v_{21} is not visible from v_2 and v_3. (b) v_3 is not visible from v_{21} and v_{22}.

Variable Generator: We construct variable generator V_i corresponding to each variable x_i in the SAT instance. A variable generator is obtained by attaching beam-machines and spikes on the arms of BVG. (It may be recalled that variable x_i occurs k_{i_1} times as x_i and k_{i_2} times as $\overline{x_i}$.) We attach k_{i_1} beam-machines $d_1, d_2, ..., d_{k_{i_1}}$ on the edge $\overline{(v_{11}, v_{12})}$ of the left-arm and k_{i_2} beam-machines $b_1, b_2, ..., b_{k_{i_2}}$ on the edge $\overline{(v_{12}, v_{13})}$ of the right-arm (Figure 3a). Corresponding to k_{i_1} beam-machines we attach k_{i_1} triangular spikes $q_1, q_2, ..., q_{k_{i_1}}$ on the edge $\overline{(v_7, v_8)}$. Similarly, we attach k_{i_2} triangular spikes $s_1, s_2, ..., s_{k_{i_2}}$ on the edge $\overline{(v_{17}, v_{18})}$.

Lemma 2: Three 45^0-lights are are necessary and sufficient to illuminate BVG.

Lemma 3: (a) $3(k_{i_1} + k_{i_2} + 1)$ 45^0-lights are necessary and sufficient to illuminate the interior of the variable generator V_i. (b) In addition to the illumination of the interior of V_i, $3(k_{i_1} + k_{i_2} + 1)$ 45^0-lights can generate k_{i_1} or k_{i_2} beams extending away from the mouth of V_i.

Formation of Shadow-region: When the background of a variable generator (say, V_i) is illuminated by the minimum number (three) of 45^0-lights, a shadow-region bounded by two illuminated region is formed in its exterior (Figure 3b). There is a critical role of such shadow-regions in the final construction of polygon P. The 45^0-lights placed at vertices v_{11} and v_{13} generate two illumination regions in the exterior of V_i, one inclined to the left and the other to the right. The region bounded by these two illumination regions is the shadow-region (shaded in the figure) induced by the background of V_i.

Polygon Construction: We start with a quadrilateral $< A, B, C, D >$ whose interior angles at B, C, and D are 90^0, 90^0, and 45^0, respectively. We attach two thin triangular spikes T_1 and T_2 on the side $\overline{(B, C)}$ such that D is the only vertex from which both of them are visible. For each variable x_i, we construct variable generator V_i containing $k_{i_1} + k_{i_2}$ beam-machines and attach them on the side $\overline{(A, B)}$. We also attach k triangular spikes on the side $\overline{(D, C)}$ corresponding to k clauses in the given SAT instance. Triangular spikes lying on the side $\overline{(D, C)}$ are the clause checkers which can be illuminated by beams comming from the variable generators. The size of the quadrilateral is made large enough (compared to the size of variable generators) so that all clause checkers lie in the intersection of shadow-regions induced by variable generators. The beam-machines are adjusted so that the beams escaping from the variable generators can be focussed on respective clause checkers to illuminate them. It is easily seen that this construction takes polynomial time.

Lemma 4: All clauses in W are satisfiable if and only if P can be illuminated by m or fewer 45^0-lights, where m can be expressed in term of the number of literals and the number of clauses.

Theorem 1: FIP is NP-hard when $\alpha = 45^0$.

4. Illumination by α-lights

In this section we show how to generalize the reduction of Section 3 when the angular aperture α is in the range $0 < \alpha \leq 360^0$. If α is greater than 45^0 then we can simply use the construction for 45^0-lights from the previous section. When α is smaller than 45^0 then we need modified structures. We prefix by α the terms/notations used in the previous section to indicate the context of illumination by α-lights. For example, beam-machines, skeleton, BVG, and variable generator V_i are referred to as α-beam-machines, α-skeleton, α-BVG, and α-V_i, respectively.

The structure of α-beam-machine is similar to the structure of 45^0-beam-machine except that (a) the angles at e and e' are α each and (b) the angle subtended by $\overline{(d, d')}$ at j and j' is slightly less than α. An α-beam-machine satisfying the properties in Table 1 and Lemma 1 (obtained by replacing 45^0 with α) can be constructed easily. To construct α-BVG, we start with the 45^0-skeleton and stretch its wings and arms by moving away vertices v_4, v_{11}, v_{13}, and v_{20} (Figure 5) so that the interior angles at v_{11} and v_{13} are reduced to α-δ each and that at v_4 and v_{20} are reduced to α-ϵ each, where ϵ and δ are small positive angles. It can be proved that α-BVG can be constructed from α-skeleton to satisfy the analogous properties listed in Table 2. Variable generator α-V_i and the final polygon P are constructed in the similar way as in the previous section. (details is omitted.)

Lemma 5: The shadow-region induced by α-lights placed at vertices v_{11} and v_{13} lies in the exterior of α-BVG.

Lemma 6: (a) $3(k_{i_1} + k_{i_2} + 1)$ α-lights are necessary and sufficient to illuminate the interior of the variable generator α-V_i. (b) In addition to the illumination of the interior of α-V_i, $3(k_{i_1} + k_{i_2} + 1)$ α-lights can generate k_{i_1} or k_{i_2} beams extending away from the mouth of α-V_i.

Theorem 2: FIP is NP-hard.

Remark 1: When an α-light L is placed on a polygon's vertex v with interior angle β ($\beta > \alpha$), the interior region of P illuminated by L depends on the orientation of L, and L can be oriented in infinitely many directions. Thus, in general, we can not check in polynomial time whether or not m α-lights can illuminate P. This implies that FIP does not belong to the class NP. Hence FIP is NP-hard and not NP-complete.

5. Discussion

We showed that the problem of illuminating a simple polygon with the minimum number of α-lights is NP-hard for any α in the range $0 < \alpha \leq 360^0$. The construction can be modified to show that the problem remains NP-hard even if α-lights are required to be flush with the edges of the polygon. It may be noted that our construction is such that, excepting the α-lights placed inside the α-beam-machines, all other α-lights are flush with the edges of the polygon. We have been able to construct a modified α-beam-machine where α-lights are forced to be flush with the edges. When the flushing condition is required we can check in polynomial time whether or not a given set of vertex α-lights can illuminate the polygon (there are at most two distinct orientations per vertex), implying that the problem is in the class NP.

Theorem 3: FIP is NP-complete even if α-lights are required to be flush with the edges of the polygon.

The FIP problem becomes the well known art gallery problem when the visibility is allowed in all directions, i.e., when $\alpha = 360^0$. Although the art gallery problem for simple polygons is known to be NP-hard [1,2], it is still open for simple orthogonal polygons. We believe that an approach based on beam-machine construction might be helpful to settle this issue. For simple orthogonal polygons, it may be possible to solve FIP in polynomial time when α-lights are required to be flush with the edges.

References

[1] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford University Press, 1987.

[2] T.C. Shermer, *Recent Results in Art Galleries*, Proceedings of the IEEE, Sept. (1992), pp. 1384-1399.

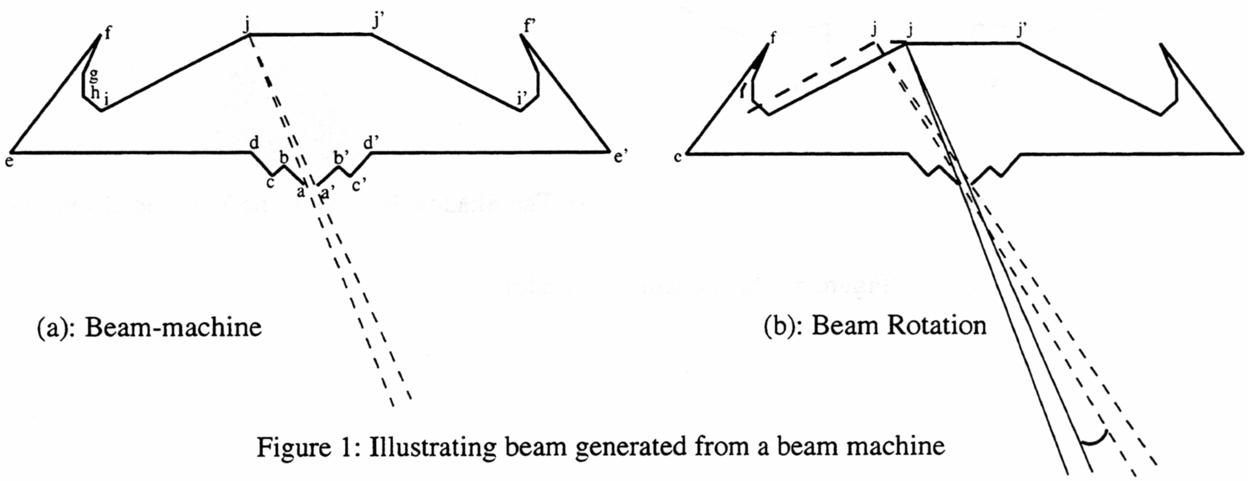
[3] V. Estivill-Castro, and J. Urrutia, *Optimal Floodlight Illumination of Art Galleries*, Proceedings of the Sixth Canadian Conference on Computational Geometry, (1994), pp. 81-86.

[4] J. O'Rourke and D. Xu, *Illumination of Polygons with* 90^0 *Vertex Lights,* Snapshots of Computational and Discrete Geometry, Edited by: D. Avis and P. Bose, McGill University, (1994), pp. 108-117.

[5] V. Estivill-Castro, J. O'Rourke, J. Urrutia, and D. Xu, Illumination of Polygons with Vertex lights, Information Processing Letters, 56 (1995) pp. 9-13.

[6] J. Bagga, L. Gewali, and D. Glasser, *Illuminating Simple Polygon by Flood-lights,* Proceedings of SPIE (Vision Geometry IV), (1995), pp. 108-115.

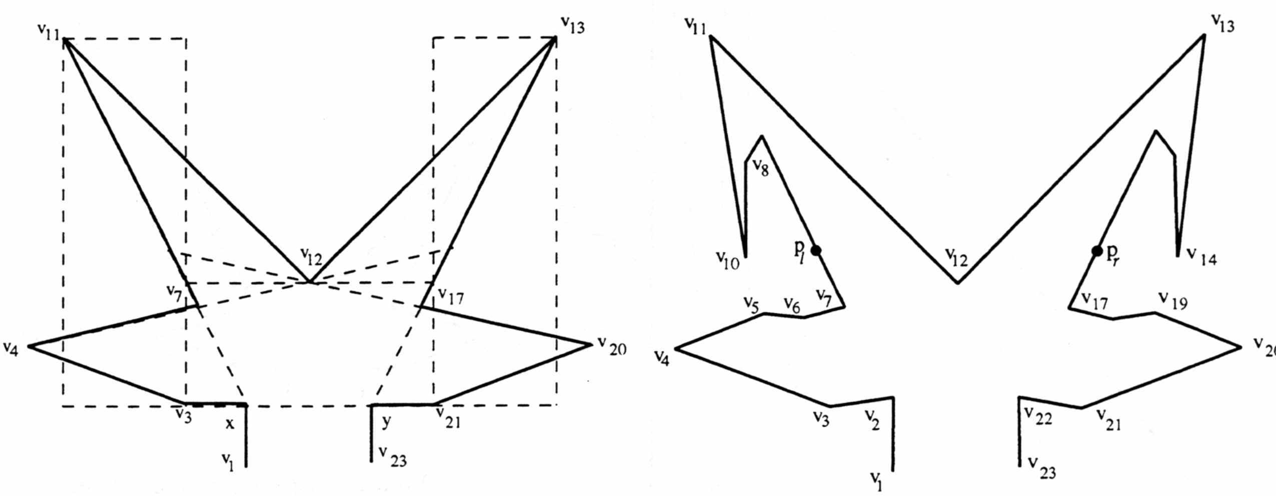
[7] J.C. Culberson and Robert A. Reckhow, *Covering Polygons is Hard,* Journal of Algorithms, 17, (1994), pp. 2-44.



(a): Beam-machine

(b): Beam Rotation

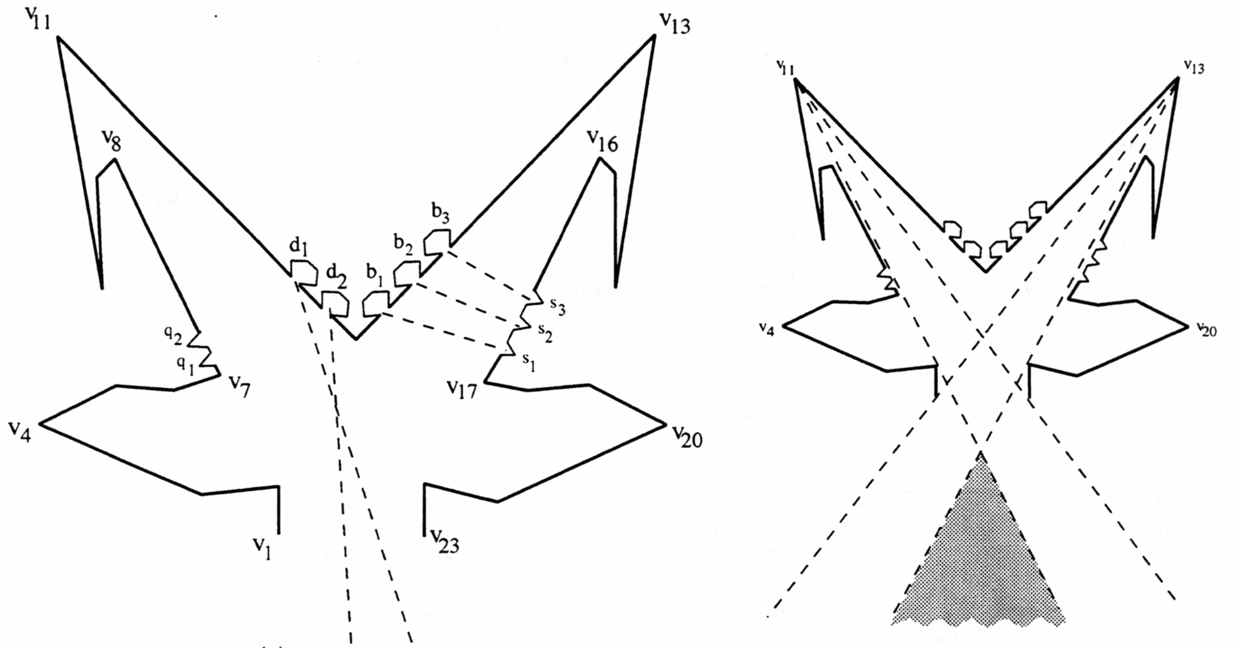
Figure 1: Illustrating beam generated from a beam machine



(a) Skeleton of BVG

(b): Backgroung of Variable Generator (BVG)

Figure 2: Illustrating the Construction of the Background of the Variable Generator



(a) (b): The Shadow Region of the Variable Generator

Figure 3: The Variable Generator

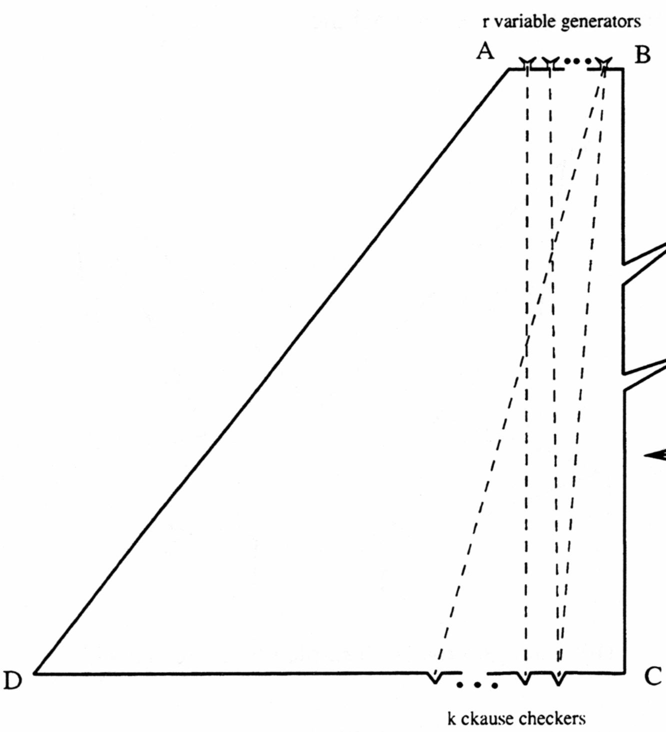


Figure 4: The Final Polygon P

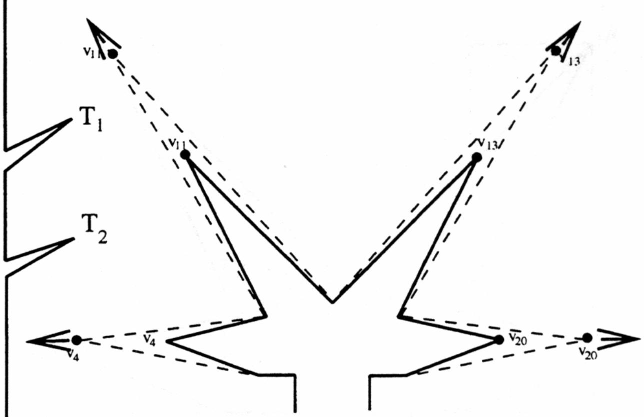


Figure 5: Stretching the Skeleton