Coequalizers and Tensor Products for Continuous Idempotent Semirings

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Abstract. We provide constructions of coproducts, free extensions, coequalizers and tensor products for classes of idempotent semirings in which certain subsets have least upper bounds and the operations are sup-continuous. Among these classes are the *-continuous Kleene algebras, the μ-continuous Chomsky-algebras, and the unital quantales.

1 Introduction

The theory of formal languages and automata has well-recognized connections to algebra, as shown by work of S. C. Kleene, H. Conway, S. Eilenberg, D. Kozen and many others. The core of the algebraic treatment of the field deals with the class of regular languages and finite automata/transducers. Here, right from the beginnings in the 1960s one finds "regular" operations + (union), · (elementwise concatenation), * (iteration, i.e. closure under 1 and ·) and equational reasoning, and eventually a consensus was reached that drew focus to so-called Kleene algebras to model regular languages and finite automata.

An early effort to expand the scope of algebraization to context-free languages was made by Chomsky and Schützenberger [3]. Somewhat later, around 1970, Gruska [5], McWhirter [19], Yntema [20] suggested to add a least-fixed-point operator μ to the regular operations. But, according to [5], "one can hardly expect to get a characterization of CFL's so elegant and simple as the one we have developed for regular expressions." Neither approach found widespread use.

After the appearance of a new axiomatization of Kleene algebras by Kozen [10] in 1990, a formalization of the theory of context-free languages within the algebra of idempotent semirings with a least-fixed-point operator was suggested in Leiß [13] and Ésik et al. [15], leading to the first complete axiomatization of the equational theory of context-free languages and the introduction of the "μ-continuous Chomsky-algebras" by Grathwohl et al. [4] in 2013. This formalism is cast in the same mould as that for the *-continuous Kleene algebra, with an infinitary identity related to the distributivity axiom below.

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Already in 2008 (see Hopkins [7, 8]), a second-order formalism for context-free languages emerged, in the guise of the category $\mathbb{D}\mathcal{C}$, as part of a larger endeavor to embody all of the Chomsky hierarchy by a family of categories $\mathbb{D}\mathcal{A}$ (discussed below). The primary models are idempotent semirings $\mathcal{A}M$ obtained by lifting the operations of monoids M to a family of $\mathcal{A}$-subsets of M, such that the lifted operations are continuous with respect to a supremum operator. The $\mathcal{A}M$ form the Kleisli subcategory of an Eilenberg-Moore category (cf. Mac Lane [18]) $\mathbb{D}\mathcal{A}$.

As shown in Hopkins [8], there is a complete lattice of suitable subfunctors $\mathcal{A}$ of the powerset functor $\mathcal{P}$; among them are $\mathcal{R}$ and $\mathcal{C}$, selecting the regular and context-free subsets of a monoid, for which $\mathbb{D}\mathcal{R}$ coincides with the *-continuous Kleene algebras and $\mathbb{D}\mathcal{C}$ with the μ-continuous Chomsky algebras. For any two functors $\mathcal{A} \leq \mathcal{B}$ in the lattice there is an adjunction $(Q^{\mathcal{B}}_{\mathcal{A}}, Q^{\mathcal{A}}_{\mathcal{B}}, \eta, \epsilon)$ where $Q^{\mathcal{A}}_{\mathcal{B}} : \mathbb{D}\mathcal{B} \to \mathbb{D}\mathcal{A}$ is the forgetful functor and $Q^{\mathcal{B}}_{\mathcal{A}} : \mathbb{D}\mathcal{A} \to \mathbb{D}\mathcal{B}$ extends $D \in \mathbb{D}\mathcal{A}$ to a certain ideal-completion $\overline{D} \in \mathbb{D}\mathcal{B}$ of D. An interesting open problem is whether and when the ideal-completion can be replaced by a more algebraic method. In particular, can we do so for $Q^{\mathcal{C}}_{\mathcal{R}} : \mathbb{D}\mathcal{R} \to \mathbb{D}\mathcal{C}$, the extension of *-continuous Kleene algebras $K \in \mathbb{D}\mathcal{R}$ to their μ-continuous completions $\overline{K} \in \mathbb{D}\mathcal{C}$? Below we provide basic algebraic and categorical constructions with this goal in mind.

By the classical theorem of Chomsky and Schützenberger [3], the context-free languages $\mathcal{C}X^*$ can be reduced to the regular languages $\mathcal{R}(X \cup Y)^*$ over an extended alphabet. The algebraic and category-theoretic constructions given below allow us to sharpen and generalize the Chomsky-Schützenberger result and construct $\mathcal{C}X^*$ from $\mathcal{R}X^*$, then $\mathcal{C}M$ from $\mathcal{R}M$ for arbitrary monoids M, and finally to provide an algebraic construction of $Q^{\mathcal{C}}_{\mathcal{R}}$ and analogous results for the Thue/Turing subsets $\mathcal{T}M$. However, this can only be indicated below; its development has to be deferred to a forthcoming publication.

Section 2 introduces the categories $\mathbb{D}\mathcal{A}$ of $\mathcal{A}$-dioids and mentions the main examples. Section 3 shows that $\mathbb{D}\mathcal{A}$ has coproducts and free extensions. Section 4 provides coequalizers for $\mathbb{D}\mathcal{A}$, shows how they relate to $\mathcal{A}$-congruences, and finally introduces a tensor product for $\mathbb{D}\mathcal{A}$. Section 5 sketches two applications, the construction of the matrix ring $D^{n\times n}$ of a $\mathcal{A}$-dioid D as a tensor product of D with the boolean matrices $\mathbb{B}^{n\times n}$ and the construction of the context-free languages $\mathcal{C}X^*$ as tensor product of $\mathcal{R}X^*$ with a regular "bracket-algebra" $C_2 \in \mathbb{D}\mathcal{R}$. Hence the context-free languages over X are the values of regular expressions in this particular (non-free) *-continuous Kleene algebra –achieving what "we can hardly expect to get" according to [5]–, which has implications for parsing theory to be worked out. Section 6 discusses potential generalizations.

2 The Category of $\mathcal{A}$-Dioids and $\mathcal{A}$-Morphisms

Let $\mathbb{M}$ be the category of monoids $(M, \cdot, 1)$ and homomorphisms between monoids. A *dioid* $(D, +, \cdot, 0, 1)$ is an idempotent semiring. Idempotency of $+$ provides a partial order $\leq$ on D, via $d \leq d'$ iff $d + d' = d'$, with 0 as least element. Distributivity makes $\cdot$ monotone with respect to $\leq$, and $+$ guarantees a least upper bound $\sum U = d_1 + \ldots + d_n$ for each finite subset $U = \{d_1, \ldots, d_n\}$ of D. Let $\mathbb{D}$

be the category of dioids with dioid homomorphisms. Following [7], a *monadic operator* is a functor $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ where for all monoids M, N, (A_0) $\mathcal{A}M$ is a set of subsets of M, (A_1) $\mathcal{A}M$ contains all finite subsets of M, (A_2) $\mathcal{A}M$ is closed under products, hence $(\mathcal{A}M, \cdot, \{1\})$ with

$$A \cdot B := \{\, a \cdot b \mid a \in A, b \in B \,\} \quad \text{for } A, B \in \mathcal{A}M$$

is itself a monoid, (A_3) $\mathcal{A}M$ is closed under unions of sets from $\mathcal{A}\mathcal{A}M$, which implies that $\mathcal{A}M$ is an idempotent semiring with

$$0 := \bigcup \emptyset, \qquad A + B := \bigcup \{A, B\}, \quad \text{for } A, B \in \mathcal{A}M,$$

and (A_4) $\mathcal{A}$ preserves homomorphisms: if $f : M \to N$ is a homomorphism, so is $\mathcal{A}f : \mathcal{A}M \to \mathcal{A}N$ –abbreviated as $\widetilde{f}$–, where for $U \subseteq M$,

$$(\mathcal{A}f)(U) := \{\, f(u) \mid u \in U \,\} =: \widetilde{f}(U).$$

We write $\mathcal{A}M$ for both the set of subsets of M and the dioid $(\mathcal{A}M, +, \cdot, 0, 1)$.

An $\mathcal{A}$-*dioid* $(D, \cdot, 1, \leq)$ is a partially ordered monoid where each $U \in \mathcal{A}D$ has a least upper bound, $\sum U \in D$, and distributivity[1] holds:

$$(\sum U)(\sum V) = \sum (UV) \qquad \text{for all } U, V \in \mathcal{A}D.$$

An $\mathcal{A}$-*morphism* $f : D \to D'$ between $\mathcal{A}$-dioids D and D' is an order-preserving homomorphism such that $f(\sum U) = \sum \widetilde{f}(U)$ for all $U \in \mathcal{A}D$. Let $\mathbb{D}\mathcal{A}$ be the category of $\mathcal{A}$-dioids and $\mathcal{A}$-morphisms between $\mathcal{A}$-dioids.

For every monoid M, $\mathcal{A}M$ is an $\mathcal{A}$-dioid, by Theorem I.1[2]. Every $\mathcal{A}$-dioid becomes a dioid, using $a + b := \sum\{a, b\}$ and $0 := \sum \emptyset$. Every $\mathcal{A}$-morphism is a dioid-homomorphism. Hence we view $\mathbb{D}\mathcal{A}$ as a subcategory of $\mathbb{D}$.

In fact, $\mathcal{A} : \mathbb{M} \to \mathbb{D}\mathcal{A}$ and the forgetful functor $\widehat{\mathcal{A}} : \mathbb{D}\mathcal{A} \to \mathbb{M}$ form an adjunction, by Theorem II.16, and combine to a monad $T_{\mathcal{A}} = (\widehat{\mathcal{A}}\mathcal{A}, \eta, \mu) : \mathbb{M} \to \mathbb{M}$ with unit $\eta : m \in M \mapsto \{m\} \in \mathcal{A}M$ and product $\mu : \mathcal{U} \in \mathcal{A}\mathcal{A}M \mapsto \bigcup \mathcal{U} \in \mathcal{A}M$.

For a partial order D, the *down-closure* $U^{\leq}$ of $U \subseteq D$ is $\{\, d \in D \mid d \leq u$ for some $u \in U \,\}$. We say $U, V \subseteq D$ are *cofinal* (in symbols: $U \simeq V$), if U and V have the same down-closure.

Example 1. If $\mathcal{F}$ assigns to each monoid M its finite subsets, then $\mathcal{F}$ is monadic and $\mathbb{D}\mathcal{F}$ is the category of dioids and dioid-homomorphisms. The power set operator $\mathcal{P}$ is monadic and $\mathbb{D}\mathcal{P}$ is the category of quantales with unit.

Example 2. For infinite cardinal κ, $\mathcal{P}_\kappa M = \{\, X \mid X \subseteq M, |X| \leq \kappa \,\}$ is a monadic operator; $\mathbb{D}\mathcal{P}_{\aleph_0}$ is the category of closed semirings [9]. For regular cardinal κ, $\mathcal{F}_\kappa M = \{\, X \mid X \subseteq M, |X| < \kappa \,\}$ is monadic; (A_3) corresponds to regularity.

[1] Distributivity is $\sum$-continuity of $\cdot$ and equivalent to $\sum(aUb) = a(\sum U)b$ for all $a, b \in D, U \in \mathcal{A}D$, an instance of which is *-continuity $\sum\{\, ac^m b \mid m \in \mathbb{N} \,\} = ac^*b$.

[2] Theorem I.1 means Theorem 1 of [7], Theorem II.1 means Theorem 1 of [8], etc.

Example 3. Regular, context-free, and Turing/Thue-subsets $\mathcal{R}M$, $\mathcal{C}M$, and $\mathcal{T}M$ of a monoid M can be defined by generalizing the grammatical approach of doing so for free monoids $M = X^*$. In this case, for $\mathcal{A} \in \{\mathcal{R}, \mathcal{C}, \mathcal{T}\}$ one puts

$$\mathcal{A}X^* := \{\, L(G) \mid G \text{ is a grammar of type } \mathcal{A} \text{ over } X \,\}.$$

Here, a grammar $G = (Q, S, H)$ of type $\mathcal{A}$ over X has a set Q disjoint from X, with distinguished element $S \in Q$, and a finite subset H of "right-linear" rules $Q \times (XQ \cup X^*)$ in case $\mathcal{A} = \mathcal{R}$, of "context-free" rules $Q \times (Q \cup X)^*$ in case $\mathcal{A} = \mathcal{C}$, and of "contextual"[3] rules $Q^+ \times (Q \cup X)^*$ in case $\mathcal{A} = \mathcal{T}$, with $Q^+ = Q^* \setminus \{1\}$. The language defined by G is $L(G) = \{\, w \in X^* \mid S \Rightarrow_G w \,\}$, where $\Rightarrow_G$ is the least reflexive, transitive relation above H that is compatible with the monoid operation on $(X \cup Q)^*$. The class of $\mathcal{A}$-languages is closed under homomorphisms:[4] If $h : X^* \to Y^*$ is a homomorphism, so is $\widetilde{h} : \mathcal{A}X^* \to \mathcal{A}Y^*$: a grammar $G = (Q, S, H)$ over X gives rise to a grammar $G^h = (Q, S, H^h)$ over Y, of the same type, with $\widetilde{h}(L(G)) = L(G^h)$.

For arbitrary monoid M, take a generating subset $X \subseteq M$ and use the canonical homomorphism $h_X : X^* \to M$, where $h_X(x) = x$ for $x \in X$, to put

$$\mathcal{A}M = \{\, \widetilde{h_X}(L(G)) \mid G \text{ a grammar of type } \mathcal{A} \text{ over } X \,\}.$$

This is independent of the choice of X, because for any generating set $Y \subseteq M$ there is a homomorphism $h : X^* \to Y^*$ with $h_X = h_Y \circ h$, so that $\widetilde{h_X}(L(G)) = \widetilde{h_Y}(L(G^h))$. We sketch why these $\mathcal{A}$ are monadic operators. Obviously, $\mathcal{A}M \subseteq \mathcal{P}M$, showing (A_0). For (A_1) and (A_2), note that any finite set $F \subseteq M$ is the language of a grammar G of type $\mathcal{A}$ over a set $X \supseteq F$ of generators, and that from grammars G_1, G_2 of type $\mathcal{A}$ one easily constructs a grammar of type $\mathcal{A}$ for $L(G_1)L(G_2)$. According to Theorem I.8, $(A_3) \wedge (A_4)$ are equivalent to (A_5): every "substitution" homomorphism $\sigma : M \to \mathcal{A}N$ lifts to a homomorphism $\sigma^* : \mathcal{A}M \to \mathcal{A}N$ by $\sigma^*(U) = \bigcup\{\, \sigma(m) \mid m \in U \,\}$, for $U \in \mathcal{A}M$. To show (A_5), assume generating subsets $X \subseteq M$, $Y \subseteq N$, homomorphisms $h_X : X^* \to M$, $h_Y : Y^* \to N$, $\sigma : M \to \mathcal{A}N$ and $U \in \mathcal{A}M$. There is a grammar G of type $\mathcal{A}$ over X with $U = \widetilde{h_X}(L(G))$. For each $x \in X$, let G_x be a grammar of type $\mathcal{A}$ over Y with $\sigma(h_X(x)) = \widetilde{h_Y}(L(G_x)) \in \mathcal{A}N$. The map $x \mapsto L(G_x) \in \mathcal{A}Y^*$ extends to a homomorphism $\hat{\sigma} : X^* \to \mathcal{A}Y^*$, and $\sigma \circ h_X = \widetilde{h_Y} \circ \hat{\sigma}$. Then

$$\sigma^*(U) = \bigcup \widetilde{\sigma}(\widetilde{h_X}(L(G))) = \bigcup \widetilde{\widetilde{h_Y}}(\widetilde{\hat{\sigma}}(L(G))) = \widetilde{h_Y}(\bigcup \widetilde{\hat{\sigma}}(L(G))) = \widetilde{h_Y}(\hat{\sigma}^*(L(G))).$$

As the $\mathcal{A}$-languages are closed under substitutions (cf. [6], Theorems 3.4, 6.2, Exercise 9.11), we get $\hat{\sigma}^* : \mathcal{A}X^* \to \mathcal{A}Y^*$, hence $\hat{\sigma}^*(L(G)) \in \mathcal{A}Y^*$ and $\sigma^*(U) \in \mathcal{A}N$.

[3] A normal form where $H \subseteq Q^+ \times (X \cup Q)^*$ instead of $H \subseteq (X \cup Q)^+ \times (X \cup Q)^*$, making rule application effective. This corrects a mistaken definition of $\mathcal{T}X^*$ in [7].

[4] This is not true for the context-sensitive languages (cf. [6], Exercise 9.14), so we have to correct Corollary I.2: there is no monadic operator $\mathcal{S}$ of context-sensitive subsets.

Remark 1. Alternative definitions of $\mathcal{R}M$ resp. $\mathcal{C}M$ can be given as the closure of $\mathcal{F}M$ under binary union, elementwise product and iteration * resp. least solutions in $\mathcal{P}M$ of polynomial inequations $x_1 \geq p_1(x_1, \dots, x_n), \dots, x_n \geq p_n(x_1, \dots, x_n)$ with parameters from $\mathcal{C}M$. $\mathbb{D}\mathcal{R}$ is the category of *-continuous Kleene algebras [9]; for a proof, see [7]. $\mathbb{D}\mathcal{C}$ is the category of μ-continuous Chomsky algebras [4]; a proof appears in [16] (in these proceedings).

Let ρ be a dioid-congruence on an $\mathcal{A}$-dioid D. The set D/ρ of congruence classes is a dioid under the operations defined by $(d/\rho)(d'/\rho) := (dd')/\rho$, $1 := 1/\rho$, $d/\rho + d'/\rho := (d + d')/\rho$, $0 := 0/\rho$. For $U \subseteq D$, $U/\rho := \{\, d/\rho \mid d \in U \,\}$.

Let $\leq$ be the partial order on D/ρ derived from $+$. An *$\mathcal{A}$-congruence* on D is a dioid-congruence ρ on D such that for all $U, U' \in \mathcal{A}D$, if $U/\rho \simeq U'/\rho$, then $(\sum U)/\rho = (\sum U')/\rho$.

For any $\rho_0 \subseteq D \times D$, there is a least $\mathcal{A}$-congruence on D above ρ_0, the intersection of all $\mathcal{A}$-congruences $\rho \supseteq \rho_0$ on D.

Lemma 1. *Let $q : D \to Q$ be an $\mathcal{A}$-morphism between $\mathcal{A}$-dioids D, Q and ρ the least $\mathcal{A}$-congruence on D above the relation $\rho_0 \subseteq D \times D$. If $q(a) = q(b)$ for all $(a, b) \in \rho_0$, then $q(a) = q(b)$ for all $(a, b) \in \rho$.*

Proof. Since A is an $\mathcal{A}$-morphism,

$$\rho_q := \{\, (a, b) \mid a, b \in D, q(a) = q(b) \,\}$$

is an $\mathcal{A}$-congruence. By assumption, $\rho_0 \subseteq \rho_q$, hence $\rho \subseteq \rho_q$. □

Proposition 1. *If D is an $\mathcal{A}$-dioid and ρ an $\mathcal{A}$-congruence on D, then D/ρ is an $\mathcal{A}$-dioid and the map $d \mapsto d/\rho$ is an $\mathcal{A}$-morphism.*

Proof. We first show that each $U' \in \mathcal{A}(D/\rho)$ has a least upper bound $\sum U'$. Since $\cdot/\rho$ is a surjective homomorphism, $U' = U/\rho$ for some $U \in \mathcal{A}D$, by Theorem I.9. If $U/\rho = \tilde{U}/\rho$ for some $\tilde{U} \in \mathcal{A}D$, then U/ρ and $\tilde{U}/\rho$ are cofinal, hence $(\sum U)/\rho = (\sum \tilde{U})/\rho$, so $(\sum U)/\rho$ depends on U' only. Clearly $d \mapsto d/\rho$ is monotone, so $(\sum U)/\rho$ is an upper bound of U'. To show that it is least, let e/ρ be any upper bound of U'. Since $\{U, \{e\}\} \in \mathcal{F}\mathcal{A}D \subseteq \mathcal{A}\mathcal{A}D$, we have $U \cup \{e\} = \bigcup\{U, \{e\}\} \in \mathcal{A}D$. By the choice of e, $(U \cup \{e\})/\rho \simeq \{e\}/\rho$ and so

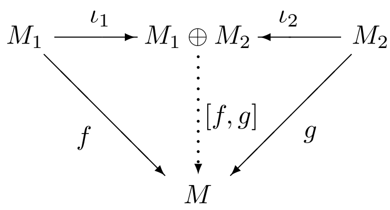
$$e/\rho + (\sum U)/\rho = (e + \sum U)/\rho = (\sum (U \cup \{e\}))/\rho = (\sum \{e\})/\rho = e/\rho.$$

This shows $(\sum U)/\rho \leq e/\rho$. Hence we can define $\sum U' := (\sum U)/\rho$. It follows that for $U \in \mathcal{A}D$, $(\sum U)/\rho = \sum\{\, d/\rho \mid d \in U \,\}$, showing that $d \mapsto d/\rho$ is an $\mathcal{A}$-morphism. Distributivity of $\sum$ can be reduced to distributivity of $\sum$ on D. □

3 Coproducts and Free Extensions

3.1 Coproducts

A *coproduct* of two objects M_1 and M_2 in a category is an object $M_1 \oplus M_2$ with two morphisms $\iota_1 : M_1 \to M_1 \oplus M_2$ and $\iota_2 : M_2 \to M_1 \oplus M_2$ such that for any two morphisms $f : M_1 \to M$ and $g : M_2 \to M$ there is a unique morphism $[f, g] : M_1 \oplus M_2 \to M$ with $f = [f, g] \circ \iota_1$ and $g = [f, g] \circ \iota_2$.



A similar definition may be applied dually with arrows reversed to yield the *product* $M_1 \times M_2$ of objects M_1 and M_2 with corresponding morphisms $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. By the universal property, the coproduct and product of M_1 and M_2 are unique up to isomorphism.

Example 4. In the category $\mathbb{M}$, a coproduct $M \oplus M'$ of M and M' can be constructed from the interleaved sequences of elements from M and M', i.e. as $(M \times M')^+/E$, the non-empty finite sequences of pairs from $M \times M'$ modulo the congruence E on $(M \times M')^+$ generated by the equations

$$\{ (m_0, 1^{M'})(m_1, m') = (m_0 m_1, m') \mid m_0, m_1 \in M, m' \in M' \}$$
$$\cup \{ (m, m'_0)(1^M, m'_1) = (m, m'_0 m'_1) \mid m \in M, m'_0, m'_1 \in M' \},$$

with unit $1 := (1^M, 1^{M'})$, injections $\iota_1(m) := (m, 1^{M'})$ and $\iota_2(m') := (1^M, m')$, product $(u/E)(v/E) := (uv)/E$ and induced function

$$[f, g]((m_0, m'_0) \cdots (m_k, m'_k)/E) := f(m_0)g(m'_0) \cdots f(m_k)g(m'_k).$$

Of course, for free monoids Γ^*, Δ^* with Γ disjoint from Δ, $\Gamma^* \oplus \Delta^* \simeq (\Gamma \cup \Delta)^*$.

Theorem 1. *The category $\mathbb{D}\mathcal{A}$ has coproducts. A coproduct*

$$\iota_1 : D_1 \to D_1 \oplus_{\mathcal{A}} D_2 \leftarrow D_2 : \iota_2$$

of $D_1, D_2 \in \mathbb{D}\mathcal{A}$ can be constructed from $\hat{\iota}_1 : M_1 \to M_1 \oplus M_2 \leftarrow M_2 : \hat{\iota}_2$, the coproduct of the monoids M_1, M_2 underlying D_1, D_2, as follows: Let $\equiv$ be the least $\mathcal{A}$-congruence on $\mathcal{A}(M_1 \oplus M_2)$ containing[5]

$$(\{(\sum A, \sum B)\}, A \times B), \text{ for all } A \in \mathcal{A}D_1, B \in \mathcal{A}D_2,$$

π the canonical map $U \mapsto U/_\equiv$ and $\eta : M_1 \oplus M_2 \to \mathcal{A}(M_1 \oplus M_2)$ be $\alpha \mapsto \{\alpha\}$. Then let $D_1 \oplus_{\mathcal{A}} D_2$ be $\mathcal{A}(M_1 \oplus M_2)/_\equiv$ and $\iota_k = \pi \circ \eta \circ \hat{\iota}_k$, for $k = 1, 2$.

Proof. $M_1 \oplus M_2$ is a monoid by Example 4, so $\mathcal{A}(M_1 \oplus M_2)$ is an $\mathcal{A}$-dioid. By Proposition 1, its quotient $D_1 \oplus_{\mathcal{A}} D_2 = \mathcal{A}(M_1 \oplus M_2)/_\equiv$ is an $\mathcal{A}$-dioid and π is an $\mathcal{A}$-morphism. Clearly, ι_1 and ι_2 are homomorphisms, since $\hat{\iota}_1, \hat{\iota}_2, \eta$ and π are. The reader may check that they are $\mathcal{A}$-morphisms, hence order-preserving.

[5] Here we use (a, b) etc. for its equivalence class in $M_1 \oplus M_2 = (M_1 \times M_2)^+/E$.

Let $f : D_1 \to D$, $g : D_2 \to D$ be $\mathcal{A}$-morphisms and M the monoid underlying D. By the universal property of $M_1 \oplus M_2$, there is a unique homomorphism $[f, g] : M_1 \oplus M_2 \to M$ with $f = [f, g] \circ \hat{\iota}_1$ and $g = [f, g] \circ \hat{\iota}_2$. It extends uniquely to an $\mathcal{A}$-morphism $[f, g]^* : \mathcal{A}(M_1 \oplus M_2) \to D$, by Theorem I.4, with

$$[f,g]^*(U) = \sum\{\, [f,g](u) \mid u \in U \,\} \qquad \text{for } U \in \mathcal{A}(M_1 \oplus M_2).$$

For the pairs $(\{(\sum A, \sum B)\}, A \times B)$ generating $\equiv$, we have

$$\begin{aligned}[f,g]^*(\{(\textstyle\sum A, \sum B)\}) &= [f,g](\textstyle\sum A, \sum B) = f(\sum A)g(\sum B) \\ = (\textstyle\sum \widetilde{f}(A))(\sum \widetilde{g}(B)) &= \textstyle\sum(\widetilde{f}(A)\widetilde{g}(B)) = \sum \widetilde{[f,g]}(A \times B) = [f,g]^*(A \times B).\end{aligned}$$

Hence by Lemma 1, $[f, g]^*$ is constant on congruence classes of $\equiv$, so we can define $[f, g]_{\mathcal{A}} : D_1 \oplus_{\mathcal{A}} D_2 \to D$ by

$$[f,g]_{\mathcal{A}}([U]) := [f,g]^*(U) \qquad \text{for } U \in \mathcal{A}(M_1 \oplus M_2).$$

Then $f = [f, g]_{\mathcal{A}} \circ \iota_1$ and $g = [f, g]_{\mathcal{A}} \circ \iota_2$, and the uniqueness of $[f, g]_{\mathcal{A}}$ can be shown from the uniqueness of $[f, g]$. □

From the uniqueness of coproducts up to isomorphism, one can derive:

Proposition 2. *$\mathcal{A}M_1 \oplus_{\mathcal{A}} \mathcal{A}M_2 \simeq \mathcal{A}(M_1 \oplus M_2)$ for monoids M_1, M_2.*

Proof. (Sketch) Let $f : (\mathcal{A}M_1 \times \mathcal{A}M_2)^+ \rightleftarrows (M_1 \times M_2)^+ : g$ map $\langle (A_i, B_i) \mid i \leq n\rangle$ to $\langle (\sum A_i, \sum B_i) \mid i \leq n\rangle$ and $\langle (a_i, b_i) \mid i \leq n\rangle$ to $\langle (\{a_i\}, \{b_i\}) \mid i \leq n\rangle$, respectively. Then $h : (\mathcal{A}M_1 \oplus_{\mathcal{A}} \mathcal{A}M_2) \rightleftarrows \mathcal{A}(M_1 \oplus M_2) : h^{-1}$, with $h([U]) := \widetilde{f}(U)$ for $U \in \mathcal{A}(\mathcal{A}M_1 \times \mathcal{A}M_2)^+$, and $h^{-1}(V) := [\widetilde{g}(V)]$ for $V \in \mathcal{A}(M_1 \oplus M_2)$, are $\mathcal{A}$-morphisms and inverse to each other. □

3.2 Freely Generated Objects and Free Extensions

An object C in a category *is freely generated by the set S* if there is a map $i : S \to C$ such that for all objects D and maps $s : S \to D$ there is a unique morphism $h_s : C \to D$ with $s = h_s \circ i$.

Example 5. In $\mathbb{M}$, the object freely generated by Q is the set Q^* of finite sequences of elements of Q with concatenation as product and the empty sequence as unit element. $i : Q \to Q^*$ maps q to the sequence of q's of length 1.

Proposition 3. *In the category $\mathbb{D}\mathcal{A}$, the object freely generated by the set Q is $\mathcal{A}Q^*$ with map $\eta \circ i : Q \to \mathcal{A}Q^*$, where $\eta : Q^* \to \mathcal{A}Q^*$ is $w \mapsto \{w\}$.*

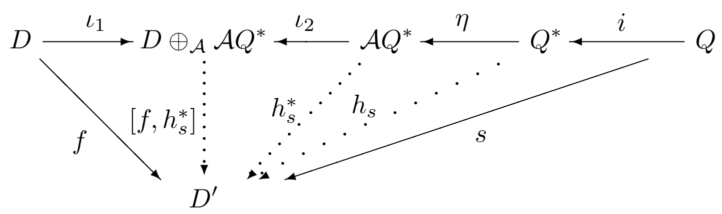
Proof. Let $D \in \mathbb{D}\mathcal{A}$ and $s : Q \to D$ be a map. As D is a monoid, there is a unique homomorphism $h_s : Q^* \to D$ with $s = h_s \circ i$. By Theorems I.3 and I.2, $\widetilde{h_s} : \mathcal{A}Q^* \to \mathcal{A}D$ and $\sum : \mathcal{A}D \to D$ are $\mathcal{A}$-morphisms. Let $h_s^* : \mathcal{A}Q^* \to D$ be their composition $\sum \circ \widetilde{h_s}$. Then $h_s(w) = \sum \widetilde{h_s}(\{w\}) = h_s^*(\{w\})$ for all $w \in Q^*$, so $s = h_s \circ i = h_s^* \circ (\eta \circ i)$. The uniqueness of h_s^* follows from that of h_s. □

A *free extension* of an object M by a set Q is an object $M[Q]$ with a morphism $\iota : M \to M[Q]$ and a map $\sigma : Q \to M[Q]$ such that for each morphism $f : M \to M'$ and map $s : Q \to M'$ there is a unique morphism $[f,s] : M[Q] \to M'$ with $f = [f,s] \circ \iota$ and $s = [f,s] \circ \sigma$. Again, the free extension of M by Q is unique up to an isomorphism.

Example 6. For monoids M, $\iota = \iota_1 : M \to M \oplus Q^* \leftarrow Q : \iota_2 \circ i = \sigma$ form a free extension of M by Q, where $\iota_1 : M \to M \oplus Q^* \leftarrow Q^* : \iota_2$ is the coproduct of M and Q^* and $i : Q \to Q^*$ the canonical embedding. Hence, $M[Q] \simeq M \oplus Q^*$.

Proposition 4. *In $\mathbb{D}\mathcal{A}$, the free extension $\iota : D \to D[Q] \leftarrow Q : \sigma$ of D by Q consists of the coproduct $D[Q] := D \oplus_{\mathcal{A}} \mathcal{A}Q^*$ of D and $\mathcal{A}Q^*$, with the embedding $\iota_1 : D \to D \oplus_{\mathcal{A}} \mathcal{A}Q^*$ as ι and the composition $\iota_2 \circ \eta \circ i$ of the maps $i : Q \to Q^*$, $\eta : Q^* \to \mathcal{A}Q^*$ with the embedding $\iota_2 : \mathcal{A}Q^* \to D \oplus_{\mathcal{A}} \mathcal{A}Q^*$ as σ.*

Proof. For an $\mathcal{A}$-morphism $f : D \to D'$ and map $s : Q \to D'$, construct an $\mathcal{A}$-morphism $[f,s] : D[Q] \to D'$ with $f = [f,s] \circ \iota$ and $s = [f,s] \circ \sigma$ from the unique homomorphism $h_s : Q^* \to D'$ with $s = h_s \circ i$, its unique extension to an $\mathcal{A}$-morphism $h_s^* : \mathcal{A}Q^* \to D'$ with $h_s = h_s^* \circ \eta$, and the unique $\mathcal{A}$-morphism $[f, h_s^*] : D \oplus_{\mathcal{A}} \mathcal{A}Q^* \to D'$ provided by the coproduct:



Putting $[f,s] := [f, h_s^*]$, $\iota := \iota_1$ and $\sigma := \iota_2 \circ \eta \circ i$, we get $f = [f,s] \circ \iota$ and $s = [f,s] \circ \sigma$. The uniqueness follows using Proposition 3 and Theorem 1. ☐

Corollary 1. $(\mathcal{A}M)[Q] = \mathcal{A}M \oplus_{\mathcal{A}} \mathcal{A}Q^* \simeq \mathcal{A}(M \oplus Q^*) = \mathcal{A}(M[Q])$.

4 Coequalizers and Tensor Products

4.1 Coequalizers and Quotients

A *coequalizer* of two morphisms $f, g : A \to B$ is an object Q with a morphism $q : B \to Q$ such that $q \circ f = q \circ g$ and for every morphism $q' : B \to Q'$ with $q' \circ f = q' \circ g$ there is a unique morphism $h_{q'} : Q \to Q'$ with $q' = h_{q'} \circ q$. By the universal property, coequalizers are unique up to isomorphism.

Example 7. In the category $\mathbb{M}$, a coequalizer of $f, g : N \to M$ consists of the quotient monoid $M/_{\equiv_{f,g}}$ with the canonical map $m \mapsto m/_{\equiv_{f,g}}$, where $\equiv_{f,g}$ is the least congruence E on M with $\{\,(f(n), g(n)) \mid n \in N\,\} \subseteq E$.

Conversely, if $E \subseteq M \times M$ is a congruence on M, the quotient monoid M/E with the canonical map $m \mapsto m/E$ is the coequalizer of the homomorphisms $f, g : N \to M$ where N is the submonoid of $M \times M$ with universe E and f, g are the restrictions of the projections $\pi_1, \pi_2 : M \times M \to M$ to N.

Likewise, in $\mathbb{D}\mathcal{A}$ coequalizers and quotients correspond to each other.

Theorem 2. *The category $\mathbb{D}\mathcal{A}$ has coequalizers.*

Proof. Let $f, g : A \to B$ be $\mathcal{A}$-morphisms between $\mathcal{A}$-dioids. Let ρ be the least $\mathcal{A}$-congruence on B above $\{\, (f(a), g(a)) \mid a \in A \,\}$, $Q := B/\rho$ and $q : B \to Q$ the canonical map, $b \mapsto b/\rho$. By Proposition 1, Q is an $\mathcal{A}$-dioid and q an $\mathcal{A}$-morphism. Clearly, $q \circ f = q \circ g$. Concerning the universal property, let $q' : B \to Q'$ be an $\mathcal{A}$-morphism with $q' \circ f = q' \circ g$. To define $h : Q \to Q'$, we put $h(b/\rho) := q'(b)$; this is well-defined by Lemma 1, since if $(b, b') = (f(a), g(a))$ for some $a \in A$, $q'(b) = q'(b')$ holds. Clearly $q' = h \circ q$, and h is an $\mathcal{A}$-morphism, because q' is; in particular, for $U \in \mathcal{A}B$,

$$h(\sum(U/\rho)) = h((\sum U)/\rho) = q'(\sum U) = \sum\{\, h(b/\rho) \mid b \in U \,\} = \sum \widetilde{h}(U/\rho).$$

As $q : B \to Q = B/\rho$ is surjective, the h with $q' = h \circ q$ is unique. □

Corollary 2. *The category $\mathbb{D}\mathcal{A}$ has colimits.*

This follows from the existence of coproducts and coequalizers, see [11], p.24.

Proposition 5. *Let D be an $\mathcal{A}$-dioid and ρ an $\mathcal{A}$-congruence on D. There are an $\mathcal{A}$-dioid N and two $\mathcal{A}$-morphisms $f, g : N \to D$ such that ρ is the least $\mathcal{A}$-congruence on D above $\{\, (f(n), g(n)) \mid n \in N \,\}$ and D/ρ with $d \mapsto d/\rho$ is a coequalizer of $f, g : N \to D$.*

Since we don't make use of this fact below, we omit the proof.

Proposition 6. *Suppose $\pi : M \to Q$ is a coequalizer of $f, g : N \to M$ in $\mathbb{M}$. Then $\mathcal{A}\pi : \mathcal{A}M \to \mathcal{A}Q$ is a coequalizer of $\mathcal{A}f, \mathcal{A}g : \mathcal{A}N \to \mathcal{A}M$ in $\mathbb{D}\mathcal{A}$.*

Proof. As coequalizers are unique up to isomorphism, we can assume Q is M/E and π is $m \mapsto m/E$, where E is the least congruence on M above $\{\, (f(n), g(n)) \mid n \in N \,\}$. We return to our abbreviation $\widetilde{f}$ for $\mathcal{A}f$ etc. Clearly, $\widetilde{\pi} \circ \widetilde{f} = \widetilde{\pi} \circ \widetilde{g}$ follows from the assumption $\pi \circ f = \pi \circ g$.

To show the universal property, let $\pi' : \mathcal{A}M \to Q'$ be an $\mathcal{A}$-morphism with $\pi' \circ \widetilde{f} = \pi' \circ \widetilde{g}$. Since π is surjective, so is $\widetilde{\pi} : \mathcal{A}M \to \mathcal{A}Q$, by Theorem I.9, and there can be at most one $\mathcal{A}$-morphism $h : \mathcal{A}Q \to Q'$ with $\pi' = h \circ \widetilde{\pi}$. As E is the closure of $\{\, (f(n), g(n)) \mid n \in N \,\}$ under reflexivity, symmetry, transitivity and monoid-congruence, one sees by induction that if $(m, m') \in E$, then $\pi'(\{m\}) = \pi'(\{m'\})$, using $\pi' \circ \widetilde{f} = \pi' \circ \widetilde{g}$ in the base case. Since $m \mapsto \{m\}$ is a homomorphism, $\{\, \{m\} \mid m \in B \,\} \in \mathcal{A}\mathcal{A}M$, so

$$\pi'(B) = \pi'(\bigcup\{\, \{m\} \mid m \in B \,\}) = \sum\{\, \pi'(\{m\}) \mid m \in B \,\}.$$

On $U = B/E = \widetilde{\pi}(B) \in \mathcal{A}Q$ with $B \in \mathcal{A}M$, put $h(B/E) := \pi'(B)$. This is well-defined, since $\pi'(\{m\}) = \pi'(\{m'\})$ for $m/E = m'/E$. Finally, for $\mathcal{U} \in \mathcal{A}\mathcal{A}(M/E)$ there is $\mathcal{V} \in \mathcal{A}\mathcal{A}M$ with $\mathcal{U} = \{\, V/E \mid V \in \mathcal{V} \,\}$, so h is an $\mathcal{A}$-morphism:

$$\begin{aligned} h(\bigcup\mathcal{U}) &= h((\bigcup\mathcal{V})/E) = \pi'(\bigcup\mathcal{V}) = \sum\{\, \pi'(V) \mid V \in \mathcal{V} \,\} \\ &= \sum\{\, h(V/E) \mid V \in \mathcal{V} \,\} = \sum \widetilde{h}(\mathcal{U}). \end{aligned}$$ □

Theorem 3. *Let E be a congruence on the monoid M, $\mathcal{A}E$ the least $\mathcal{A}$-congruence on $\mathcal{A}M$ above $\{ (\{m\}, \{m'\}) \mid (m, m') \in E \}$. Then $\mathcal{A}M/\mathcal{A}E \simeq \mathcal{A}(M/E)$.*

Proof. By Example 7, there are a monoid N and homomorphisms $f, g : N \to M$ such that $\pi : M \to M/E$ is the coequalizer of $f, g : N \to M$ and E is the least congruence on M above $\{ (f(n), g(n)) \mid n \in N \}$. We show that the canonical map $c : \mathcal{A}M \to \mathcal{A}M/\mathcal{A}E$ is a coequalizer of $\widetilde{f}, \widetilde{g} : \mathcal{A}N \to \mathcal{A}M$. Then, by the uniqueness of coequalizers and Proposition 6, $\mathcal{A}M/\mathcal{A}E \simeq \mathcal{A}(M/E)$. Write $[U]$ for the $\mathcal{A}E$-congruence class $c(U)$ of $U \in \mathcal{A}M$.

First, $c \circ \widetilde{f} = c \circ \widetilde{g}$: By Proposition 1, c is an $\mathcal{A}$-morphism, so for $A \in \mathcal{A}N$, from $\{ \{n\} \mid n \in A \} \in \mathcal{A}\mathcal{A}N$ we get

$$(c \circ \widetilde{f})(A) = \bigcup \{ (c \circ \widetilde{f})(\{n\}) \mid n \in A \} = \bigcup \{ [\{f(n)\}] \mid n \in A \}.$$

Therefore, it is sufficient to show $[\{f(n)\}] = [\{g(n)\}]$ for each $n \in N$. But since $(f(n), g(n)) \in E$, we have $(\{f(n)\}, \{g(n)\}) \in \mathcal{A}E$.

Second, $c : \mathcal{A}M \to \mathcal{A}M/\mathcal{A}E$ has the universal property for coequalizers of $\widetilde{f}, \widetilde{g}$: Let $q : \mathcal{A}M \to Q$ be an $\mathcal{A}$-morphism with $q \circ \widetilde{f} = q \circ \widetilde{g}$. We have to show that q uniquely factors through c. By Proposition 6, $\widetilde{\pi} : \mathcal{A}M \to \mathcal{A}(M/E)$ is a coequalizer of $\widetilde{f}, \widetilde{g} : \mathcal{A}N \to \mathcal{A}M$. As $q \circ \widetilde{f} = q \circ \widetilde{g}$, there is a unique $\mathcal{A}$-morphism h_q with $q = h_q \circ \widetilde{\pi}$. We show that $\widetilde{\pi}$ and hence q are constant on congruence classes of $\mathcal{A}E$, so that by

$$h([U]) := q(U), \quad \text{for } U \in \mathcal{A}M,$$

$h : \mathcal{A}M/\mathcal{A}E \to Q$ is well-defined. By Lemma 1, $\widetilde{\pi}$ is constant on $\mathcal{A}E$-congruence classes, if $\widetilde{\pi}(U) = \widetilde{\pi}(U')$ for all $(U, U') \in \{ (\{m\}, \{m'\}) \mid (m, m') \in E \}$. But in this case, $(m, m') \in E$ gives $\widetilde{\pi}(U) = U/E = \{m/E\} = \{m'/E\} = U'/E = \widetilde{\pi}(U')$.

Finally, as c is surjective, for every $\mathcal{V} \in \mathcal{A}(\mathcal{A}M/\mathcal{A}E)$ there is $\mathcal{U} \in \mathcal{A}\mathcal{A}M$ with $\mathcal{V} = \{ [U] \mid U \in \mathcal{U} \}$; therefore, h is an $\mathcal{A}$-morphism:

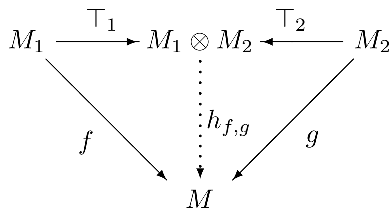
$$h(\sum \mathcal{V}) = h([\bigcup \mathcal{U}]) = q(\bigcup \mathcal{U}) = \sum \{ q(U) \mid U \in \mathcal{U} \} = \sum \widetilde{h}(\mathcal{V}).$$

Since c is surjective, h is the unique $\mathcal{A}$-morphism with $q = h \circ c$. □

4.2 Tensor Products

Two monoid-homomorphisms $f : M_1 \to M$ and $g : M_2 \to M$ are *relatively commuting*, if for all $m_1 \in M_1$ and $m_2 \in M_2$, $f(m_1)g(m_2) = g(m_2)f(m_1)$.

In a category whose objects have a monoid structure, a *tensor product* of two objects M_1 and M_2 is an object $M_1 \otimes M_2$ with two relatively commuting morphisms $\top_1 : M_1 \to M_1 \otimes M_2$ and $\top_2 : M_2 \to M_1 \otimes M_2$ such that for any pair of relatively commuting morphisms $f : M_1 \to M$ and $g : M_2 \to M$ there is a unique morphism $h_{f,g} : M_1 \otimes M_2 \to M$ with $f = h_{f,g} \circ \top_1$ and $g = h_{f,g} \circ \top_2$:



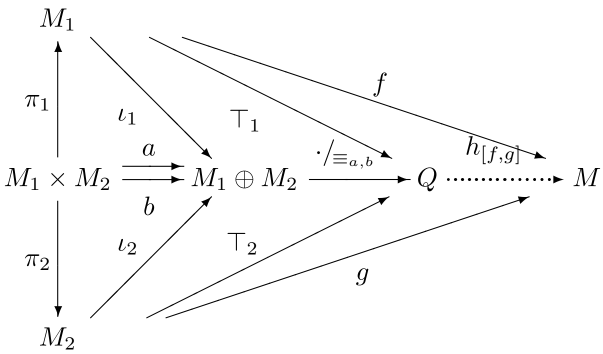
Intuitively, a tensor product is a free extension of both objects in which elements of one commute with elements of the other.

Example 8. A tensor product $\top_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \top_2$ of two monoids M_1 and M_2 can be constructed as the coequalizer $(M_1 \oplus M_2)/_{\equiv_{a,b}}$ of the homomorphisms $a, b : M_1 \times M_2 \to M_1 \oplus M_2$ defined by

$$a(m_1, m_2) = (m_1, 1)(1, m_2), \qquad b(m_1, m_2) = (1, m_2)(m_1, 1),$$

with the embeddings $\top_1(m_1) = (m_1, 1)/_{\equiv_{a,b}}$, $\top_2(m_2) = (1, m_2)/_{\equiv_{a,b}}$.

Proof. By the universal property of the coproduct $\iota_1 : M_1 \to M_1 \oplus M_2 \leftarrow M_2 : \iota_2$, there is a unique homomorphism $[f, g] : M_1 \oplus M_2 \to M$ with $f = [f, g] \circ \iota_1$ and $g = [f, g] \circ \iota_2$. Since f and g are relatively commuting, $[f, g] \circ a = [f, g] \circ b$. Hence, by the universal property of the coequalizer $\cdot/_{\equiv_{a,b}} : M_1 \oplus M_2 \to Q$ of a, b there is a unique homomorphism $h_{[f,g]} : Q \to M$ such that $[f, g] = h_{[f,g]} \circ (\cdot/_{\equiv_{a,b}})$.



It follows that $f = [f, g] \circ \iota_1 = h_{[f,g]} \circ (\cdot/_{\equiv_{a,b}}) \circ \iota_1 = h_{[f,g]} \circ \top_1$, likewise $g = h_{[f,g]} \circ \top_2$. Thus, $h_{f,g} := h_{[f,g]}$ is the induced homomorphism for f, g. □

To obtain a tensor product for $\mathcal{A}$-dioids, we can lift this construction to $\mathbb{D}\mathcal{A}$.

Theorem 4. *A tensor product $\top_1 : D_1 \to D_1 \otimes_{\mathcal{A}} D_2 \leftarrow D_2 : \top_2$ of $\mathcal{A}$-dioids D_1 and D_2 can be obtained from the tensor product $\hat{\top}_1 : M_1 \to M_1 \otimes M_2 \leftarrow M_2 : \hat{\top}_2$ of the monoids M_1, M_2 underlying D_1, D_2 by taking $D_1 \otimes_{\mathcal{A}} D_2 := \mathcal{A}(M_1 \otimes M_2)/_{\equiv}$ and $\top_k = \pi \circ \eta \circ \hat{\top}_k$, $k = 1, 2$, where π is the canonical map $U \mapsto U/_{\equiv}$ and $\equiv$ is the least $\mathcal{A}$-congruence on $\mathcal{A}(M_1 \otimes M_2)$ containing*

$$(\{(\sum A, \sum B)\}, A \times B), \text{ for all } A \in \mathcal{A}D_1, B \in \mathcal{A}D_2.$$

Proof. The proof is analogous to the proof of Theorem 1. We write $[U]$ for $\pi(U)$, where $U \in \mathcal{A}(M_1 \otimes M_2) = \mathcal{A}(M_1 \times M_2)$. $\top_1 : D_1 \to D_1 \otimes_{\mathcal{A}} D_2$ is an $\mathcal{A}$-morphism, and so is $\top_2$, because $\top_1$ is obviously a homomorphism and for $A \in \mathcal{A}D_1$,

$$\begin{aligned}\top_1(\sum A) &= [\{\hat{\top}_1(\sum A)\}] = [\{(\sum A, 1)\}] = [A \times \{1\}] \\ &= \sum\{\, [\{(a,1)\}] \mid a \in A\,\} = \sum\{\, [\hat{\top}_1(a)] \mid a \in A\,\} = \sum \widetilde{\top_1}(A).\end{aligned}$$

Let $f : D_1 \to D$ and $g : D_2 \to D$ be relatively commuting $\mathcal{A}$-morphisms. By the universal property for the tensor product in $\mathbb{M}$, there is a unique homomorphism $\hat{h}_{f,g} : M_1 \otimes M_2 \to D$ such that $f = \hat{h}_{f,g} \circ \hat{\top}_1$ and $g = \hat{h}_{f,g} \circ \hat{\top}_2$. By Theorem I.4, $\hat{h}_{f,g}$ extends uniquely to an $\mathcal{A}$-morphism $\hat{h}^*_{f,g} : \mathcal{A}(M_1 \otimes M_2) \to D$ with $\hat{h}_{f,g} = \hat{h}^*_{f,g} \circ \eta$, where for $U \in \mathcal{A}(M_1 \otimes M_2)$,

$$\hat{h}^*_{f,g}(U) = \sum \widetilde{\hat{h}_{f,g}}(U) = \sum\{\, \hat{h}_{f,g}(m_1, m_2) \mid (m_1, m_2) \in U\,\}.$$

Define $h_{f,g} : D_1 \otimes_{\mathcal{A}} D_2 \to D$ by

$$h_{f,g}([U]) := \hat{h}^*_{f,g}(U) = \sum \widetilde{\hat{h}_{f,g}}(U), \qquad \text{for } U \in \mathcal{A}(M_1 \otimes M_2).$$

As for $[f,g]^*$ in the proof of Theorem 1 one sees that $\hat{h}^*_{f,g}$ is constant on $\equiv$-classes, so $h_{f,g}$ is well-defined. Then for $d \in D_1$,

$$(h_{f,g} \circ \top_1)(d) = h_{f,g}([\{\hat{\top}_1(d)\}]) = \hat{h}_{f,g}(\hat{\top}_1(d)) = f(d),$$

and likewise $h_{f,g} \circ \top_2 = g$. The uniqueness of $h_{f,g}$ follows from the surjectivity of π and the uniqueness properties of $\hat{h}^*_{f,g}$ and $\hat{h}_{f,g}$. □

For $d_1 \in D_1$ and $d_2 \in D_2$, let $d_1 \otimes d_2$ be $[\{(d_1, d_2)\}] \in D_1 \otimes_{\mathcal{A}} D_2$. The elements of $D_1 \otimes_{\mathcal{A}} D_2$ can be written as

$$\sum\{\, d_1 \otimes d_2 \mid (d_1, d_2) \in U\,\}, \quad U \in \mathcal{A}(D_1 \times D_2).$$

Remark 2. A slightly different tensor product for $\mathbb{D}\mathcal{P}$, the quantales with unit, has been constructed by Liang [17], admitting lattice operations and using bimorphisms rather than relatively commuting morphisms.

Proposition 7. *$\mathcal{A}M_1 \otimes_{\mathcal{A}} \mathcal{A}M_2 \simeq \mathcal{A}(M_1 \otimes M_2)$ for monoids M_1, M_2.*

Proof. (Sketch) Define $h : (\mathcal{A}M_1 \otimes_{\mathcal{A}} \mathcal{A}M_2) \rightleftarrows \mathcal{A}(M_1 \otimes M_2) : h^{-1}$ by

$$\begin{aligned}[U] &\mapsto \{\, (\sum A, \sum B) \mid (A,B) \in U\,\}, \quad U \in \mathcal{A}(\mathcal{A}M_1 \times \mathcal{A}M_2) \\ V &\mapsto [\{\, (\{a\},\{b\}) \mid (a,b) \in V\,\}], \quad V \in \mathcal{A}(M_1 \otimes M_2).\end{aligned}$$ □

5 Applications

We note that if the dioid $D^{n\times n}$ of $n \times n$ square matrices of an $\mathcal{A}$-dioid D is an $\mathcal{A}$-dioid, it is isomorphic to the tensor product $D \otimes_{\mathcal{A}} \mathbb{B}^{n\times n}$.

Proposition 8. *If $D \in \mathbb{D}\mathcal{A}$ and $D^{n\times n} \in \mathbb{D}\mathcal{A}$, then $D^{n\times n} \simeq D \otimes_{\mathcal{A}} \mathbb{B}^{n\times n}$.*

Proof. (Sketch) One shows that $I_D : D \to D^{n\times n} \leftarrow \mathbb{B}^{n\times n} : Id$ has the properties of a tensor product, where $I_D(d) := dI_n$ and $Id(B) = B$, for $d \in D$ and $B \in \mathbb{B}^{n\times n}$. If $f : D \to D' \leftarrow \mathbb{B}^{n\times n} : g$ are relatively commuting $\mathcal{A}$-morphisms,

$$h_{f,g}(A) := \sum\{\, f(A_{i,j})g(E_{(i,j)}) \mid i, j < n \,\}, \quad A \in D^{n\times n},$$

is the induced $\mathcal{A}$-morphism with $f = h_{f,g} \circ I_D$ and $g = h_{f,g} \circ Id$. Here, $E_{(i,j)} \in \mathbb{B}^{n\times n}$ is the matrix with 1 only in line i, row j. □

In general, to have $D^{n\times n} \in \mathbb{D}\mathcal{A}$, every $U \in \mathcal{A}D^{n\times n}$ must have a supremum, with $(\sum U)_{i,j} := \sum U_{i,j}$, where $U_{i,j} = \{\, A_{i,j} \mid A \in U \,\}$ for $i, j < n$. Thus, $\sum U$ exists if $U_{i,j} \in \mathcal{A}D$ for each $U \in \mathcal{A}D^{n\times n}$ and $i, j < n$.

Example 9. For the operators $\mathcal{A}$ of Example 1 ($\mathcal{F}$, $\mathcal{P}$) and Example 2 ($\mathcal{F}_\kappa$, $\mathcal{P}_\kappa$), $\mathbb{D}\mathcal{A}$ is closed under matrix ring formation, i.e. for each $D \in \mathbb{D}\mathcal{A}$ and $n \in \mathbb{N}$, $D^{n\times n} \in \mathbb{D}\mathcal{A}$. This also holds for operators $\mathcal{R}$ and $\mathcal{C}$ of Example 3 (cf. [10], [14]).

Finally, we return to the earlier attempt by Chomsky and Schützenberger to expand the programme of algebraization beyond the regular languages.

Example 10. Let $X = \{u, v\}$, $Y = \{b, d, p, q\}$ and view b, d and p, q as two pairs of brackets. Let $D \subseteq (X \cup Y)^*$ be the Dyck-language of strings with balanced brackets and $h : (X \cup Y)^* \to X^*$ the bracket-erasing homomorphism. By a theorem of Chomsky/Schützenberger [3], $\mathcal{C}X^* = \{\, h(R \cap D) \mid R \in \mathcal{R}(X \cup Y)^* \,\}$. The theorem admits a form relating $\mathcal{C}$ with $\otimes_{\mathcal{R}}$, which we indicate by an example.

Let $C = \mathcal{R}Y^*/\{bd = 1 = pq, bq = 0 = pd\}$ be the *polycyclic* $\mathcal{R}$-dioid, in which matching brackets reduce to 1 and bracket mismatches to 0; in the semiring equations $bq = 0$ etc., we use $y \in Y$ for $\{y\} \in \mathcal{R}Y^*$. Then the tensor product $\mathcal{R}X^* \otimes_{\mathcal{R}} C$ contains all context-free languages $L \in \mathcal{C}X^*$; for example, if $L = \{\, u^n v^n \mid n \in \mathbb{N} \,\}$, then L belongs to $\mathcal{R}X^* \otimes_{\mathcal{R}} C$, since by *-continuity

$$b(up)^*(qv)^*d = b(\sum_n (up)^n)(\sum_m (qv)^m)d = \sum_{n,m} u^n v^m b p^n q^m d = \sum_n u^n v^n,$$

where b, d, p, q and u, v stand for their images $1 \otimes \{b\}$, $\{u\} \otimes 1$ etc. in $\mathcal{R}X^* \otimes_{\mathcal{R}} C$.

The reduction of $\mathcal{C}X^*$ to $\mathcal{R}(X \cup Y)^*$ by Chomsky and Schützenberger can be improved to a reduction of $\mathcal{C}M$ to $\mathcal{R}M$ for arbitrary monoids M as follows:

Theorem 5. *For any monoid M, $\mathcal{C}M \simeq Z_{C_2}(\mathcal{R}M \otimes_{\mathcal{R}} C_2)$, where*

$$C_2 = \mathcal{R}\{b, d, p, q\}^*/\{bd = 1 = pq, bq = 0 = pd, db + qp = 1\}$$

and $Z_{C_2}(\mathcal{R}M \otimes_{\mathcal{R}} C_2)$ is the centralizer of C_2 in $\mathcal{R}M \otimes_{\mathcal{R}} C_2$, the set of those elements that commute with C_2 in $\mathcal{R}M \otimes_{\mathcal{R}} C_2$.

This generalizes to a construction of $\mathbb{DC}$ from $\mathbb{DR}$. The proofs have to be deferred to a future publication.

To give an idea, in the classical theorem of [3], each $L \in \mathcal{C}X^*$ has a *rational kernel* $\hat{L} \in \mathcal{R}(X \cup Y)^*$ in the extended alphabet $X \cup Y$, where intuitively, $\hat{L} \cap D$ consists of the strings in L with begin- and end-markers of phrases according to a context-free grammar for L (or the push- and pop-actions of a push-down automaton for L) inserted. In the general case, $L \in \mathcal{C}M$ is elementwise embedded in $\mathcal{R}M \otimes_{\mathcal{R}} C_2$ and gives a copy $\sum\{ \{m\} \otimes 1 \mid m \in L \} \in \mathcal{R}M \otimes_{\mathcal{R}} C_2$. It can be obtained from a regular kernel $\hat{L} \in \mathcal{R}(M \times Y^*) \simeq \mathcal{R}M \otimes_{\mathcal{R}} \mathcal{R}Y^*$ by performing in $\mathcal{R}Y^*$ analogs of the intersection with D and the bracket-erasure via calculations modulo the defining relations of C_2. The equation $db + qp = 1$ not present in C is needed to encode stack operations in C_2.

A similar reduction of $\mathcal{T}M$ to $\mathcal{R}M \otimes_{\mathcal{R}} C_2 \otimes_{\mathcal{R}} C_2$ can be made. To this end, a second copy of C_2 is used to enable computations of a 2-stack or Turing machine.

Open Problems

1. Can an operator $\mathcal{S}$ of context-sensitive subsets be provided by restricting to a subcategory of $\mathbb{M}$ in which erasing homomorphisms are excluded?
2. Are all categories $\mathbb{DA}$ closed under matrix ring formation, with a uniform proof?
3. Concerning transductions between monoids M and M', is the image of a set $A \in \mathcal{A}M$ under a relation $T \in \mathcal{A}(M \times M')$ a set in $\mathcal{A}M'$?

6 Conclusion

We have studied functors $\mathcal{A} : \mathbb{M} \to \mathbb{D}$ between the finite-subset functor $\mathcal{F}$ and the power set functor $\mathcal{P}$ that give rise to subcategories $\mathbb{DA}$ of $\mathbb{D}$ and are left adjoints of adjunctions $(\mathcal{A}, \widehat{\mathcal{A}}, \eta, \epsilon)$ between $\mathbb{M}$ and $\mathbb{DA}$, where $\widehat{\mathcal{A}} : \mathbb{DA} \to \mathbb{M}$ is the forgetful functor. Each $D \in \mathbb{DA}$ has a subset $\mathcal{A}D$ between $\mathcal{F}D$ and $\mathcal{P}D$ whose elements $U \in \mathcal{A}D$ have a least upper bound $\sum U \in D$ satisfying the distributivity property $(\sum U)(\sum V) = \sum(UV)$. Based on a notion of $\mathcal{A}$-congruence we have provided constructions of coproduct, coequalizers, quotient and tensor product by lifting corresponding constructions from $\mathbb{M}$ to $\mathbb{DA}$.

Do our results hold more generally or are they better expressed at a higher level of abstraction like universal algebra or category theory? A reviewer suggested that our $\mathbb{DA}$'s are infinitary quasi-varieties (or *prevarieties* in Bergman [2]) and, for example, the existence of coproducts follows from known results. As far as we see, many of our classes $\mathbb{DA}$ indeed are infinitary quasi-varieties, but probably not all. An *infinitary quasi-variety* is a class of algebras M that can be axiomatized by a set of equational implications

$$\forall \boldsymbol{x} \in M^I (\bigwedge \{ a_j(\boldsymbol{x}) = b_j(\boldsymbol{x}) \mid j \in J \} \to c(\boldsymbol{x}) = d(\boldsymbol{x})),$$

where I and J are sets and a_j, b_j, c, d are derived operations (resp. terms) of arity I. The idea is to to express the least-upper-bound property of $\sum U$ by

$$\bigwedge\{\, u \leq \sum U \mid u \in U \,\} \;\wedge\; \forall y \in M(\bigwedge\{\, u \leq y \mid u \in U \,\} \to \sum U \leq y)$$

and replace quantification over sets $U \in \mathcal{A}M$ by quantification over families $\boldsymbol{x} \in M^I$ of individuals, for a set I of size $|I| \geq |U|$. For an algebra signature with operations $\sum$ and projections π_i of arbitrary large arities I the least-upper-bound property of $\sum \boldsymbol{x}$ becomes an infinite equational implication. By quantifying over families $\boldsymbol{x}$ instead of subsets U, the classes of size-bounded subsets, $\mathbb{D}\mathcal{P}_\kappa$ and $\mathbb{D}\mathcal{F}_\kappa$ of Example 2, as well as $\mathbb{D}\mathcal{P}$ of Example 1 become infinitary quasi-varieties (in signatures of class size). Clearly, $\mathbb{D}\mathcal{F} = \mathbb{D}\mathcal{F}_{\aleph_0} = \mathbb{D}$, the idempotent semirings, form a variety (in the signature $+, \cdot, 0, 1$).

For $\mathbb{D}\mathcal{R}$ of Example 3, one can use the equational implications of Kozen's [9] axioms for Kleene algebra (in $0, 1, +, \cdot, ^*$), plus *-continuity of $\cdot$ in the form

$$\forall a, b, c, y[\bigwedge\{\, ab^n c \leq ab^* c \mid n \in \mathbb{N} \,\} \wedge (\bigwedge\{\, ab^n c \leq y \mid n \in \mathbb{N} \,\} \to ab^* c \leq y)].$$

Likewise, for $\mathbb{D}\mathcal{C}$ one can use the μ-continuity condition of [4] (say, with $|\boldsymbol{y}|$-ary Skolem functions $f_{\boldsymbol{x},\boldsymbol{p}}$ for each system of polynomial inequations $\boldsymbol{x} \geq \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$, and first-order terms $f_{\boldsymbol{p}(\boldsymbol{x},\boldsymbol{y})}(\boldsymbol{y})$ for $\mu \boldsymbol{x}\boldsymbol{p}$). Thus, $\mathbb{D}\mathcal{R}$ and $\mathbb{D}\mathcal{C}$ form infinitary quasi-varieties. For $\mathbb{D}\mathcal{T}$ we know of no such axiomatization.

For arbitrary $\mathcal{A}$, the $\mathcal{A}$-subsets $\mathcal{A}D$ on which $\sum$ is defined are not images under arbitrary maps of fixed sets I given by a signature. Hence it seems more appropriate to view the $\mathcal{A}$-dioids as two-sorted algebras $(D, +, \cdot, 0, 1\,;\, \mathcal{A}D, \sum_D)$ or as T-algebras $(D, \sum_D : TD \to D)$ of the Eilenberg-Moore category $\mathbb{M}^T$ of the monad $T = T_{\mathcal{A}}$ of the adjunction $(\mathcal{A}, \hat{\mathcal{A}}, \eta, \epsilon) : \mathbb{M} \to \mathbb{D}\mathcal{A}$. Embeddings that are T-algebra morphisms would give the right notion of $\mathcal{A}$-subdioids, but we don't know if there are Birkhoff-type theorems for T-algebras (cf. [1]) which show that the $\mathbb{D}\mathcal{A}$ are quasi-varieties in a generalized sense, and if so, whether this implies some of our results. Our lifting of closure properties of $\mathbb{M}$ to closure properties of the categories $\mathbb{D}\mathcal{A}$ is based on taking quotients of $\mathcal{A}$-dioids under $\mathcal{A}$-congruences. It may be possible to perform these liftings from a base category $\mathbb{C}$ with suitable notion of congruence to the category $\mathbb{C}^T$ under more general conditions.

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References

1. M. Barr. HSP subcategories of Eilenberg-Moore algebras. *Theory and Applications of Categories*, 18:461–468, 2002.
2. G. M. Bergman. On coproducts in varieties, quasivarieties and prevarieties. *Algebra and Number Theory 3*, pages 847–879, 2009.

3. N. Chomsky and M. Schuetzenberger. The algebraic theory of context free languages. In P. Braffort and D. Hirschberg, editors, *Computer Programming and Formal Systems*, pages 118–161, 1963.
4. N. B. B. Grathwohl, F. Henglein, and D. Kozen. Infinitary axiomatization of the equational theory of context-free languages. In D. Baelde and A. Carayol, editors, *Fixed Points in Computer Science (FICS 2013)*, volume 126 of *EPTCS*, pages 44–55, 2013.
5. J. Gruska. A characterization of context-free languages. *Journal of Computer and System Sciences*, 5:353 – 364, 1971.
6. J. Hopcroft and J. D. Ullman. *Formal Languages and their Relation to Automata.* Addison Wesley, Reading, Mass., 1979.
7. M. Hopkins. The algebraic approach I: The algebraization of the Chomsky hierarchy. In R. Berghammer, B. Möller, and G. Struth, editors, *Relational Methods in Computer Science/Applications of Kleene Algebra*, LNCS 4988, pages 155–172, Berlin Heidelberg, 2008. Springer Verlag.
8. M. Hopkins. The algebraic approach II: Dioids, quantales and monads. In R.Berghammer, B. Möller, and G. Struth, editors, *Relational Methods in Computer Science/Applications of Kleene Algebra*, LNCS 4988, pages 173–190, Berlin Heidelberg, 2008. Springer Verlag.
9. D. Kozen. On Kleene algebras and closed semirings. In B. Rovan, editor, *15th Int. Symp. on Mathematical Foundations of Computer Science, Banská Bystrica, 1990*, LNCS 452, pages 26–47. Springer Verlag, 1990.
10. D. Kozen. A completeness theorem for Kleene algebras and the algebra of regular events. *Information and Computation*, 110(2):366–390, 1994.
11. J. Lambek and P. Scott. *Introduction to Higher Order Categorical Logic.* Cambridge University Press, 1986.
12. S. Lang. *Algebra.* Addison-Wesley, 1965.
13. H. Leiß. Towards Kleene algebra with recursion. In E. Börger, G. Jäger, H. Kleine-Büning, and M. M. Richter, editors, *CSL '91. 5th Workshop on Computer Science Logic*, LNCS 626, pages 242–256. Springer Verlag, 1991.
14. H. Leiß. The matrix ring of a μ-continuous Chomsky algebra is μ-continuous. In L. Regnier and J.-M. Talbot, editors, *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, Leibniz International Proceedings in Informatics, pages 1–16. Leibniz-Zentrum für Informatik, Dagstuhl Publishing, 2016.
15. H. Leiß and Z. Ésik. Algebraically complete semirings and Greibach normal form. *Annals of Pure and Applied Logic*, 133:173–203, 2005.
16. H. Leiß and M. Hopkins. C-dioids and μ-continuous Chomsky algebras. In J. Desharnais, W. Guttmann, and S. Joosten, editors, *17th Int. Conf. on Relational and Algebraic Methods in Computer Science (RAMiCS-17).* Springer Verlag, 2018.
17. S. Liang. The tensor product of unital quantales. *Studies in Mathematical Sciences*, 7(1):15–21, 2013.
18. S. Mac Lane. *Categories for the Working Mathematician.* Springer-Verlag, New York Inc., 1971.
19. I. P. McWhirter. Substitution expressions. *Journal of Computer and System Sciences*, 5:629–637, 1971.
20. M. K. Yntema. Cap expressions for context-free languages. *Information and Control*, 8:311–318, 1971.