2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

Connecting the numerical scale model to the unbalanced linguistic term sets

Yucheng Dong, Cong-Cong Li, and Francisco Herrera

***Abstract*— Herrera and Martínez initiated a 2-tuple fuzzy linguistic representation model for computing with words (CWW). In addition to the Herrera and Martínez model, two different models based on linguistic 2-tuples (i.e., the model of Herrera et al. and the numerical scale model) have been developed to deal with term sets that are not uniformly and symmetrically distributed, i.e., unbalanced linguistic term sets (ULTSs). Both the model of Herrera et al. and the numerical scale model can deal with ULTSs, so a challenge is naturally proposed to analysts: how to compare these two different models. In this study, we provide a connection between the model of Herrera et al. and the numerical scale model. The results show that the model of Herrera et al. provides a new approach to set a numerical scale. Furthermore, we prove the equivalence of the linguistic computational models between the model of Herrera et al. and the numerical scale model, if the numerical scale is set based on the model of Herrera et al.**

I. INTRODUCTION

Using linguistic information in decision making problems implies the need for computing with words (CWW)[10], [17], [18], [25], [26], [38], [39], [40]. Several different linguistic computational models for CWW have been presented in [19], [23], [27], [33], [34], [37]. In particular, Herrera and Martínez[13] initiated the 2-tuple fuzzy linguistic representation model. The Herrera and Martínez model is well suited for dealing with linguistic term sets that are uniformly and symmetrically distributed, and with the result of this model matching the elements in the initial linguistic terms. The Herrera and Martínez model has been successfully used in a wide range of applications (e.g.,[1], [5], [6], [20], [22], [24], [28], [30]).

In recent years, two different models based on linguistic 2-tuples have been developed to deal with term sets that are not uniformly and symmetrically distributed, i.e., unbalanced linguistic term sets (ULTSs). The first model has been presented in Herrera et al.[12]. The model of Herrera et al. is based on the use of a linguistic hierarchy[4], [9], [14], [32] and the Herrera and Martínez model[13], and is also applied to information retrieval system[16], consensus models[2], [3], aggregation operators[15], [27], and olive oil sensory evaluation[21]. The second model has been presented and developed in Wang and Hao[29], [31] and Dong et al.[7], [8], which is called the numerical scale model in this study.

Yucheng Dong and Cong-Cong Li are with the Business School, Sichuan University, Chengdu 610065, China (email: ycdong@scu.edu.cn, congcongliscu@gmail.com).

Francisco Herrera is with the Department of Computer Science and Artificial Intelligence, University of Granada, Granada 18071, Spain. (email: herrera@decsai.ugr.es).

This work was supported by a grant (No. 71171160) from NSF of China.

Wang and Hao[29], [31] proposed a generalized version (i.e., the proportional 2-tuple fuzzy linguistic representation model) of the 2-tuple fuzzy linguistic representation model to deal with ULTSs. The Wang and Hao model is based on the concepts of symbolic proportion and the canonical characteristic values (CCVs) of linguistic terms. Traditional linguistic 2-tuples and proportional 2-tuples are used in the Herrera and Martínez model [13] and the Wang and Hao model[29], [31], respectively. By defining the concept of *numerical scale*, Dong et al.[7] proposed the numerical scale model based on traditional linguistic 2-tuples to deal with ULTSs. Dong et al.[7] proved that setting certain numerical scale in Dong et al.'s model[7] yields the Wang and Hao model.

As mentioned above, both the model of Herrera et al. and the numerical scale model can deal with ULTSs. So, a challenge is naturally proposed to analysts: how to compare these two different models. One aim of this study is to provide a connection between these two different models. The analytical results in this study show the equivalence of these two linguistic computational models (in some sense).

This paper is organized as follows. Section II introduces basic knowledge regarding linguistic 2-tuples setting numerical scales, and the model of Herrera et al. to deal with ULTSs. Section III shows the equivalence (in some sense) between the model of Herrera et al. and the numerical scale model in dealing with ULTSs. Based on this, Section IV proposes an illustrative example for the equivalence. Finally, Section V concludes the study.

II. PRELIMINARIES

This section introduces the basic knowledge regarding the linguistic 2-tuples setting numerical scales, and the model of Herrera et al. to deal with ULTSs.

A. The 2-tuple fuzzy linguistic representation model

The basic notations and symbolic operational laws of linguistic variables are introduced in [10], [11], [23], [25], [37]. Let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set with odd cardinality. The term s_i represents a possible value for a linguistic variable. The set S is ordered: $s_i > s_j$ if and only if $i > j$.

If the semantics of the elements in the linguistic term set are given by fuzzy numbers (defined in the [0,1] interval), then the middle term represents an assessment of "approximately 0.5". Similar to the study presented in Herrera et al.[12], this paper assumes that in the investigated linguistic

978-1-4799-2072-3/14/$31.00

term set exists a middle term . This paper denotes $s^* \in S$ as the middle term of S.

Herrera and Martínez[13] proposed the 2-tuple fuzzy linguistic representation model.

Definition 1: [13] Let $\beta \in [0,g]$ be a number in the granularity interval of the linguistic term set $S = \{s_0, ..., s_g\}$ and let $i = round(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0,g]$ and $\alpha \in [-0.5, 0.5)$. Then α is called a *symbolic translation*, with *round* being the usual *rounding* operation.

The Herrera and Martínez model represents the linguistic information by 2-tuples (s_i, α), where $s_i \in S$ and $\alpha \in [-0.5, 0.5)$. This linguistic model defines a function with the purpose of making transformations between linguistic 2-tuples and numerical values.

Definition 2: [13] Let $S = \{s_0, ..., s_g\}$ be a linguistic term set and $\beta \in [0,g]$ a value representing the result of a symbolic aggregation operation, then the 2-tuple that expresses the equivalent information to β is obtained with the following function: $\Delta : [0,g] \to S \times [-0.5, 0.5)$, where

$$\Delta(\beta) = (s_i, \alpha), with \begin{cases} s_i, i = round(\beta) \\ \alpha = \beta - i, \alpha \in [-0.5, 0.5) \end{cases}.$$

Clearly, Δ is a one to one mapping function. For convenience, its range is denoted as $\overline{S}$. Then, Δ has an inverse function with $\Delta^{-1} : \overline{S} \to [0,g]$ with $\Delta^{-1}((s_i, \alpha)) = i + \alpha$.

A computational model has been developed for the Herrera and Martínez model, in which exists the following:

1) A 2-tuple comparison operator: Let (s_k, α) and (s_l, γ) be two 2-tuples. Then:

(1) if $k < l$, then (s_k, α) is smaller than (s_l, γ).

(2) if $k = l$, then

a) if $\alpha = \gamma$, then (s_k, α), (s_l, γ) represents the same information.

b) if $\alpha < \gamma$, then (s_k, α) is smaller than (s_l, γ).

2) A 2-tuple negation operator:

$$Neg((s_i, \alpha)) = \Delta(g - (\Delta^{-1}(s_i, \alpha))). \quad (1)$$

3) Several 2-tuple aggregation operators have been developed (see [13], [23]). For example, let $L = \{l_1, ..., l_m\}$, where $l_i \in \overline{S}$ be a set of terms to aggregate, and let $w = \{w_1, w_2, ..., w_m\}$ be an associated weighting vector that satisfies $w_i \geq 0$ and $\sum_{i=1}^{m} w_i = 1$. Then, based on the ordered weighted averaging (OWA) operator[36], the 2-tuple ordered weighted averaging (TOWA) operator in the Herrera and Martínez model is computed as

$$TOWA_w(l_1, ..., l_m) = \Delta\left(\sum_{i=1}^{m} w_i y_i^*\right), \quad (2)$$

where y_i^* is the ith largest of the y_i values, and $y_i = \Delta^{-1}(l_i)$.

Let $s_i, s_j \in S$ be two simple terms. Xu [35] defined the deviation measure between s_i and s_j as follows: $d(s_i, s_j) = \frac{|i-j|}{g+1}$. For linguistic 2-tuples $(s_i, \alpha), (s_j, \gamma) \in \overline{S}$, Dong et al.[5] similarly defined the deviation measure between (s_i, α) and (s_j, γ) as follows: $d((s_i, \alpha), (s_j, \gamma)) = \frac{|\Delta^{-1}((s_i, \alpha)) - \Delta^{-1}((s_j, \gamma))|}{g+1}$. If only one pre-established linguistic label set is used in a decision making model, Dong et al.[5] simply considered

$$d((s_i, \alpha), (s_j, \gamma)) = |\Delta^{-1}((s_i, \alpha)) - \Delta^{-1}((s_j, \gamma))|. \quad (3)$$

B. Numerical scale model

The Herrera and Martínez model is well suited for dealing with linguistic term sets that are uniformly and symmetrically distributed. By defining the concept of *numerical scale*, Dong et al.[7] proposed an extension of the 2-tuple fuzzy linguistic representation model to deal with ULTSs.

Definition 3: [7] Let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be a linguistic term set, and R be the set of real numbers. The function: $NS : S \to R$ is defined as a numerical scale of S, and $NS(s_i, 0)$ is called the numerical index of s_i. If the function NS is strictly monotone increasing, then NS is called an ordered numerical scale.

Note 1. In this paper, for notation simplicity, the numerical index of s_i, $NS(s_i, 0)$, will be simplified as $NS(s_i)$. Similarly, in the following, $\Delta^{-1}(s_i, 0)$ will also be simplified as $\Delta^{-1}(s_i)$.

Definition 4: [7] Let $S, \overline{S}$ and NS on S be as before. For $(s_i, \alpha) \in \overline{S}$, the numerical scale $\overline{NS}$ on $\overline{S}$ is defined by

$$\overline{NS}((s_i, \alpha)) = \begin{cases} NS(s_i) + \alpha \times (NS(s_{i+1}) - NS(s_i)), & \alpha \geq 0 \\ NS(s_i) + \alpha \times (NS(s_i) - NS(s_{i-1})), & \alpha < 0 \end{cases}. \quad (4)$$

For notation simplicity, $\overline{NS}$ will also be denoted as NS in this study.

Proposition 1: [7] Setting $NS(s_i) = i$ $(i = 0, 1, ..., g)$ yields the Herrera and Martínez model.

In general, the semantics of the elements in a linguistic term set are given by fuzzy numbers (defined in the [0,1] interval), which are described by linear triangular membership functions or linear trapezoidal membership functions. For instance, the linear trapezoidal membership function is achieved by the 4-tuple (a, b, c, d), b and c indicate the interval in which the membership value is 1, and a and d are the left and right limits of the definition domain of a trapezoidal membership function. Fuzzy numbers with trapezoidal membership functions are denoted by $T[a, b, c, d]$. Wang and Hao [29] proposed an interesting generalized version of the 2-tuple fuzzy linguistic representation model. The semantics of linguistic terms used in the Wang and Hao model are defined by symmetrical trapezoidal fuzzy numbers. If the semantic of s_i is defined by $T[b_i - \sigma_i, b_i, c_i, c_i + \sigma_i]$, In the Wang and Hao model the canonical characteristic value (CCV) of s_i is $\frac{b_i + c_i}{2}$, i.e., $CCV(s_i) = \frac{b_i + c_i}{2}$.

Proposition 2: [7] Setting $NS(s_i) = CCV(s_i)$ $(i = 0, 1, ..., g)$ yields the Wang and Hao model.

Propositions 1 and 2 provide good linkage of the numerical scale to the Herrera and Martínez model and also the Wang and Hao model.

Note 2. Traditional 2-tuples and proportional 2-tuples are used in the Herrera and Martínez model and the Wang and Hao model, respectively. By setting the numerical scale $NS(s_i) = CCV(s_i)$ $(i = 0, 1, ..., g)$, Dong et al.[7] showed the Wang and Hao model can be redescribed as a linguistic model based on traditional 2-tuples. For notation simplicity, we use traditional 2-tuples throughout this study.

C. The model of Herrera et al. to deal with ULTSs

The model of Herrera et al.[12] to deal with ULTSs is based on linguistic hierarchy and the Herrera and Martínez model. Linguistic hierarchy has been presented and developed in[4], [9], [14], [32]. Over the linguistic hierarchy a computational symbolic model based on the 2-tuple is defined to accomplish processes of CWW[14]. A linguistic hierarchy, LH, is the union of all levels t: $LH = \bigcup_t l(t, n(t))$, where each level t of a LH corresponds to a linguistic term set with a granularity of uncertainty of $n(t)$ denoted as: $l(t, n(t)) = S^{n(t)} = \{s_0^{n(t)}, ..., s_{n(t)-1}^{n(t)}\}$. Furthermore, the linguistic term set of the level $t+1$ is obtained from its predecessor as $l(t, n(t)) \rightarrow l(t+1, 2n(t)-1)$.

Transformation functions between terms from different levels to make processes of CWW in multigranular linguistic information contexts were presented as Definition 5.

Definition 5: [14] Let $LH = \bigcup_t l(t, n(t))$ be a linguistic hierarchy whose linguistic term sets are denoted as $S^{n(t)} = \{s_0^{n(t)}, ..., s_{n(t)-1}^{n(t)}\}$, and let us consider the 2-tuple fuzzy linguistic representation. The transformation function from a linguistic term in level t to a term in level t' is defined as $TF_{t'}^t : l(t, n(t)) \rightarrow l(t', n(t'))$ such that

$$TF_{t'}^t(s_i^{n(t)}, \alpha) = \Delta\left(\frac{\Delta^{-1}(s_i^{n(t)}, \alpha) \times (n(t') - 1)}{n(t) - 1}\right). \quad (5)$$

Example 1. Let $LH = l(1,3) \cup l(3,9) \cup l(4,17) = \{(s_0^3, s_1^3, s_2^3) \cup (s_0^5, s_1^5, ..., s_4^5) \cup (s_0^9, s_1^9, ..., s_8^9) \cup (s_0^{17}, s_1^{17}, ..., s_{16}^{17})\}$. Then, $TF_2^3(s_6^9, 0.4) = \Delta\left(\frac{\Delta^{-1}(s_6^9, 0.4) \times (5-1)}{9-1}\right) = \Delta(3.2) = (s_3^5, 0.2)$.

Generally, in the computational model defined for the linguistic hierarchy LH, "any" level in the LH may be selected to unify the multigranular linguistic information in the computational model defined for the linguistic hierarchy LH. In this study, the maximum level t_m in the LH is used, i.e., $l(t_m, n(t_m)) = S^{n(t_m)} = \{s_0^{n(t_m)}, ..., s_{n(t_m)-1}^{n(t_m)}\}$.

Let $S = \{s_i | i = 0, 1, 2, ..., g\}$ be an ULTS. In the model of Herrera et al.[12] to deal with ULTSs, the process of CWW is based on the Herrera and Martínez model and the use of linguistic hierarchies.

(1) Representation in the linguistic hierarchy: The representation algorithm uses a linguistic hierarchy, LH, to model the unbalanced terms in S. Therefore, the first step to accomplish processes of CWW is to transform the unbalanced terms in S into their correspondent terms in the LH, $s_k^{n(t)} \in LH = \bigcup_t l(t, n(t))$, by using the transformation function, Ψ, that associates with each unbalanced linguistic 2-tuple (s_i, α) its respective linguistic 2-tuple in $LH(\overline{S})$, i.e.,

$$\Psi : \overline{S} \rightarrow LH(\overline{S}), \quad (6)$$

such that $\Psi(s_i, \alpha) = (s_{I(i)}^{G(i)}, \alpha)$, for $\forall\ (s_i, \alpha) \in \overline{S}$.

(2) Computational phase: It accomplishes the process of CWW by using the computational model defined for the linguistic hierarchy. Firstly, using the Eq. (5) transforms $(s_{I(i)}^{G(i)}, \alpha)$ $(i = 0, 1, ..., g)$ into linguistic 2-tuples, denoted as $(s_{I'(i)}^{n(t_m)}, \lambda)$, in $\overline{S^{n(t_m)}}$. Without loss of generality, if $G(i) = n(t')$, then Eq. (7) is obtained. Next, the computational model developed for the Herrera and Martínez model is used over $\overline{S^{n(t_m)}}$. Then, the result is obtained, denoted as $(s_r^{n(t_m)}, \lambda_r) \in \overline{S^{n(t_m)}}$.

(3) Retranslation process: A retranslation process is used to transform the result $(s_r^{n(t_m)}, \lambda_r) \in \overline{S^{n(t_m)}}$ into the unbalanced term in $\overline{S}$, by the transformation function, Ψ^{-1}, i.e.,

$$\Psi^{-1} : LH(\overline{S}) \rightarrow \overline{S}. \quad (8)$$

such that $\Psi^{-1}(s_r^{n(t_m)}, \lambda_r) = (s_{result}, \lambda_{result}) \in \overline{S}$.

Note 3. Without loss of generality, if $G(i) = n(t')$, let $s_{I'(i)}^{n(t_m)} = TF_{t_m}^{t'}(s_{I(i)}^{G(i)})$. Throughout this study, we denote $s_{I(i)}^{G(i)}$ and $s_{I'(i)}^{n(t_m)}$ as the corresponding terms in $S^{G(i)}$ and $S^{n(t_m)}$, associated with $s_i \in S$, respectively. We also denote $(s_J^{n(t_m)}, \lambda)$ as the corresponding 2-tuple term in $\overline{S^{n(t_m)}}$, associated with $(s_i, \alpha) \in \overline{S}$. Moreover, in the model of Herrera et al.[12], the boolean function $Brid$: $S \rightarrow \{False, True\}$ has been defined. If $Brid(s_i) = True$, the semantic representation of s_i is achieved from two levels in LH.

III. Connecting the numerical scale model to the ULTS model

In this section, we will provide a connection between the model of Herrera et al.[12] and the numerical scale model[7]. Specifically, ULTSs are redefined in Section III.A, a revised retranslation process for the model of Herrera et al. is proposed in Section III.B, and the equivalence between the model of Herrera et al. and the numerical scale model are analyzed in Section III.C.

A. Definition of unbalanced linguistic term sets

In the model of Herrera et al.[12] and the Wang and Hao model[29], ULTSs are defined in different ways. Specifically, the concept of the middle term and the concept of equally informative CCVs are used in these two models, respectively. For unified notation, inspired by the middle term used in Herrera et al.[12] and equally informative CCVs presented in Wang and Hao[29], this paper redefines ULTSs based on numerical scales (see Definition 6).

Definition 6: Let $S = \{s_0, s_1, ..., s_g\}$, s^* and NS on S be as before. S is a linguistic term set that is uniformly and symmetrically distributed, if the following two conditions are satisfied:

(1) There exists a unique constant $\lambda > 0$ such that $NS(s_i) - NS(s_j) = \lambda(i - j)$ for all $i, j = 0, 1, ..., g$.

(2) Let $\overline{\overline{S}} = \{s | s \in S, s > s^*\}$ and $= \{s | s \in S, s < s^*\}$. Let $\#\overline{\overline{S}}$ and $\#(\underline{\underline{S}})$ be the cardinality of $\overline{\overline{S}}$ and $\underline{\underline{S}}$, respectively. Then, $\#\overline{\overline{S}} = \#(\underline{\underline{S}})$.

If S is an uniformly and symmetrically distributed term set, then S is called a balanced linguistic term set (with respect

$$(s_J^{n(t_m)}, \lambda) = TF_{t_m}^{t'}(s_{I(i)}^{G(i)}, \alpha) = \Delta \left(\frac{\Delta^{-1}(s_{I(i)}^{G(i)}, \alpha).(n(t_m) - 1)}{G(i) - 1} \right). \quad (7)$$

to NS). Otherwise, S is called an unbalanced linguistic term set (ULTS).

Clearly, the ULTSs in both the model of Herrera et al.[12] and the Wang and Hao model[29] satisfy this new definition.

B. The revised retranslation process in the model of Herrera et al.

In the model of Herrera et al.[12], a retranslation process is used to transform the terms in LH into the terms in the ULTS S. Here, we provide a revised retranslation process, providing a basis for connecting the model of Herrera et al. to the numerical scale model. Meanwhile, we will show that the results, obtained by the revised retranslation process, are same to the ones obtained by the original retranslation process.

Let S, $\overline{S}$, and $s_{I'(i)}^{n(t_m)}$ be as Section II.C. Let $(s_x^{n(t)}, \alpha)$ be any 2-tuple term in LH, and let $(s_r^{n(t_m)}, \lambda) = TF_{t_m}^{t}(s_x^{n(t)}, \alpha)$.

The main idea of the revised retranslation process is based on the use of the deviation measure (i.e., Eq. (3)). Without loss of generality, if $s_{I'(k)}^{n(t_m)} \leq (s_r^{n(t_m)}, \lambda) \leq s_{I'(k+1)}^{n(t_m)}$, then the revised retranslation process $\Psi^{-1'}$ can be described as Eqs. (9)-(11):

$$(s_{result^*}, \lambda_{result^*}) = \Psi^{-1'}(s_x^{n(t)}, \alpha), \quad (9)$$

where, s_{result^*} is as Eq. (10) and

$$\lambda_{result^*} = \begin{cases} \dfrac{d\left((s_{I'(k)}^{n(t_m)}), (s_r^{n(t_m)}, \lambda)\right)}{d\left(s_{I'(k)}^{n(t_m)}, s_{I'(k+1)}^{n(t_m)}\right)}, & s_{result^*} = s_k \\ -\dfrac{d\left(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)\right)}{d\left((s_{I'(k)}^{n(t_m)}), s_{I'(k+1)}^{n(t_m)}\right)}, & s_{result^*} = s_{k+1} \end{cases}. \quad (11)$$

Note 4. The revised retranslation process will be more convenient to connect the model of Herrera et al. to the numerical scale model, which is discussed in Section III.C. So, in the rest of this study the revised retranslation process is adopted. For notation simplicity, the revised retranslation process $\Psi^{-1'}$ is still denoted as Ψ^{-1}.

Note 5. We argue that the result, obtained by the revised retranslation process, is same to the one obtained by the original retranslation process. In the future, we will provide a detailed proof for the equivalence between the original retranslation process and the revised retranslation process.

C. Equivalence between the numerical scale model and the model of Herrera et al.

Before connecting the model of Herrera et al. and the numerical scale model, we propose an approach to set the numerical scale based on the model of Herrera et al. Let S, $\overline{S}$, and $s_{I'(i)}^{n(t_m)}$ be as before. This approach to set the numerical scale is described as Eq. (12):

$$NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right), i = 1, 2, ..., g. \quad (12)$$

If the numerical scale is set as Eq. (12), this section will show the equivalence of the linguistic computational models between the model of Herrera et al. and the numerical scale model. Because the comparison operators, defined in the model of Herrera et al. and the numerical scale model, are same, we only analyze the equivalence for the aggregation operators and negation operators in the rest of this section.

1) Equivalence for aggregation operators: When analyzing the equivalence of the aggregation operators for the model of Herrera et al. and the numerical scale model, we only consider the OWA operator. The results for the other aggregation operators are similar.

In the model of Herrera et al. and the numerical scale model, the OWA operators can be defined as Definitions 7 and 8, respectively.

Definition 7: Let S, $\overline{S}$, $S^{(t(m))}$ and $\overline{S^{(t(m))}}$ be as before. Let $L = \{l_1, ..., l_m\}$, where $l_i \in \overline{S}$ be a set of terms to aggregate. Let $(s_{i'}^{t(m)}, \lambda_{i'})$ be the corresponding 2-tuple term in $\overline{S^{t(m)}}$, associated with l_i. Let $w = \{w_1, w_2, ..., w_m\}$ be an associated weighting vector that satisfies $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$. The 2-tuple ordered weighted averaging (TOWA) operator in the model of Herrera et al. is computed as

$$TOWA_w^{LH}(l_1, l_2, ..., l_m) = \Psi^{-1}\left(\Delta\left(\sum_{i=1}^m w_i y_i^*\right)\right), \quad (13)$$

where y_i^* is the ith largest of the y_i values, and $y_i = \Delta^{-1}\left(s_{i'}^{t(m)}, \lambda_{i'}\right)$.

Definition 8: Let S and $\overline{S}$ be as before. Let $L = \{l_1, ..., l_m\}$, where $l_i \in \overline{S}$ be a set of terms to aggregate. Let NS be an ordered numerical scale over $\overline{S}$, and $w = \{w_1, w_2, ..., w_m\}$ be an associated weighting vector that satisfies $w_i \geq 0$ and $\sum_{i=1}^m w_i = 1$. The 2-tuple ordered weighted averaging (TOWA) operator under NS is computed as

$$TOWA_w^{NS}(l_1, l_2, ..., l_m) = NS^{-1}\left(\sum_{i=1}^m w_i y_i^*\right), \quad (14)$$

where y_i^* is the ith largest of the y_i values, and $y_i = NS(l_i)$.

Before analyzing the equivalence of $TOWA_w^{LH}$ and $TOWA_w^{NS}$, we provide Lemmas 1-3.

Lemma 1: Let NS be an ordered numerical scale, i.e., $NS(s_i) < NS(s_{i+1})$. For $\forall y \in [NS(s_0), NS(s_g)]$, if $NS(s_i) \leq y \leq NS(s_{i+1})$, then the inverse operation of

$$s_{result^*} = \begin{cases} s_k, & d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) < d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \\ s_{k+1}, & d(s_{I'(k)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \geq d(s_{I'(k+1)}^{n(t_m)}, (s_r^{n(t_m)}, \lambda)) \end{cases}, \quad (10)$$

NS is

$$NS^{-1}(y) = \begin{cases} \left(s_i, \frac{y-NS(s_i)}{NS(s_{i+1})-NS(s_i)}\right), \ y < \frac{NS(s_{i+1})+NS(s_i)}{2} \\ \left(s_{i+1}, \frac{y-NS(s_{i+1})}{NS(s_{i+1})-NS(s_i)}\right), \ y \geq \frac{NS(s_{i+1})+NS(s_i)}{2} \end{cases} \quad (15)$$

The proof of Lemma 1 is provided in Appendix.

Lemma 2: Let $(s_J^{n(t_m)}, \lambda)$ be the corresponding 2-tuple term in $\overline{S^{n(t_m)}}$, associated with $(s_i, \alpha) \in \overline{S}$. If the numerical scale is set as Eq. (12), i.e., $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$ $(i = 1, 2, ..., g)$. Then

$$NS(s_i, \alpha) = \Delta^{-1}\left(s_J^{n(t_m)}, \lambda\right). \quad (16)$$

The proof of Lemma 2 is provided in Appendix.

Lemma 3: For any $s \in \overline{S^{n(t_m)}}$, $NS^{-1}(\Delta^{-1}(s)) = \Psi^{-1}(s)$ if $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$ $(i = 1, 2, ..., g)$.

The proof of Lemma 3 is provided in Appendix.

Using Lemmas 1-3 yields Proposition 3.

Proposition 3: Let S and $\overline{S}$ be as before. Let $L = \{l_1, ..., l_m\}$, where $l_i \in \overline{S}$ be a set of terms to aggregate, and $w = \{w_1, w_2, ..., w_m\}$ be an associated weighting vector. Then,

$$TOWA_w^{LH}(l_1, l_2, ..., l_m) = TOWA_w^{NS}(l_1, l_2, ..., l_m), \quad (17)$$

under the condition that the numerical scale is set as Eq. (12), i.e., $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$, $i = 1, 2, ..., g$.

The proof of Proposition 3 is provided in Appendix.

Proposition 3 guarantees the equivalence of the OWA operators, used in the model of Herrera et al. and the numerical scale model, if the numerical scale is set as Eq. (12).

2) Equivalence for negation operators: In the model of Herrera et al. and the numerical scale model, the negation operators can be defined as Definitions 9 and 10, respectively.

Definition 9: Let S, $\overline{S}$, $S^{(t(m))}$ and $\overline{S^{(t(m))}}$ be as before. Let $s \in \overline{S}$, and let s' be the corresponding 2-tuple term in $\overline{S^{(t(m))}}$, associated with s. Then, the negation operator in the model of Herrera et al. is defined as

$$Neg^{LH}(s) = \Psi^{-1}(Neg(s')). \quad (18)$$

Definition 10: Let S and $\overline{S}$ be as before, and NS be an ordered numerical scale over $\overline{S}$. For any $s \in \overline{S}$, the negation operator under NS is defined as Eq. (19), where $null$ denotes undefined elements.

Lemma 4: If the numerical scale is set as Eq. (12), i.e., $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$ $(i = 1, 2, ..., g)$, then NS is an ordered numerical scale, such as $NS(s_0) = 0$, $NS(s_{\frac{g}{2}}) = \frac{n(t_m)-1}{2}$, and $NS(s_g) = n(t_m) - 1$.

The proof of Lemma 4 is provided in Appendix.

Proposition 4: Let S and $\overline{S}$ be as before. For any $s \in \overline{S}$, $Neg^{LH}(s) = Neg^{NS}(s)$.

The proof of Proposition 4 is provided in Appendix.

Proposition 4 guarantees the equivalence of the negation operators, used in the model of Herrera et al. and the numerical scale model.

Note 6. Setting the numerical scale of linguistic term sets is key task for CWW in the numerical scale framework. In the existing studies, Dong et al.[7] defined the concept of the transitive calibration matrix and its consistent index. By maximizing the consistency level, Dong et al.[7] developed an optimization based approach to compute the numerical scale of a linguistic term set. Meanwhile, Wand and Hao[29], and Dong et al.[8] developed the CCV approach to set a numerical scale. The results in this section show the model of Herrera et al. provides an novel numerical scale approach (i.e., Eq. (12)). If the numerical scale is set as Eq. (12), we analytically prove the equivalence of the linguistic computational models between the model of Herrera et al. and the numerical scale model.

IV. Illustrative example

Herrera et al.[12] proposed an example to evaluate students knowledge from different tests to obtain a global evaluation. In this example, an ULTS is used,

$$S = \{s_0 = F, s_1 = D, s_2 = C, s_3 = B, s_4 = A\}.$$

A student, Martina Grant, has completed six different tests to demonstrate his knowledge. The evaluations of tests are assessed using the ULTS S. The unbalanced linguistic assessments are listed in Table 1.

In this example, we set $t_m = 3$. According to the model of Herrera et al., the values for $s_{I(i)}^{G(i)}$, $Brid(s_i)$ and $s_{I'(i)}^{n(t_m)}$ are listed in Table 2.

First, we illustrate the equivalence of the OWA operators, used in the model of Herrera et al. and the numerical scale model. In this example, the tests are equally important, i.e., $w = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)^T$. Based on Eq. (12), we set the numerical scale:

$$NS(s_0) = 0, \ NS(s_1) = 4, \ NS(s_2) = 6,$$
$$NS(s_3) = 7, \ NS(s_4) = 8.$$

According to Definition 8, we have

$$TOWA_w^{NS}(s_4, s_1, s_1, s_2, s_3, s_4)$$
$$= NS^{-1}\left(\frac{8+4+4+6+7+8}{6}\right) = NS^{-1}\left(\frac{37}{6}\right).$$

Based on Eq. (15), we have $NS^{-1}\left(\frac{37}{6}\right) = (s_2, 0.16) = (C, 0.16)$, i.e.,

$$TOWA_w^{NS}(s_4, s_1, s_1, s_2, s_3, s_4) = (C, 0.16).$$

$$Neg^{NS}(s) = \begin{cases} NS^{-1}(2NS(s^*) - NS(s)), \ NS(s_0) \leq 2NS(s^*) - NS(s) < NS(s_g) \\ null, \ others \end{cases}, \quad (19)$$

Table 1. Unbalanced linguistic assessments in each exam

	T_1	T_2	T_3	T_4	T_5	T_6
M. Grant	s_4	s_1	s_1	s_2	s_3	s_4

Table 2. The values for $s_{I(i)}^{G(i)}$, $Brid(s_i)$ and $s_{I'(i)}^{n(t_m)}$ $(i = 0, 1, ..., 4)$

s_i	$s_{I(i)}^{G(i)}$	$Brid(s_i)$	$s_{I'(i)}^{n(t_m)}$
$s_0 = F$	$s_{I(0)}^{G(0)} = s_0^3$	False	$s_{I'(i)}^{n(t_m)} = s_0^9$
$s_1 = D$	$s_{I(1)}^{G(1)} = s_1^3$	True	$s_{I'(i)}^{n(t_m)} = s_4^9$
$s_2 = C$	$s_{I(2)}^{G(2)} = s_3^5$	True	$s_{I'(i)}^{n(t_m)} = s_6^9$
$s_3 = B$	$s_{I(3)}^{G(3)} = s_7^9$	False	$s_{I'(i)}^{n(t_m)} = s_7^9$
$s_4 = A$	$s_{I(4)}^{G(4)} = s_8^9$	False	$s_{I'(i)}^{n(t_m)} = s_8^9$

In [12], Herrera et al. have shown $TOWA_w^{LH}(s_4, s_1, s_1, s_2, s_3, s_4) = (C, 0.16)$. So

$$TOWA_w^{LH}(s_4, s_1, s_1, s_2, s_3, s_4) =$$

$$TOWA_w^{NS}(s_4, s_1, s_1, s_2, s_3, s_4) = (C, 0.16).$$

Next, we illustrate the equivalence of the negation operators, used in the model of Herrera et al. and the numerical scale model. For example, based on Eq. (18),

$$Neg^{LH}(s_2) = \Psi^{-1}(Neg(s_6^9)) = \Psi^{-1}(s_2^9).$$

Furthermore, since $s_0^9 = s_{I'(0)}^{n(t_3)} \leq s_2^9 \leq s_{I'(1)}^{n(t_3)} = s_4^9$ and $d(s_0^9, s_2^9) = d(s_2^9, s_4^9)$, using Eqs. (9)-(10) obtains $\Psi^{-1}(s_2^9) = (s_1, -0.5)$, i.e., $Neg^{LH}(s_2) = (s_1, -0.5) = (D, -0.5)$.

Based on Eq. (19),

$$Neg^{LH}(s_2) = NS^{-1}(8 - NS(s_2)) = NS^{-1}(2).$$

According to Eq. (15), $NS^{-1}(2) = (s_1, -0.5)$, i.e., $Neg^{LH}(s_2) = (s_1, -0.5) = (D, -0.5)$. So,

$$Neg^{LH}(s_2) = Neg^{NS}(s_2) = (D, -0.5).$$

V. Conclusions

Herrera and Martínez [13] initiated the 2-tuple fuzzy linguistic representation model. The Herrera and Martínez model is well suited for dealing with linguistic term sets that are uniformly and symmetrically distributed, with the results of this model having the capacity to match the elements in the initial linguistic terms. In recent years, two different models (i.e., the model of Herrera et al.[12] and the numerical scale model[7]) based on linguistic 2-tuples have been developed to deal with term sets that are not uniformly and symmetrically distributed, i.e., ULTSs. The model of Herrera et al. is based on the use of a linguistic hierarchy and the Herrera and Martínez model, and the numerical scale model has been presented and developed in Wang and Hao[29], [31] and Dong et al.[7], [8]. Because both the model of Herrera et al. and the numerical scale model can deal with ULTSs, a challenge is naturally proposed to analysts: how to compare these two different models. In this study, we provide an interesting connection between the model of Herrera et al. and the numerical scale model. The results show that the model of Herrera et al. provides a new approach to set a numerical scale. Moreover, we show the equivalence of the linguistic computational models between the model of Herrera et al. and the numerical scale model (in some sense).

In order to complete this study, in the future we will provide a detailed proof for the equivalence between the original retranslation process and the revised retranslation process.

Appendix

The proof of Lemma 1. For any $(s_i, \alpha) \in \overline{S}$, $NS^{-1}(NS(s_i, \alpha)) = (s_i, \alpha)$. So, the inverse operation of NS is Eq. (15).

The proof of Lemma 2. Here, we only consider the case of $\alpha \geq 0$. The proof for the case of $\alpha < 0$ is similar. Since $\alpha \geq 0$, using Eq. (4) obtains

$$NS(s_i, \alpha) = NS(s_i) + \alpha \times (NS(s_{i+1}) - NS(s_i)). \quad (20)$$

Let $s_{I(i)}^{G(i)}$ and $s_{I(i+1)}^{G(i+1)}$ be as before. When $\alpha \geq 0$, according to the model of Herrera et al.,

$$G(i) = G(i+1) \quad (21)$$

and

$$I(i+1) = I(i) + 1. \quad (22)$$

Note 7. In the model of Herrera et al., if $Brid(s_i) = False$ and $Brid(s_{i+1}) = False$, Eqs. (21) and (22) can be obtained directly. For the case of $Brid(s_i) = True$ (or $Brid(s_{i+1}) = True$), Eqs. (21) and (22) also holds. For example, if $Brid(s_i) = True$ and $s_i > s^*$, then $s_{I(i)}^{G(i)}$ is selected from two terms, denoted by $\{s_x^{n(t)}, s_{2x}^{n(t+1)}\}$, according to different cases. In $\{s_x^{n(t)}, s_{2x}^{n(t+1)}\}$, $n(t) = G(i+1)$ or $n(t+1) = G(i+1)$. In this case, if $n(t) = G(i+1)$, we set $s_{I(i)}^{G(i)} = s_x^{n(t)}$; otherwise, we set $s_{I(i)}^{G(i)} = s_{2x}^{n(t+1)}$. As a result, Eqs. (21) and (22) holds.

Moreover, let $s \in S^{n(t)}$, and let $s^{'} = TF_{t_m}^{t}(s)$. According to Eq. (5),

$$\Delta^{-1}(s^{'}) = \frac{n(t_m)-1}{n(t)-1}\Delta^{-1}(s). \tag{23}$$

Since $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$ $(i = 1, 2, ..., g)$, using Eq. (20) obtains Eq. (24).

Without loss of generality, if $G(i) = G(i+1) = n(t)$. According to Eq. (23), we have Eqs. (25) and (26)

Based on (22), (24), (25) and (26), we have that

$$NS(s_i, \alpha) = \frac{n(t_m)-1}{n(t)-1}(I(i)+\alpha) = \frac{n(t_m)-1}{n(t)-1}(\Delta^{-1}(s_{I(i)}^{n(t)}+\alpha)). \tag{27}$$

Since $(s_J^{n(t_m)}, \lambda) = TF_{t_m}^{t}(s_i, \alpha) = \Delta\left(\frac{n(t_m)-1}{n(t)-1}(\Delta^{-1}(s_{I(i)}^{n(t)} + \alpha))\right)$, we obtain $NS(s_i, \alpha) = \Delta^{-1}\left(s_J^{n(t_m)}, \lambda\right)$. This completes the proof of Lemma 2.

The proof of Lemma 3. Let $(s_{LH}, \alpha_{LH}) = \Psi^{-1}(s)$ and $(s_{NS}, \alpha_{NS}) = NS^{-1}(\Delta^{-1}(s))$. Let $y = \Delta^{-1}(s)$. Without loss of generality, we assume that $NS(s_i) \leq y \leq NS(s_{i+1})$. We consider two cases.

Case A: $y < \frac{NS(s_{i+1})+NS(s_i)}{2}$. In this case, since $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$ $(i = 1, 2, ..., g)$ and $NS(s_i) \leq y \leq NS(s_{i+1})$, we have $s_{I'(i)}^{n(t_m)} \leq s \leq s_{I'(i+1)}^{n(t_m)}$ and $d(s_{I'(i)}^{n(t_m)}, s) < d(s_{I'(i+1)}^{n(t_m)}, s)$. Furthermore, according to Eqs. (10) and (15), we have

$$s_{LH} = s_{NS} = s_i. \tag{28}$$

According to Eqs. (11) and (15), we obtain

$$\alpha_{LH} = \alpha_{NS} = \frac{y - NS(s_i)}{NS(s_{i+1}) - NS(s_i)} = \frac{d\left(s_{I'(i)}^{n(t_m)}, s\right)}{d\left(s_{I'(i)}^{n(t_m)}, s_{I'(i+1)}^{n(t_m)}\right)}. \tag{29}$$

Case B: $y \geq \frac{NS(s_{i+1})+NS(s_i)}{2}$. In this case, since $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right)$ $(i = 1, 2, ..., g)$ and $NS(s_i) \leq y \leq NS(s_{i+1})$, we have $s_{I'(i)}^{n(t_m)} \leq s \leq s_{I'(i+1)}^{n(t_m)}$ and $d(s_{I'(i)}^{n(t_m)}, s) \geq d(s_{I'(i+1)}^{n(t_m)}, s)$. Furthermore, according to Eqs. (10) and (15), we have

$$s_{LH} = s_{NS} = s_{i+1}. \tag{30}$$

According to Eqs. (11) and (15), we obtain

$$\alpha_{LH} = \alpha_{NS} = \frac{y - NS(s_{i+1})}{NS(s_{i+1}) - NS(s_i)} = -\frac{d\left(s_{I'(i+1)}^{n(t_m)}, s\right)}{d\left(s_{I'(i)}^{n(t_m)}, s_{I'(i+1)}^{n(t_m)}\right)}. \tag{31}$$

As a result, $(s_{LH}, \alpha_{LH}) = (s_{NS}, \alpha_{NS})$, i.e., $NS^{-1}(\Delta^{-1}(s)) = \Phi^{-1}(s)$. This completes the proof of Lemma 3.

The proof of Proposition 3. Let $(s_{i'}^{t(m)}, \lambda_{i'})$ be the corresponding 2-tuple term in $\overline{S^{t(m)}}$, associated with l_i. Let $y_i = \Delta^{-1}\left(s_{i'}^{t(m)}, \lambda_{i'}\right)$, and y_i^* is the ith largest of the y_i values. Let $z_i = NS\left(l_i\right)$ and z_i^* is the ith largest of the z_i values. Since $NS(s_i) = \Delta^{-1}\left(s_{I'(i)}^{n(t_m)}\right), i = 1, 2, ..., g$, based on Lemma 2, we have $y_i = z_i$. As a result,

$$\sum_{i=1}^{m} w_i y_i^* = \sum_{i=1}^{m} w_i z_i^*. \tag{32}$$

Let $l = \Delta(\sum_{i=1}^{m} w_i y_i^*) \in S^{n(t_m)}$. Then, according to Definitions 7 and 8,

$$TOWA_w^{LH}(l_1, l_2, ..., l_m) = \Psi^{-1}(l) \tag{33}$$

and

$$TOWA_w^{NS}(l_1, l_2, ..., l_m) = NS^{-1}(\Delta^{-1}(l)). \tag{34}$$

Based on Lemma 3, we have $TOWA_w^{LH}(l_1, l_2, ..., l_m) = TOWA_w^{NS}(l_1, l_2, ..., l_m)$. This completes the proof of Proposition 3.

The proof of Lemma 4. According to the representation algorithm used in the model of Herrera et al., Lemma 4 can be obtained.

The proof of Proposition 4. Because $s^{'}$ be the corresponding 2-tuple term in $\overline{S^{(t(m))}}$, associated with s, based on Lemma 2 we have that

$$NS(s) = \Delta^{-1}\left(s^{'}\right). \tag{35}$$

From (1), we have

$$Neg(s^{'}) = \Delta^{-1}(n(t_m) - 1 - \Delta^{-1}(s^{'}). \tag{36}$$

Meanwhile, based on Lemma 4,

$$2NS(s^*) - NS(s) = n(t_m) - 1 - \Delta^{-1}\left(s^{'}\right) = \Delta^{-1}(Neg(s^{'})). \tag{37}$$

Based on Lemma 3 and (37), we have

$$\Psi^{-1}(Neg(s^{'})) = NS^{-1}(2NS(s^*) - NS(s)). \tag{38}$$

Based on (38) and Definitions 9 and 10, we have $Neg^{LH}(s) = Neg^{NS}(s)$. This completes the proof of Proposition 4.

REFERENCES

[1] S. Alonso, F.J. Cabrerizo, F. Chiclana, F. Herrera, and E. Herrera-Viedma, "Group decision-making with incomplete fuzzy linguistic preference relations," *Int. J. Intell. Syst.,* vol. 24, no. 2 pp. 201-222, 2009.

[2] F.J. Cabrerizo, S. Alonso, and E. Herrera-Viedma, "A consensus model for group decision making problems with unbalanced fuzzy linguistic information," *Int. J. Inf. Tech. Decis Making.,* vol. 8, no. 1, pp. 109-131, 2009.

$$NS(s_i,\alpha) = \Delta^{-1}\left(s^{n(t_m)}_{I'(i)}\right) + \alpha \times \left(\Delta^{-1}\left(s^{n(t_m)}_{I'(i+1)}\right) - \Delta^{-1}\left(s^{n(t_m)}_{I'(i)}\right)\right). \quad (24)$$

$$\Delta^{-1}\left(s^{n(t_m)}_{I'(i)}\right) = I'(i) = \frac{n(t_m)-1}{n(t)-1}\Delta^{-1}(s^{n(t)}_{I(i)}) = \frac{n(t_m)-1}{n(t)-1}I(i). \quad (25)$$

$$\Delta^{-1}(s^{n(t_m)}_{I'(i+1)}) = I'(i+1) = \frac{n(t_m)-1}{n(t)-1}\Delta^{-1}(s^{n(t)}_{I(i+1)}) = \frac{n(t_m)-1}{n(t)-1}I(i+1). \quad (26)$$

[3] F.J. Cabrerizo, I.J. Pérez, and E. Herrera-Viedma, "Managing the consensus in group decision making in an unbalanced fuzzy linguistic context with incomplete information," *Knowl.-Based Syst.,* vol. 23, pp. 169-181, 2010.

[4] O. Cordón, F. Herrera, and I. Zwir, "Linguistic modeling by hierarchical systems of linguistic rules," *IEEE Trans. Fuzzy Syst.,* vol. 10, no. 1, pp. 2-20, 2001.

[5] Y.C. Dong, W.C. Hong, Y.F. Xu, and S. Yu, "Selecting the individual numerical scale and prioritization method in the analytic hierarchy process: A 2-tuple fuzzy linguistic approach," *IEEE Trans. Fuzzy Syst.,* vol. 19, pp. 13-25, 2011.

[6] Y.C. Dong, W.C. Hong, Y.F. Xu, and S. Yu, "Numerical scales generated individually for analytic hierarchy process," *Eur. J. Oper. Res.,* vol. 229, pp. 654-662, 2013.

[7] Y.C. Dong, Y.F. Xu, and S. Yu, "Computing the numerical scale of the linguistic term set for the 2-tuple fuzzy linguistic representation model," *IEEE Trans. Fuzzy Syst.,* vol. 17, pp. 1366-1378, 2009.

[8] Y.C. Dong, G.Q. Zhang, W.C. Hong, and Y.F. Xu, "Linguistic computational model based on 2-tuples and intervals," *IEEE Trans. Fuzzy Syst.,* vol. 21, pp. 1006-1018, 2013.

[9] M. Espinilla, J. Liu, and L. Martínez, "An extended hierarchical linguistic model for decision-making problems," *Comput. Intell.,* Vol. 27, pp. 489-512, 2011.

[10] F. Herrera, S. Alonso, F. Chiclana, and E. Herrera-Viedma, "Computing with words in decision making: foundations, trends and prospects," *Fuzzy Optim Decis Making.,* vol. 8, no. 4, pp. 337-364, 2009.

[11] F. Herrera, and E. Herrera-Viedma, "Linguistic decision analysis: steps for solving decision problems under linguistic information," *Fuzzy Sets Syst.,* vol. 115, pp. 67-82, 2000.

[12] F. Herrera, E. Herrera-Viedma, and L. Martínez, "A fuzzy linguistic methodology to deal with unbalanced linguistic term sets," *IEEE Trans. Fuzzy Syst.,* vol. 16, no. 2, pp. 354-370, 2008.

[13] F. Herrera, and L. Martínez, "A 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Trans. Fuzzy Syst.,* vol. 8, no. 6, pp. 746-752, 2000.

[14] F. Herrera, and L. Martínez, "A model based on linguistic 2-tuples for dealing with multigranularity hierarchical linguistic contexts in multiexpert decision-making," *IEEE Trans. Syst., Man Cybern. B, Cybern.,* vol. 31, no. 2, pp. 227-234, 2001.

[15] E. Herrera-Viedma, F.J. Cabrerizo, I.J. Pérez and M.J. Cobo, S. Alonso, and F. Herrera, "Applying linguistic OWA operators in consensus models under unbalanced linguistic information," R.R. Yager et al. (Eds.): *Recent Developments in the OWA Operators,* STUDFUZZ 265, pp. 167-186, 2011.

[16] E. Herrera-Viedma, and A.G. López-Herrera, "A model of information retrieval system with unbalanced fuzzy linguistic information," *Int. J. Intell. Syst.,* vol. 22, no. 11, pp. 1197-1214, 2007.

[17] J. Kacprzyk and S. Zadrozny, "Computing with words in decision making: Through individual and collective linguistic choice rules," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.,* vol. 9, pp. 89-102, 2001.

[18] J. Lawry, "An alternative approach to computing with words," *Int. J. Uncertain., Fuzz. Knowl.-Based Syst.,* vol. 9(Suppl), pp. 3-16, 2001.

[19] D.F. Li, "Multiattribute group decision making method using extended linguistic variables," *Int. J. Uncertain., Fuzz. Know.-based Syst.,* vol. 17, no. 6, pp. 793-806, 2009.

[20] L. Martínez, "Sensory evaluation based on linguistic decision analysis," *Int. J. Approx. Reason.,* vol. 44, pp. 148-164, 2007.

[21] L. Martínez, M. Espinilla, J. Liu, L.G. Pérez, and P.J. Sánchez, "An evaluation model with unbalanced linguistic information applied to olive oil sensory evaluation," information retrieval system *J. Multi.-Valued Logic and Soft Comp.,* vol. 15, pp. 229-251, 2009.

[22] L. Martínez, M. Espinilla, and L.G. Pérez,"A linguistic multigranular sensory evaluation model for olive oil," *Int. J. Comp. Intell. Syst.,* vol. 1, no. 2, pp. 148-158, 2008.

[23] L. Martínez, and F. Herrera, "An overview on the 2-tuple linguistic model for computing with words in decision making: Extensions, applications and challenges," *Inf. Sci.,* vol. 207, pp. 1-18, 2012.

[24] L. Martínez, J. Liu, D. Ruan, and J.B. Yang, "Dealing with heterogeneous information in engineering evaluation processes," *Inf. Sci.,* vol. 177, no. 7, pp. 1533-1542, 2007.

[25] L. Martínez, D. Ruan, and F. Herrera, "Computing with words in decision support systems: An overview on models and applications," *Int. J. Comp. Intell. Syst.,* vol. 3, no. 4, pp. 382-395, 2010.

[26] J.M. Mendel, L.A. Zadeh, R.R.Yager, J. Lawry, H. Hagras, and S. Guadarrama, "What computing with words means to me," *IEEE Comput. Intell. Mag.,* vol. 5, no. 1, pp. 20-26, Feb. 2010.

[27] D. Meng and Z. Pei, "On weighted unbalanced linguistic aggregation operators in group decision making," *Inf. Sci.,* vol. 223, pp. 31-41, 2013.

[28] S.-Y. Wang, "Applying 2-tuple multigranularity linguistic variables to determine the supply performance in dynamic environment based on product-oriented strategy," *IEEE Trans. Fuzzy Syst.,* vol. 16, no. 1, pp. 29-39, 2008.

[29] J.H. Wang, and J.Y. Hao, "A new version of 2-tuple fuzzy linguistic representation model for computing with words," *IEEE Trans. Fuzzy Syst.,* vol. 14, no. 3, pp. 435-445, 2006.

[30] J.H. Wang, and J.Y. Hao, "Fuzzy linguistic PERT," *IEEE Trans. Fuzzy Syst.,* vol. 15, no. 2, pp. 133-144, 2007.

[31] J.H. Wang, and J.Y. Hao, "An approach to computing with words based on canonical characteristic values of linguistic labels," *IEEE Trans. Fuzzy Syst.,* vol. 15, no. 4, pp. 593-604, 2007.

[32] D. Wu and J.M. Mendel, "Computing with words for hierarchical decision making applied to evaluating a weapon system," *IEEE Trans. Fuzzy Syst.,* vol. 18, no. 3, pp. 441-460, 2010.

[33] Z. Xu, "A method based on linguistic aggregation operators for group decision making with linguistic preference relations," *Inf. Sci.,* vol. 166, pp. 19-30, 2004.

[34] Z. Xu, "Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment," *Inf. Sci.,* vol. 168, pp. 171-184, 2004.

[35] Z. Xu, "Deviation measures of linguistic preference relations in group decision making," *Omega.,* vol. 33, pp. 249-254, 2005.

[36] R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decision making," *IEEE Trans. Syst., Man, Cybern.,* vol. 18, pp. 183-190, 1988.

[37] R. Yager, "An approach to ordinal decision making," *Int. J. Approx. Reason.,* vol. 12, pp. 237-261, 1995.

[38] L.A. Zadeh, "The concept of a linguistic variable and its applications to approximate reasoning, Part I," *Inf. Sci.,* vol. 8, pp. 199-249, 1975.

[39] L.A. Zadeh, "Fuzzy logic = computing with words," *IEEE Trans. Fuzzy Syst.,* vol. 4, pp. 103-111, 1996.

[40] S.-M. Zhou, R.I. John, F. Chiclana and J. M. Garibaldi, "On aggregating uncertain information by type-2 OWA operators for soft decision making," *Int. J. Intell. Syst.,* vol. 25, no. 6, pp. 540-558, 2010.