The Distance Geometry of Deep Rhythms and Scales

Erik D. Demaine* Francisco Gomez-Martin† Henk Meijer‡ David Rappaport‡
Perouz Taslakian§ Godfried T. Toussaint§ Terry Winograd¶ David R. Wood‖

Abstract We characterize which sets of k points chosen from n points spaced evenly around a circle have the property that, for each $i = 1, 2, \ldots, k-1$, there is a nonzero distance along the circle that occurs as the distance between exactly i pairs from the set of k points. Such a set can be interpreted as the set of onsets in a rhythm of period n, or as the set of pitches in a scale of n tones, in which case the property states that, for each $i = 1, 2, \ldots, k-1$, there is a nonzero time [tone] interval that appears as the temporal [pitch] distance between exactly i pairs of onsets [pitches]. Rhythms with this property are called *Erdős-deep*. The problem is a discrete, one-dimensional (circular) analog to an unsolved problem posed by Erdős in the plane.

1 Introduction

Musical rhythms and scales can both be seen as two-way infinite sequence of bits where each 1 bit represents an onset played in a rhythm or a pitch included in a scale. Here we suppose that all time intervals between onsets in a rhythm are multiples of a fixed time unit, and that all tone intervals between pitches in a scale are multiples of a fixed tonal unit (in logarithm of frequency). It is also generally assumed that the two-way infinite bit sequence is periodic with some period n, so that the information can be compacted down to an n-bit string. To properly represent the cyclic nature of this string, we imagine assigning the bits to n points equally spaced around a circle of circumference n. A rhythm or scale can therefore be represented as a subset of these n points. Let k denote the size of this subset, i.e., the number of onsets or pitches. Time intervals between onsets in a rhythm and tone intervals between pitches in a scale can thus be measured as (integral) distances around the circle between two points in the subset.

A musical scale or rhythm is *Winograd-deep* if every possible distance from 1 to $\lfloor n/2 \rfloor$ has a unique multiplicity (number of occurrences). This notion was introduced by Winograd in an oft-cited but unpublished class project from 1966 [10], disseminated and further developed by the class instructor Gamer in 1967 [3, 4], and considered further in numerous papers and books, e.g., [1, 5]. Equivalently, a scale is *Winograd-deep* if the number of tones it has in common with each of its cyclic shifts (rotations) is unique. This equivalence is the Common Tone Theorem of [5, p. 42], and is originally described by Winograd [10], who in fact uses this definition as his primary definition of "deep". Deepness is one property of the ubiquitous Western *diatonic* 12-tone major scale $\{0, 2, 4, 5, 7, 9, 11\}_{12}$, and it captures some of the rich structure that perhaps makes this scale so attractive. More generally, Winograd [10], and independently Clough et al. [1], characterize all Winograd-deep scales: up to rotation, they are the scales generable as the first $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$ multiples (modulo n) of a value m that is relatively prime to n, plus one exceptional scale $\{0, 1, 2, 4\}_6$.

Every scale can be reinterpreted as a rhythm. In particular, the diatonic major scale, which translates into box-like notation as [x . x . x x . x . x . x], is internationally the most well known of all African rhythms. It is traditionally played on an iron bell, and is known on the world scene mainly by its Cuban name *Bembé* [8]. However, the notion of Winograd-deepness is rather restrictive for rhythms, because it requires exactly half of the pulses in a period (rounded to a nearest integer) to be onsets. For example, the popular Bossa-Nova 16-pulse rhythm $\{0, 3, 6, 10, 13\}_{16}$ = [x . . x . . x . . . x . . x . .] has only five onsets [7]. Nonetheless, if we focus just on distances that appear at least once between two onsets, then the frequencies of occurrence are all unique and form an interval starting at 1: distance 4 occurs once, distance 7 occurs twice, distance 6 occurs thrice, and distance 3 occurs four times.

We therefore define an *Erdős-deep* rhythm (or scale) to be a rhythm with the property that, for every $i = 1, 2, \ldots, k-1$, there is a nonzero distance determined by the points on the circle that occurs exactly i times.

*Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, edemaine@mit.edu

†Departament de Matemática Aplicada, Universidad Politècnica de Madrid, fmartin@eui.upm.es

‡School of Computing, Queen's University, {henk,daver}@cs.queensu.ca

§School of Computer Science, McGill University, {perouz, godfried}@cs.mcgill.ca

¶Department of Computer Science, Stanford University, winograd@cs.stanford.edu

‖Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, david.wood@upc.edu

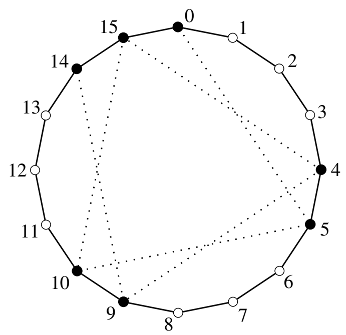

Figure 1: An Erdős-deep rhythm with $k = 7$ onsets and period $n = 16$. Distances ordered by multiplicity from 1 to 6 are 2, 7, 4, 1, 6, and 5. The dotted line shows how the rhythm is generated by multiples of $m = 5$.

The same definition is made in [9]. Figure 1 shows another example. It turns out that every Winograd-deep rhythm is also Erdős deep. Furthermore, we prove a similar (but more general) characterization of Erdős-deep rhythms: up to rotation and scaling, they are the scales generable as the first k multiples (modulo n) of a value m that is relatively prime to n, plus the same exceptional scale $\{0, 1, 2, 4\}$ of period $n = 6$. The key difference is that k is now a free parameter, instead of being forced to be either $\lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor + 1$. Our proof follows Winograd's proof of his characterization, but differs in one case (the second case of Theorem 3).

The property of Erdős deepness involves only the distances between points in a set, and is thus an issue of *distance geometry*, in this case in the discrete space of n points spaced equally around a circle. In 1989, Paul Erdős [2] considered the analogous question in the plane, asking whether there exist n points in the plane (no three on a line and no four on a circle) such that, for every $i = 1, 2, \ldots, n-1$, there is a distance determined by these points that occurs exactly i times. Solutions have been found for $2 \le n \le 8$, but in general the problem remains open. Palásti [6] considered a variant of this problem with further restrictions—no three points form a regular triangle, and no one is equidistant from three others—and solves it for $n = 6$.

2 Definitions

Although the rest of the paper speaks about rhythms, the results apply equally well to scales. We define a *rhythm* of period n to be a subset of $\{0, 1, \ldots, n-1\}$, representing the set of pulses that are onsets in each repetition. For clarity, we write the period n as a subscript after the subset: $\{\ldots\}_n$. Geometrically, we can view such a rhythm as a subset of n points equally spaced clockwise around a circle of circumference n. Let $k = |R|$ denote the number of onsets in rhythm R. The *successor* of an onset i in R is the smallest onset $j \ge i$ in R, if such a j exists, or else the overall smallest onset j in R.

The *oriented distance* from onset i to onset j in R is $j - i$ if $i < j$, and $n + j - i$ if $i > j$, i.e., the length of the counterclockwise arc starting at i and ending at j on the circle of circumference n. The *distance* between two onsets i and j in R is $d(i, j) = \min\{|i - j|, n - |i - j|\}$, i.e., the minimum of the oriented distance from i to j and the oriented distance from j to i, i.e., the length of the shortest arc connecting points i and j on the circle of circumference n. Every distance is between 0 and $\lfloor n/2 \rfloor$. The *distance multiset* of a rhythm R is the multiset of all nonzero pairwise distances, i.e., $\{d(i, j) \mid i, j \in R, i \ne j\}$. The distance multiset has cardinality $\binom{k}{2} = \frac{k(k-1)}{2}$. The *multiplicity* of a distance d is the number of occurrences of d in the distance multiset.

A rhythm is *Erdős-deep* if it has (exactly) one distance of multiplicity i, for $i = 1, 2, \ldots, k-1$. Note that these frequencies sum to $\sum_{i=1}^{k-1} i = \frac{k(k-1)}{2} = \binom{k}{2}$, which is the cardinality of the distance multiset, and hence these are all the occurrences of distances in the rhythm. A rhythm is *Winograd-deep* if every two possible distances from $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$ have different multiplicity.

A *shelling* of a Erdős-deep rhythm R is a sequence $s_1, s_2, \ldots, s_k$ of onsets in R such that $R - \{s_1, s_2, \ldots, s_i\}$ is a Erdős-deep rhythm for $i = 0, 1, \ldots, k$. (The definition of Erdős-deep rhythm includes all rhythms with 0, 1, or 2 onsets.)

3 Characterization

Our characterization of Erdős-deep rhythms is in terms of two families of rhythms. The main rhythm family consists of the rhythms $D_{k,n,m} = \{im \bmod n \mid i = 0, 1, \ldots, k-1\}_n$ of period n, for certain values of k, n, and m. The one exceptional rhythm is $F = \{0, 1, 2, 4\}_6$ of period 6.

Fact 1 *F is Erdős-deep.*

Lemma 2 *$D_{k,n,m}$ is Erdős-deep if $k \le \lfloor n/2 \rfloor + 1$ and m and n are relatively prime.*[1]

Proof: By definition of $D_{k,n,m}$, the multiset of oriented distances is $\{(jm - im) \bmod n \mid i < j\} = \{(j - i)m \bmod n \mid i < j\}$. There are $k - p$ choices of i and j such that $j - i = p$, so there are exactly p occurrences of the oriented distance $(pm) \bmod n$ in the multiset. Each of these oriented distances corresponds to a nonoriented distance—either $(pm) \bmod n$ or $(-pm) \bmod n$, whichever is smaller (at most $n/2$). We claim that these distances are all distinct. Then the multiplicity of each distance $(\pm pm) \bmod n$ is exactly p, establishing that the rhythm is Erdős-deep.

[1] Two numbers m and n are *relatively prime* if their greatest common divisor is 1.

For two distances to be equal, we must have $\pm pm \equiv \pm qm \pmod{n}$ for some (possibly different) choices for the $\pm$ symbols, and for some $p \neq q$. By (possibly) multiplying both sides by -1, we obtain two cases: (1) $pm \equiv qm \pmod{n}$ and (2) $pm \equiv -qm \pmod{n}$. Because m is relatively prime to n, m has a multiplicative inverse modulo n. Dividing both sides of the congruence by m, we obtain (1) $p \equiv q \pmod{n}$ and (2) $p \equiv -q \pmod{n}$. Because $0 \leq i < j < k \leq \lfloor n/2 \rfloor + 1$, $0 \leq p = j - i < \lfloor n/2 \rfloor + 1$, and similarly for q: $0 \leq p, q \leq \lfloor n/2 \rfloor$. Thus, the first case of $p \equiv q \pmod{n}$ can happen only when $p = q$, and the second case of $p + q \equiv 0 \pmod{n}$ can happen only when $p = q = 0$ or when $p = q = n/2$. Either case contradicts that $p \neq q$. Therefore the distances arising from different values of p are indeed distinct, proving the lemma. ☐

We now state and prove our characterization of Erdős-deep rhythms, which is up to rotation and scaling. The *rotation* of a rhythm R by an integer $\Delta \geq 0$ is the rhythm $\{(i + \Delta) \bmod n \mid i \in R\}$ of the same period n. Rotation preserves the distance multiset and therefore Erdős-deepness (and Winograd-deepness). The *scaling* of a rhythm R of period n by an integer $\alpha \geq 1$ is the rhythm $\{\alpha i \mid i \in R\}$ of period αn. Scaling maps each distance d to αd, and thus preserves multiplicities and therefore Erdős-deepness (but not Winograd-deepness).

Theorem 3 *A rhythm is Erdős-deep if and only if it is a rotation of a scaling of either the rhythm F or the rhythm $D_{k,n,m}$ for some k, n, m with $k \leq \lfloor n/2 \rfloor + 1$, $1 \leq m \leq \lfloor n/2 \rfloor$, and m and n are relatively prime.*

Proof: Because a rotation of a scaling of an Erdős-deep rhythm is Erdős-deep, the "if" direction of the theorem follows from Fact 1 and Lemma 2.

Consider an Erdős-deep rhythm R with k onsets. By definition of Erdős-deepness, R has one nonzero distance with multiplicity i for each $i = 1, 2, \ldots, k-1$. Let m be the distance with multiplicity $k-1$. Because m is a distance, $1 \leq m \leq \lfloor n/2 \rfloor$. Also, $k \leq \lfloor n/2 \rfloor + 1$ (for any Erdős-deep rhythm R), because all nonzero distances are between 1 and $\lfloor n/2 \rfloor$ and therefore at most $\lfloor n/2 \rfloor$ nonzero distances occur. Thus k and m are suitable parameter choices for $D_{k,n,m}$.

Consider the graph $G_m = (R, E_m)$ with vertices corresponding to onsets in R and with an edge between two onsets of distance m. By definition of distance, every vertex i in G_m has degree at most 2: the only onsets at distance exactly m from i are $(i - m) \bmod n$ and $(i + m) \bmod n$. Thus, the graph G_m is a disjoint union of paths and cycles. The number of edges in G_m is the multiplicity of m, which we supposed was $k-1$, which is 1 less than the number of vertices in G_m. Thus, the graph G_m consists of exactly one path and any number of cycles.

The cycles of G_m have a special structure because they correspond to subgroups generated by single elements in the cyclic group $C_n = (\mathbb{Z}/(n), +)$. Namely, the onsets corresponding to vertices of a cycle in G_m form a regular (n/a)-gon, with a distance of $a = \gcd(m, n)$ between consecutive onsets. (a is called the index of the subgroup generated by m.) In particular, every cycle in G_m has the same length $r = n/a$. Because G_m is a simple graph, every cycle must have at least 3 vertices, so $r \geq 3$.

The proof partitions into four cases depending on the length of the path and on how many cycles the graph G_m has. The first two cases will turn out to be impossible; the third case will lead to a rotation of a scaling of rhythm F; and the fourth case will lead to a rotation of a scaling of rhythm $D_{k,n,m}$.

First suppose that the graph G_m consists of a path of length at least 1 and at least one cycle. We show that this case is impossible because the rhythm R can have no distance with multiplicity 1. Suppose that there is a distance with multiplicity 1, say between onsets i_1 and i_2. If i is a vertex of a cycle, then both $(i + m) \bmod n$ and $(i - m) \bmod n$ are onsets in R. If i is a vertex of the path, then one or two of these are onsets in R, with the case of one occurring only at the endpoints of the path. If $(i_1 + m) \bmod n$ and $(i_2 + m) \bmod n$ were both onsets in R, or $(i_1 - m) \bmod n$ and $(i_2 - m) \bmod n$ were both onsets in R, then we would have another occurrence of the distance between i_1 and i_2, contradicting that this distance has multiplicity 1. Thus, i_1 and i_2 must be opposite endpoints of the path. If the path has length ℓ, then the oriented distance between i_1 and i_2 is $(\ell m) \bmod n$. This oriented distance (and hence the corresponding distance) appears in every cycle, of which there is at least one, so the distance has multiplicity more than 1, a contradiction. Therefore this case is impossible.

Second suppose that the graph G_m consists of a path of length 0 and at least two cycles. We show that this case is impossible because the rhythm R has two distances with the same multiplicity. Pick any two cycles C_1 and C_2, and let d be the smallest positive oriented distance from a vertex of C_1 to a vertex of C_2. Thus i is a vertex of C_1 if and only if $(i + d) \bmod n$ is a vertex of C_2. Because the cycles are disjoint, $d < a$. Because $r \geq 3$, $d < n/3$, so oriented distances of d are also true distances of d. The number of occurrences of distance d between a vertex of C_1 and a vertex of C_2 is either r or $2r$, the case of $2r$ arising when $d = a/2$ (i.e., C_2 is a "half-rotation" of C_1). The number of occurrences of distance $d' = \min\{d + m, n - (d + m)\}$ is the same—either r or $2r$, in the same cases. (Note that $d < a \leq n - m$, so $d + m < n$, so the definition of d' correctly captures a distance modulo n.) The same is true of distance $d'' = \min\{d - m, n - (d - m)\}$. If other pairs of cycles have the same smallest positive oriented distance d, then the number of occurrences of d, d', and d'' between those cycles are also equal. Because the cycles are disjoint, distance d and thus $d+m$ and $d-m$ cannot be $(pm) \bmod n$ for any p, so these distances cannot occur between two vertices of the same cycle. Finally, the sole vertex x of the path has distance d to onset i (which must be a vertex of some cycle) if and only if x has distance d' to onset $(i+m) \bmod n$ (which must be a vertex of the same cycle) if and only if x has distance d'' to onset $(i-m) \bmod n$ (which also must be a vertex of the same cycle). Therefore the frequencies of distances d, d', and d'' must be equal. Because R is Erdős-deep, we must have $d = d' = d''$. To have $d = d'$, either $d = d+m$ or $d = n-(d+m)$, but the first case is impossible because $d > 0$ by nonoverlap of cycles, so $2d+m = n$. Similarly, to have $d = d''$, we must have $2d - m = n$. Subtracting these two equations, we obtain that $2m = 0$, contradicting that $m > 0$. Therefore this case is also impossible.

Third suppose that the graph G_m consists of a path of length 0 and exactly one cycle. We show that this case forces R to be a rotation of a scaling of rhythm F because otherwise two distances m and m' have the same multiplicity. The number of occurrences of distance m in the cycle is precisely the length r of the cycle. Similarly, the number of occurrences of distance $m' = \min\{2m, n-2m\}$ in the cycle is r. The sole vertex x on the path cannot have distance m or m' to any other onset (a vertex of the cycle) because then x would then be on the cycle. Therefore the frequencies of distances m and m' must be equal. Because R is Erdős-deep, m must equal m', which implies that either $m = 2m$ or $m = n - 2m$. The first case is impossible because $m > 0$. In the second case, $3m = n$, i.e., $m = \frac{1}{3}n$. Therefore, the cycle has $r = 3$ vertices, say at $\Delta, \Delta + \frac{1}{3}n, \Delta + \frac{2}{3}n$. The fourth and final onset x must be midway between two of these three onsets, because otherwise its distance to the three vertices are all distinct and therefore unique. No matter where x is so placed, the rhythm R is a rotation by $\Delta + c\frac{1}{3}n$ (for some $c \in \{0, 1, 2\}$) of a scaling by $n/6$ of the rhythm F.

Finally suppose that G_m has no cycles, and consists solely of a path. We show that this case forces R to be a rotation of a scaling of a rhythm $D_{k,n',m'}$ with $1 \le m' \le \lfloor n'/2 \rfloor$ and with m' and n' relatively prime. Let b be the onset such that $(b-m) \bmod n$ is not an onset (the "beginning" vertex of the path). Consider rotating R by $-i$ so that 0 is an onset in the resulting rhythm $R - i$. The vertices of the path in $R - i$ form a subset of the subgroup of the cyclic group C_n generated by the element m. Therefore the rhythm $R - i = D_{k,n,m} = \{(im) \bmod n \mid i = 0, 1, \ldots, k-1\}$ is a scaling by a of the rhythm $D_{k,n/a,m/a} = \{(im/a) \bmod (n/a) \mid i = 0, 1, \ldots, k-1\}$. The rhythm $D_{k,n/a,m/a}$ has an appropriate value for the third argument: m/a and n/a are relatively prime ($a = \gcd(m,n)$) and $1 \le m/a \le \lfloor n/2 \rfloor / a \le \lfloor (n/a)/2 \rfloor$. Also, $k \le \lfloor (n/a)/2 \rfloor + 1$ because the only occurring distances are multiples of a and therefore the number $k-1$ of distinct distances is at most $\lfloor (n/a)/2 \rfloor$. Therefore R is a rotation by i of a scaling by a of $D_{k,n/a,m/a}$ with appropriate values of the arguments. □

An interesting consequence of this characterization is the following:

Corollary 4 *Every Erdős-deep rhythm has a shelling.*

Proof: If the Erdős-deep rhythm is $D_{k,n,m}$, we can remove the last onset from the path, resulting in $D_{k-1,n,m}$, and repeat until we obtain the empty rhythm $D_{0,n,m}$. At all times, k remains at most $\lfloor n/2 \rfloor + 1$ (assuming it was originally) and m remains between 1 and $\lfloor n/2 \rfloor$ and relatively prime to n. On the other hand, $F = \{0, 1, 2, 4\}_6$ has the shelling $4, 2, 1, 0$ because $\{0, 1, 2\}_6$ is Erdős-deep. □

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