On the conversion of ordinary Voronoi diagrams into Laguerre diagrams

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Abstract

We present some geometric relationships between the ordinary Voronoi diagram, and the Voronoi diagram in the Laguerre geometry. We derive from these properties an algorithm for the conversion of ordinary Voronoi diagrams into Voronoi diagrams in the Laguerre geometry.

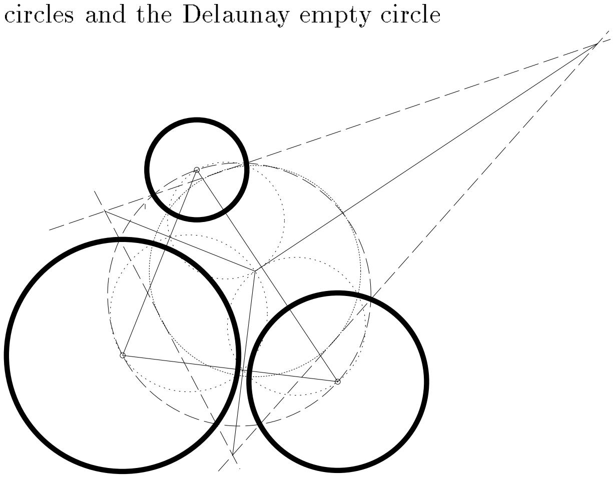
1 Introduction

Voronoi diagrams are one of the most extensively studied structure in computational geometry (see Okabe [8] for a general survey). The first type of Voronoi diagrams to be considered was the ordinary Voronoi diagram for sets of points. The Voronoi diagram has been generalized to different sets of sites (lines, circles, curves, polygons), and different metrics or geometries (L_p metrics, convex distances, Laguerre geometry). The generalization of the Voronoi diagram to the Laguerre geometry is known as Laguerre diagram [9]. They are weighted Voronoi diagrams based on a metric different from the Euclidean distance: the power of a point relatively to a weight circle. This Laguerre metric is intrinsically connected to the inversive geometry, and extends to the power of a point relatively to a sphere in three dimensions (or relatively to hypersheres in higher dimensions). The tessellation formed by the Laguerre diagram is analogous to the tessellation formed by the Power Voronoi diagram. The difference is that in the Laguerre geometry the radii of the weight circles correspond to the elevation of the sites in the three-dimensional space (see Imai and *al.* [7]), meanwhile in the Power Voronoi diagram, they correspond to the square root of the weight of the site (see Aurenhammer [1]).

The work of Edelsbrunner and Shah [5] provides an algorithm for the incremental construction of Power Voronoi diagrams in the d-dimensional space by lifting the sites to the d+1-dimensional space, constructing the convex hull in the d+1-dimensional space, and projecting it back to the d-dimensional space. Their algorithm runs in $O\left(n\,log\,n + n^{\lceil \frac{d}{2} \rceil}\right)$ expected time. The projection of this convex hull onto the d-dimensional space is the Power Voronoi diagram of the initial set of sites. The algorithm proposed by Gavrilova [6] converts incrementally ordinary Voronoi diagrams into Power Voronoi diagrams in $O\left(n^2\right)$ worst-case running time without lifting to the d+1-dimensional space. Her work [6] does not provide a geometric construction for this conversion. Moreover her assumption (Assumption 3.1.1 in [6]), that the weight circles do not intersect is very limitative, since in the applications the circles are not guaranteed to be disjoint.

In this paper, we will study an algorithm for the geometric conversion of the ordinary Voronoi diagram into the Laguerre diagram. We exhibit a property of the Voronoi diagram in the Laguerre geometry analogous to the empty circumcircle property of the dual of the ordinary Voronoi diagram (the Delaunay triangulation [10]). This works extends the applicability of an "empty circle criterion" without assuming that the circles do not intersect. Moreover, we give a geometric construction and a geometric interpretation of the "empty circle criterion". The algorithm for the conversion of ordinary Voronoi diagrams into Laguerre diagrams uses the fact that the Laguerre vertex corresponding to a triple of points can be constructed knowing only the circumcircle of these three points (called here the Delaunay circle, and represented in Figure 1 as a dashed circle) and their "weight" circles (represented on Figure 1 as thick plain circles), a "flipping" condition associated with the "Laguerre empty circle" (represented on Figure 1 as a densely dotted circle), and a linear time traversal of the Delaunay triangulation.

Figure 1: The "Laguerre empty circle", the "weight" circles and the Delaunay empty circle



This paper is organized as follows. In section 2, we present some preliminaries about radical axis, inversive geometry, and the Voronoi diagram in the Laguerre geometry. In section 3, we present an algorithm and its geometric interpretation for the conversion from the ordinary Voronoi vertex to the Laguerre vertex. In section 4, we present a flipping criterion for the Laguerre diagram, that works even if the weight circles intersect. This flipping criterion is related to the Laguerre vertex of a triple of circles. We introduce an algorithm for the conversion from ordinary Voronoi diagrams to Laguerre diagrams.

2 Preliminaries

Let R^2 be the Euclidean plane. Let m be a point of R^2. Let $S(a,r)$ be a circle of centre a and radius r. Its equation is given by $S(x)=0$ where $S(x)=d(a,x)^2-r^2$ (see [4]). The power of the point m with respect to the circle S is defined as [4]: $S(m)=d(a,m)^2-r^2$. If $S(m)=0$, the point m is on $S(a,r)$, if $S(m)<0$, the point m is inside $S(a,r)$, if $S(m)>0$, m is outside $S(a,r)$. If we take any line l passing through m and intersecting $S(a,r)$, at the intersection points t and t', we have the following property [4]: $\overrightarrow{mt}\cdot\overrightarrow{mt'}=S(m)$. Now, let's consider two circles $S(a,r)$ and $S'(a',r')$. The locus of points whose power with respect to $S(a,r)$ equals their power with respect to $S'(a',r')$ is called the radical axis or chordale of $S(a,r)$ and $S'(a',r')$. This locus is defined if, and only if $a\neq a'$ (see Berger [2]). In the case where it is defined, its equation is: $S(x)=S'(x)$, and it is orthogonal to $\overrightarrow{aa'}$ [2].

Two circles $S(a,r)$ and $S'(a',r')$ are said to be orthogonal if, and only if, one of the following conditions holds:

1. $S'(a)=r^2$
2. $S(a')=r'^2$
3. $d(a,a')^2=r^2+r'^2$ [2].

Let $\mathcal{S}=\{S_i\}$ be the set of generators (circles), where S_i is the circle whose centre is the projection $\overrightarrow{p_i}$ of the site onto the two-dimensional plane Oxy, and its radius (the elevation of the site) z_i [7]. The Laguerre region of S_i is the set of points of the Euclidean plane whose power with respect to S_i is smaller than or equal to their power with respect to the other spheres [7]. It is defined by:

$\mathcal{V}(S_i)=\{M\in R^2:\forall j\neq i, S_i(M)\leq S_j(M)\}$.

The bisector H_{ij} between S_i and S_j is the locus of points whose power with respect to S_i equals their power with respect to S_j. Therefore, H_{ij} is the radical axis or chordale of S_i and S_j.

3 Ordinary Voronoi vertex and Laguerre vertex

In this section, we will study the geometric transformation from the ordinary Voronoi vertex of a triple of points $\{a,b,c\}$ of the set of sites $\mathcal{P}$ into the Laguerre vertex of the triple of circles $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ of the set of circles $\mathcal{S}$. We do mean that the set of sites $\mathcal{P}$ for the ordinary Voronoi diagram is the set composed of the centres of the circles of $\mathcal{S}$. The ordinary Voronoi vertex of the triple of points $\{a,b,c\}$ is the point of the plane equidistant from these three points, and therefore, it is the centre of the circle passing through these three points. If this circle is a valid Delaunay empty circumcircle, then a, b, and c are not aligned. In this section we suppose that a, b, and c are not aligned. We do not assume as in [6], that the weight circles do not intersect. The circle passing through the three points is called the Delaunay circle in this section. Let's assume that the Delaunay circle is the circle $D(d,\delta)$. Due to the limitations on the length of the abstract, we will present some of the lemmas and theorems without proof.

Lemma 1 : *Let H_{AD} denote the chordale of A and D, H_{BD} denote the chordale of B and D, and H_{CD} denote the chordale of C and D. Then, they intersect two by two at three points i, j, and k.*

Proof: By definition, the radical axis of two circles is orthogonal to the line joining the centres of the two circles. Therefore, $H_{AD}\perp\langle a,d\rangle$, and $H_{BD}\perp\langle b,d\rangle$, and $H_{CD}\perp\langle c,d\rangle$. Since we supposed that a, b, and c are not aligned, $\langle a,d\rangle$, $\langle b,d\rangle$, and $\langle c,d\rangle$, which all contain d, are three distinct lines with distinct directions. Therefore, since $H_{AD}\perp\langle a,d\rangle$, and $H_{BD}\perp\langle b,d\rangle$, and $H_{CD}\perp\langle c,d\rangle$; H_{AD}, H_{BD}, and H_{CD} are three distinct lines with distinct directions. Therefore, they intersect two by two at three points i, j, and k. Q.E.D.

Lemma 2: *Let i denote the intersection of H_{AD} with H_{BD}, j denote the intersection of H_{BD} with H_{CD}, and k denote the intersection of H_{CD} with H_{AD}. Then, the line orthogonal to $\langle a,b\rangle$ and containing i, the line orthogonal to $\langle b,c\rangle$ and containing j, and the line orthogonal to $\langle c,a\rangle$ and containing k, intersect at a common point p.*

Proof: Since i pertains to H_{AD}, $A(i) = D(i)$. Since i pertains also to H_{BD}, $D(i) = B(i)$. Therefore, $A(i) = D(i) = B(i)$. Therefore i pertains to the radical axis H_{AB} of A, and B. Similarly, j pertains to the radical axis H_{BC} of B, and C; and k pertains to the radical axis H_{AC} of A, and C. By definition, the radical axis of two circles is orthogonal to the line joining the centres. Therefore, the line orthogonal to $\langle a,b\rangle$ and containing i is the radical axis H_{AB} of A, and B, and its equation is $A(x) = B(x)$. Using the same argument based on the definition of the radical axis of two circles, we can state that the line orthogonal to $\langle b,c\rangle$ and containing j is the radical axis H_{BC} of B, and C, and its equation is $B(x) = C(x)$; and the line orthogonal to $\langle a,c\rangle$ and containing k is the radical axis H_{AC} of A, and C, and its equation is $A(x) = C(x)$. Since a, b, and c are not aligned, $\langle a,b\rangle$ and $\langle b,c\rangle$ are not parallel. Therefore, since $\langle a,b\rangle \perp H_{AB}$ and $\langle b,c\rangle \perp H_{BC}$, H_{AB} and H_{BC} are not parallel. Thus, they have an intersection point. Let's call it p. Since p pertains to H_{AB}, whose equation is $A(x) = B(x)$, $A(p) = B(p)$ holds. Since p pertains to H_{BC}, whose equation is $B(x) = C(x)$, $B(p) = C(p)$ holds. Therefore, $A(p) = B(p) = C(p)$. Since $A(p) = C(p)$, p belongs to H_{AC} (whose equation has been shown to be $A(x) = C(x)$). Therefore, p belongs to H_{AB}, H_{BC}, and H_{AC}. Therefore, p belongs to the line orthogonal to $\langle a,b\rangle$ and containing i; the line orthogonal to $\langle b,c\rangle$ and containing j, and the line orthogonal to $\langle c,a\rangle$ and containing k. Q.E.D.

Proposition 1 (Construction of the Laguerre vertex from the Delaunay circle and the weight circles): *The Laguerre vertex of $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ is the point p intersection of the three lines mentioned in Lemma 2.*

Proof: In the proof of Lemma 2, we have seen that $A(p) = B(p) = C(p)$. Therefore, p has the same power with respect to the circles A, B, and C. Therefore it is the Laguerre vertex of A, B, and C. Q.E.D.

Theorem 1: *The Laguerre vertex of $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ is the intersection of the internal bisectors of the three radical axes H_{AD}, H_{BD}, and H_{CD}.*

Proof: The radical axis H_{AD} is orthogonal to $\langle a,d\rangle$. The radical axis H_{BD} is orthogonal to $\langle b,d\rangle$. Therefore the angle $\overline{H_{AD},H_{BD}}$ formed by the chordales H_{AD} and H_{BD} is the same as the angle $\overline{\overrightarrow{da},\overrightarrow{db}}$ formed by the lines $\langle d,a\rangle$ and $\langle d,b\rangle$. Since d is the circumcentre of a, b, and c, the triangle $\{a,d,b\}$ is isocele in d ($da = db$). Therefore, the line l' orthogonal to $\langle a,b\rangle$ and containing d is also the bisector of the angle $\overline{\overrightarrow{da},\overrightarrow{db}}$. We have proved in proposition 1, that the Laguerre vertex pertains to the line l orthogonal to $\langle a,b\rangle$ and containing i. This line l is thus parallel to l' (they are both orthogonal to $\langle a,b\rangle$), and thus l bisects internally the $\overline{H_{AD},H_{BD}}$ in i. Therefore the Laguerre vertex pertains to the internal bisector of the angle $\overline{H_{AD},H_{BD}}$. From Proposition 1, we know that the Laguerre vertex pertains also to the line orthogonal to $\langle b,c\rangle$ and containing j, and to the line orthogonal to $\langle c,a\rangle$ and containing k. Therefore, using a similar reasoning, we can prove that the Laguerre vertex pertains also to the internal bisector of the angle $\overline{H_{AD},H_{BD}}$, and the internal bisector of the angle $\overline{H_{AD},H_{BD}}$. Therefore, the Laguerre vertex is the intersection of the internal bisectors of the three radical axes H_{AD}, H_{BD}, and H_{CD}. Q.E.D.

Theorem 2: *There exists a (possibly pure imaginary) circle centered at the Laguerre vertex of $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ and orthogonal to both $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$. We call it the Laguerre empty circle of $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$.*

Proof: We know that $A(p) = B(p) = C(p)$. Therefore $d(p,a)^2 - \alpha^2 = d(p,b)^2 - \beta^2 = d(p,c)^2 - \gamma^2$. Let ρ be $d(p,a)^2 - \alpha^2 = d(p,b)^2 - \beta^2 = d(p,c)^2 - \gamma^2$. Then, if we denote by ι the real or pure imaginary number such as $\iota^2 = \rho$, the following relationships hold: $d(p,a)^2 = \alpha^2 + \iota^2$, $d(p,b)^2 = \beta^2 + \iota^2$, and $d(p,c)^2 = \gamma^2 + \iota^2$. Therefore the circle centered at p and of radius ι is orthogonal to both $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$. This circle is a pure imaginary circle [3] if, and only if, $A(p) = B(p) = C(p) < 0$. Q.E.D.

Theorem 3: *This circle is a true circle if, and only if, the intersection of the disk bounded by A, and the disk bounded by B, and the disk bounded by C is empty.*

4 The conversion from the ordinary Voronoi diagram to the Laguerre diagram

Lemma 3: *Let e be the vertex of the triple $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$, and $E(e,\epsilon)$ be the circle orthogonal to $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$. Then, there exists a circle $D(d,\delta)$ of $\mathcal{S}$ for which the vertex e is also the vertex of the triples $\{A,B,D\}$ and $\{A,C,D\}$ if, and only if, $D(d,\delta)$ is orthogonal to $E(e,\epsilon)$.*

Proof: (only if): If such circle $D(d,\delta)$ exists, then $A(e) = B(e) = D(e)$ and $A(e) = C(e) = D(e)$. Therefore, $A(e) = B(e) = C(e) = D(e)$. Since $E(e,\epsilon)$ is orthogonal to $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$, according to section 2, $A(e) = \epsilon^2$. Since $A(e) = D(e)$, we have $D(e) = \epsilon^2$, which expresses that $D(d,\delta)$ is orthogonal to $E(e,\epsilon)$.

Proof: (if): If $D(d,\delta)$ is orthogonal to $E(e,\epsilon)$, then $D(e) = \epsilon^2$. Since $E(e,\epsilon)$ is orthogonal to $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$, the following relationships hold: $A(e) = \epsilon^2$, $B(e) = \epsilon^2$, and $C(e) = \epsilon^2$. Since we had already $D(e) = \epsilon^2$, we get $A(e) = B(e) = C(e) = D(e)$. And, we can conclude that the vertex e is also the vertex of the triples $\{A,B,D\}$ and $\{A,C,D\}$ if, and only if, $D(d,\delta)$ is orthogonal to $E(e,\epsilon)$. Q.E.D.

Theorem 4: *Let e be the vertex of the triple $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$. Then, the vertex e is valid (the triangle $\{A,B,C\}$ pertains to the dual of the Laguerre diagram) if, and only if, the power of e with respect to any circle $D(d,\delta)$ of $\mathcal{S}$ is greater than any of the power of e with respect to $A(a,\alpha)$, $B(b,\beta)$, or $C(c,\gamma)$.*

Proof: The vertex e is valid (the triangle $\{A,B,C\}$ pertains to the dual of the Laguerre diagram) if, and only if, $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ are the three sites closest to e in the Laguerre metric. Since, e has the same power with respect to $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$; $A(a,\alpha)$, $B(b,\beta)$, and $C(c,\gamma)$ are the three sites closest to e in the Laguerre metric if, and only if, there is no other site $D(d,\delta)$ of $\mathcal{S}$, that is closer to, or at the same distance from e than $A(a,\alpha)$, $B(b,\beta)$, or $C(c,\gamma)$ in the Laguerre metric. Therefore e is valid if, and only if, the power of e with respect to any circle $D(d,\delta)$ of $\mathcal{S}$ is greater than any of the power of e with respect to $A(a,\alpha)$, $B(b,\beta)$, or $C(c,\gamma)$. Q.E.D.

The proofs that have been given in this section are much simpler than the proof of the INCIRCLE test in Gavrilova [6]. Lemma 3 gives a necessary and sufficient condition and a geometric interpretation for the degenerate case where a Laguerre vertex is closer to four circles. Theorem 4 gives a way to test if a Laguerre vertex is valid. Its geometric interpretation is that the vertex e is valid (the triangle $\{A,B,C\}$ pertains to the dual of the Laguerre diagram) if, and only if, no circle $D(d,\delta)$ of $\mathcal{S}$, is orthogonal to a circle centered at e and of radius smaller than or equal to ϵ. The assumption that no three circle centres are aligned guarantees that the bisectors (the chordales) are defined and intersect, and that each Laguerre vertex is three-valent. It is possible to convert an ordinary Voronoi diagram into a Voronoi diagram for the Laguerre metric by performing the test given in Theorem 4. In the worst case, $O(n^2)$ tests have to be performed. This test can be performed by using a queue storing the triangles that have to be checked, in a way similar to Gavrilova [6]. After each triangle flip, the old triangles are dequeued, and the new ones are queued.

5 Conclusions

We have presented an algorithm for the conversion from the ordinary Voronoi diagram to the Laguerre diagram. We have presented some geometric properties related to this algorithm, its geometric construction and its geometric interpretation. The conversion from the Voronoi vertex to the Laguerre vertex and the flipping criterion do not suppose that the weight circles do not intersect, and therefore they generalize the results in [6].

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