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Aggregating fuzzy implications based on OWA-operators

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***Abstract*—This paper presents the fuzzy (S,N)- QL- and D-subimplication classes, which are obtained by OWA operators performed over the families of triangular subnorms and subconorms along with fuzzy negations. Since these classes of subimplications are explicitly represented by such connectives, the corresponding (S,N)- QL- and D-subimplicatios are characterized by the generalized associativity and distributive properties together with extensions of the exchange and neutrality principles. As the main results, these families of subimplications extend related implications by preserving their corresponding properties.**

***Index Terms*—OWA operators; fuzzy t-sub(co)norm; fuzzy (sub)implication;**

I. Introduction

Despite potential areas for applications of aggregation operators, this paper deals with the current status of the theory of aggregation operators and also considers some of their main properties. It is a large domain, modelling uncertainty in distinct fields as social, engineering or economical problems which are based on fuzzy logic (FL) [1], [2], [3].

They have been applied to many fields of approximate reasoning, e.g. image processing, data mining, pattern recognition, fuzzy relational equations and fuzzy morphology (see [4], [5], [6], [7], [8] and [9]).

Moreover, many other extensions of fuzzy logic make use of aggregation operators, e.g. Interval-valued Fuzzy Logic ([10], [11], [12], [13], [14], [15] and [16]), Intuitionistic Fuzzy Logic ([10], [17], [18], [19], [20], [21] and [22]) and Hesitant Fuzzy Logic [23], [24] and [25].

Distinguished classes of aggregation operators have been studied in the literature. We focus on the OWA (Ordered Weighted Average) operators, which are applied into a family of fuzzy connectives to generate new fuzzy connectives, preserving the same properties verified by the corresponding family.

Following the studies presented in [26], by relaxing the neutral element property related to triangular (co)norms (t-(co)norms), the class of *t*-sub(co)norms is considered. Additionally, the fuzzy (S,N)-subimplication class, explicitly represented by fuzzy negations and such class of fuzzy t-subconorms, is also reported. Generalizations of the product t-norm and probabilistic sum are taken into account providing interesting examples.

Since this study considers the *n*-ary OWA operator, generalized associativity, exchange principle and distributivity properties also need to be considered. As the main contribution, this paper introduces the class $\mathcal{I}_D$ of fuzzy D-subimplications, which is obtained by the OWA operator performed over a family of t-sub(co)norms $\mathcal{T}(\mathcal{S})$ along with fuzzy negations.

These results state the following constructions as equivalent:

- (⇒) Firstly, we can aggregate all the *t*-sub(co)norms and after that, we are able to generate a class $\mathcal{J}$ of (D-) QL-subimplications; or the converse order,
- (⇐) Each (D-) QL-subimplication can be firstly obtained by composition of a *t*-sub(co)norm and a fuzzy negation and, in the sequence, the new implication is given by aggregating all the (D-) QL-subimplications related to the OWA operator.

By extending the main results described in [26] and [27] and related to (S,N)-implication and R-implication classes, this paper provides new members in the classes of D- and QL-subimplications which are generated by the OWA operator, generalizing the arithmetic mean and median aggregations performed over t-sub(co)norms and fuzzy negations.

The paper innovation refers to a **new operator which is able to aggregate not only pairs but also families of implications**. Moreover, such operator ensures:

- (i) **invariance in the implication class**, in the sense of the main properties of each fuzzy implication in the aggregated family are preserved by the new operator;
- (ii) **closure in the implication class**, meaning that the new implication also belongs to the family of aggregated implications;
- (iii) **invariance of the aggregation operator**, by preserving same results in extensions of the corresponding aggregation class, from the arithmetic mean and the median to the OWA operator;
- (iv) **extension in the implication class** since relevant classes of functions such as (S, N)- QL-and D-implications are verified by such operator.

A. Related works and possible applications

In in multi-criteria decision [17], [19], the relevance of an evaluated criterion often need to be well established, making use of extensions of the usual non weighted operators. Despite causing the loss of neutrality from the decision system, these weights frequently provide better performance [28], [29].

OWA operators are applied to adjust the terms AND and OR, making easier semantic interpretation of the linguistic quantifiers [5]. Fuzzy implications obtained by aggregation operators in discrete cases can also model rules with respect to a fuzzy inference systems [30].

The image processing research area deals with aggregation in order to increase detection of patterns that must be rejected. And so, it is able to infer satisfactory decision boundaries [31], [32].

B. Outline Paper

The paper is organized as follows. The preliminaries in Section II are concerned with fuzzy connectives and their algebraic properties. Section III reports concepts of aggregation functions, their main properties and examples. Focusing on the OWA operator and the two classes of t-subconorm and t-subnorm we analyse the corresponding properties. Section IV considers both classes, (S,N)-(sub)implications and QL-(sub)implications. Main results concerned with aggregating D- and QL-subimplications by applying the OWA operator are described in SectionVI. it is also shown that the OWA operator preserves (S,N)- QL- and D-implication classes. Lastly, the conclusion and final remarks are presented.

II. Fuzzy Connectives

In the following, basic concepts of fuzzy negation and fuzzy subimplications are reported [12], [33].

A. Fuzzy negations

Let $U = [0,1]$ be the unit interval. A **fuzzy negation** (FN) $N\colon U \to U$ satisfies:

N1 : $N(0) = 1$ and $N(1) = 0$;
N2 : If $x \geq y$ then $N(x) \leq N(y)$, $\forall\ x, y \in U$.

Fuzzy negations satisfying the involutive property are called **strong** FNs:

N3 : $N(N(x)) = x$, $\forall\ x \in U$.

The standard negation $N_S(x) = 1 - x$ is a strong fuzzy negation.

Let N be a FN and $f\colon U^n \to U$ be a real function. Then, for all $\vec{x} = (x_1, x_2, \ldots, x_n) \in U^n$, the $N-$**dual function** of f is given by the expression:

$$f_N(\vec{x}) = N(f(N(x_1), N(x_2), \ldots, N(x_n))) = N(f(N(\vec{x}))). \quad (1)$$

Notice that, when N is involutive, $(f_N)_N = f$, that is the N-dual function of f_N coincides with f. In addition, if $f = f_N$ then it is clear that f is a self-dual function. Many other properties of fuzzy negations and related main extensions can be founded in [34] and [35].

B. Fuzzy Subimplications

A function $I : U^2 \to U$ is a **fuzzy subimplicator** if it satisfies the conditions:

I0 : $I(1,1) = I(0,1) = I(0,0) = 1$;

When a fuzzy subimplicator $I : U^2 \to U$ also satisfies this boundary condition:

I1 : $I(1,0) = 0$;

I is called **fuzzy implicator**. And, a fuzzy (sub)implicator I satisfying the properties:

I2 : If $x \leq z$ then $I(x,y) \geq I(z,y)$ (left antitonicity);
I3 : If $y \leq z$ then $I(x,y) \leq I(x,z)$ (right isotonicity);
I4 : $I(0,y) = 1$ (left boundary property);

is called a **fuzzy (sub)implication** [10, Def. 6][36].

III. Aggregation functions

Based on [5] and [13], the general meaning of an aggregation function in FL is to assign an n-tuple of real numbers belonging to U^n to a single real number on U, such that it is a non-decreasing and idempotent (i.e., it is the identity when an n-tuple is unary) function satisfying boundary conditions. In [6, Def. 2], an n-ary *aggregation* function $A\colon U^n \to U$ demands, for all $\vec{x} = (x_1, x_2, \ldots, x_n)$, $\vec{y} = (y_1, y_2, \ldots, y_n) \in U^n$, the following conditions:

A1: **Boundary Conditions**
$A(\vec{0}) = A(0,0,\ldots,0) = 0$ and
$A(\vec{1}) = A(1,1,\ldots,1) = 1$;

A2: **Monotonicity** If $\vec{x} \leq \vec{y}$ then $A(\vec{x}) \leq A(\vec{y})$ where $\vec{x} \leq \vec{y}$ iff $x_i \leq y_i$, for all $0 \leq i \leq n$.

Some extra usual properties for aggregation functions are the following:

A3: **Symmetry**
$A(\overrightarrow{x_\sigma}) = A(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = A(\vec{x})$, when $\sigma\colon \mathbb{N}^n \to \mathbb{N}^n$ is a permutation;

A4 : **Compensation - Pareto Property**
$\min_{i=0}^n(x_i) \leq A(\vec{x}) \leq \max_{i=0}^n(x_i)$

A5: **Idempotency**
$A(x, x, \ldots, x) = x$, for all $x \in U$;

A6: **Continuity**
If for each $i \in \{1,\ldots,n\}$, $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots x_n \in U$ and convergent sequence $\{x_{ij}\}_{j\in\mathbb{N}}$ we have that $\lim_{j\to\infty} A(x_1, \ldots, x_{i-1}, x_{ij}, x_{i+1}, \ldots, x_n) = A(x_1, \ldots, x_{i-1}, \lim_{j\to\infty} x_{ij}, x_{i+1}, \ldots x_n)$;

A7: **k-homogeneity**
For all $k \in]0,\infty[$ and $\alpha \in [0,\infty[$ such that $\alpha^k\vec{x} = (\alpha^k x_1, \alpha^k x_2, \ldots, \alpha^k x_n) \in U^n$,
$A(\alpha^k\vec{x}) = \alpha^k A(\vec{x})$;

A8: **Distributivity of an aggregation** $A : U^n \to U$ **related to a function** $F : U^2 \to U$
$A(F(x,y_1), \ldots, F(x,y_n)) = F(x, A(y_1, \ldots, y_n))$, for all $x, y_1, \ldots, y_n \in U$.

Proposition 1. *Let $\sigma\colon \mathbb{N}^n \to \mathbb{N}^n$ be a permutation function ordering the elements: $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)}$. Let $w_1, w_2, \ldots, w_n$ be non negative weights ($w_i \leq 0$) such that their sum equals one ($\sum_{i=0}^n w_i = 1$). For all $\vec{x} \in U^n$, the n-ary aggregation function $M\colon U^n \to U$ called OWA and given as:*

$$M(\vec{x}) = \sum_{i=0}^{n} w_i x_{\sigma(i)} \quad (2)$$

verifies Property **Ak**, *for $k \in \{3,4,5\}$.*

Proof. *Straightforward.*

A. Triangular sub(co)norms

A **triangular sub(co)norm** (t-sub(co)norm)[4] is a binary aggregation function $(S)T: U^2 \to U$ such that, for all $x, y \in U$, the following holds:

T0: $T(x,y) \leq \min(x,y)$ **S0**: $S(x,y) \geq \max(x,y)$

and also verifying the commutativity, associativity and monotonicity properties which are, respectively, given by:

T1: $T(x,y) = T(y,x)$; **S1**: $S(x,y) = S(y,x)$;
T2: $T(x,T(y,z))=T(T(x,y),z)$; **S2**: $S(x,S(y,z))=S(S(x,y),z)$;
T3: $T(x,z) \leq T(y,z)$, if $x \leq y$; **S3**: $S(x,z) \leq S(y,z)$, if $x \leq y$.

A **t-(co)norm** is a t-sub(co)norm satisfying the condition:

T4: $T(x,1) = x$; **S4**: $S(x,0) = x$.

Remark 1. *Based on Properties **S0** and **T0**, we have that:*

$S(0,0) \geq 0$; $S(0,1) = 1$; $S(1,0) = 1$; $S(1,1) = 1$.
$T(1,1) \leq 1$; $T(1,0) = 0$; $T(0,1) = 0$; $T(0,0) = 0$.

Proposition 2. *For $i \geq 1$ and $x, y \in U$, $T_i\,(S_i) : U^2 \to U$ is a t-sub(co)norm given by*

$$T_i(x,y) = \frac{1}{i}xy, \qquad S_i(x,y) = 1 - \frac{1}{i}(1 - x - y + xy), \quad (3)$$

Proof. *Straightforward.*

The families of all t-subnorms T_i and of all t-conorms S_i are referred as $\mathcal{T}$ and $\mathcal{S}$, respectively.

Remark 2. *Observe that, for $i = 1$, Eq.(3)a and Eq.(3)b are named as the product t-norm and the the algebraic sum, respectively, and corresponding expression can be given as*

$$T_P(x,y) = xy \quad S_P(x,y) = x + y - xy \quad (4)$$

Moreover, T_P and S_P constitute a pair of N_S-mutual dual functions.

IV. (Sub)implication Classes

The main results considered in this section are reported from [37], [26], [38] and [37].

A. Fuzzy (S,N)-(sub)implication class

A function $I_{S,N}: U^2 \to U$ is called an **(S,N)-(sub)implication** if there exists a t-(sub)conorm S and a fuzzy negation N such that

$$I_{S,N}(x,y) = S(N(x),y), \quad (5)$$

for all $x, y \in U$. If N is a strong FN, then I is called an **S-(sub)implication**. Clearly, a fuzzy implication $I_{S,N}$ is also a fuzzy (sub)implication.

The family of all (S,N)-subimplicators is referred as $\mathcal{I}_{(S,N)}$.

Proposition 3. *[26, Proposition 4.10] The following statements are equivalent:*
1. $I: U^2 \to U$ is an (S,N)-implication underlying a continuous FN N and a t-subconorm S at point 0;
*2. I is continuous at point $x = 1$ in the first component, satisfying **I3** and the two additional conditions:*

I5 : ***Exchange Principle***:
$I(x, I(y,z)) = I(y, I(x,z))$, for all $x, y, z \in U$;

I6 : ***Contrapositive Symmetry***:
$I(x,y) = I(N(y), N(x))$, for all $x, y \in U$.

Proposition 4. *The binary function $I_i : U \to U$, defined as*

$$I_i(x,y) = 1 - \frac{1}{i}(x - xy), \quad \forall i \geq 1, \quad (6)$$

is a fuzzy (S,N)-subimplication.

Proof. *Taking $S_i(x,y) = 1 - \frac{1}{i}(1 - x - y + xy)$, for $i \leq 1$, we have that*

$$S_i(N_S(x),y) = 1 - \frac{1}{i}(1 - (1-x) - y + (1-x)y) = 1 - \frac{1}{i}(x - xy).$$

Consequently, for all $x, y \in U$, it holds that

$$I_i(x,y) = S_i(N_S(x),y).$$

Therefore I_i is an (S_i,N_S)-implication.

See in Fig. 1 the three members I_1, I_2, I_3 of $\mathcal{I}$. In particular, I_1 is referred as the Reichenbach's implication and related to $i = 1$ in Eq.(6). It shows that $I_1(1,0) = 0$ while in other two subimplications I_2 and I_2 we have that $I_2(1,0) = 0.5$ and $I_3(1,0) = 0.3$, respectively.

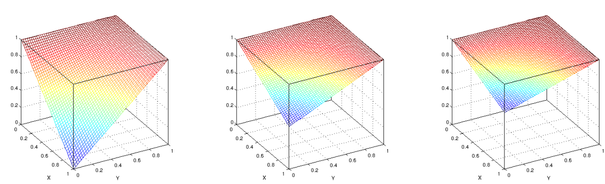

Fig. 1. Fuzzy (S,N)-subimplications of family $\mathcal{I} = \{I_1, I_2, I_3\}$.

Proposition 5. *An (S,N)-subimplication in $\mathcal{I}$ verifies Property $\mathbf{I_k}$, for $\mathbf{k} \in \{0,2,3,4,5,6\}$.*

Proof. *The following holds.*

I0: *Straightforward from definition of I_i by Eq. (6) in Proposition 4.*

I2: *If $x_1 \leq x_2$, for all $x_1, y, x_2 \in U$, it holds that*
$I_i(x_1,y) = 1 - \frac{1}{i}(x_1 - x_1y) \geq x_1 - \frac{1}{i}(x_1 - x_1y) = I_i(x_2,y)$.

I3: *If $y_1 \leq y_2$, for all $y_1, y_2, x \in U$ it holds that*

$$I_i(x,y_1) = 1 - \frac{1}{i}(x - xy_1) \leq x - \frac{1}{i}(x - xy_2) = I_i(x,y_2).$$

I4: $I_i(0,y) = 1 - \frac{1}{i} \cdot 0 = 1$, *for all $y \in U$.*

I5: *For all $x, y, z \in U$, it holds that*

$$I_i(x, I_i(y,z)) = 1 - \frac{1}{i}x - x(1 - \frac{1}{i}(y - yz)) = 1 - \frac{x}{i^2}(y - yz).$$

Therefore, we obtain the following:

$$\begin{aligned} I_i(x, I_i(y,z)) &= 1 - \frac{y}{i^2}(x - xz) \\ &= 1 - \frac{1}{i}y - y(1 - \frac{1}{i}(x - xz)) \\ &= I_i(y, I_i(x,z)). \end{aligned}$$

I6: *For all $x, y \in U$, the following is held:*

$$I_i(N_S(y), N_S(x)) = 1\frac{1}{i}(1 - y - (1 - y - x + xy))$$
$$= 1 - \frac{1}{i}(x - xy) = I_i(x, y).$$

Concluding, Proposition 5 is verified.

Theorem 1. *The operator $I_i : U^2 \to U$ given by Eq. (6) is an (S_i, N_S)-implication underlying the continuous negation N_S and the continuous t-subconorm S_i at point 0.*

Proof. *Straightforward from Propositions 3, 4 and 5.*

B. Fuzzy QL-(sub)implication class

This section reviews the main properties of fuzzy QL-(sub)implication class. See [35], [39] and [33] for more information.

Definition 1. *A function $I_{S,N,T}: U^2 \to U$ is called a **QL-(sub)implicator** if, for all $x, y \in U$, there exist a t-(sub)conorm S, a t-norm T and a fuzzy negation N such that:*

$$I_{S,N,T}(x, y) = S(N(x), T(x, y)). \quad (7)$$

In this case, a QL-subimplication $I_{S,N,T}$ indicates the underlying t-(sub)conorm, t-norm and fuzzy negation as S, T and N, respectively. The family of all fuzzy QL-subimplicators is referred as $\mathcal{J}$.

Proposition 6. *The binary function $J_i : U^2 \to U$, given by the expression*

$$J_i(x, y) = 1 - \frac{1}{i}(x - x^2 y), \forall x, y \in U, \quad (8)$$

is a fuzzy QL-subimplication.

Proof. *Consider the functions $T_P(x, y) = xy$, $S_{Pi}(x, y) = 1 - \frac{1}{i}(1 - x - y + xy)$ and $N_S(x) = 1 - x$. If $T_P(x, y) = xy$, $S_{Pi}(x, y) = 1 - \frac{1}{i}(1 - x - y + xy)$ and $N_S(x) = 1 - x$, it holds that*

$$I_{S_{Pi},N_S,T_P}(x, y) = S_i(N_S(x), T(x, y))$$
$$= 1 - \frac{1}{i}(1 - (1 - x) - xy + (1 - x)xy)$$
$$= 1 - \frac{1}{i}(x - x^2 y)$$
$$= J_i(x, y).$$

Concluding, $J_i \in \mathcal{J}$ which means it is a fuzzy QL-subimplication.

The following proposition is an extension of Proposition 4.2 in [35] by considering the main algebraic properties which characterize the fuzzy QL-subimplication class.

Proposition 7. *A QL-subimplicator $I_{S,N,T} \in \mathcal{J}$ verifies **Ik** for **k** $\in \{0, 2, 4\}$ together with the additional two properties:*

- **I9** : *If $S(N(x), x) = 1$ then $I(x, 1) \le 1$, for all $x \in U$;*
- **I10***a* : *if $x_1 \ge x_2$ then $I(x_1, 0) \le I(x_2, 0)$, for all $x_1, x_2 \in U$.*
- **I10***b* : *if $y_1 \ge y_2$ then $I(1, y_1) \le I(1, y_2)$, for all $y_1, y_2 \in U$.*

Proof. *For $x_1, x_2, x, y_1, y_2, y \in U$, the following is verified.*

- **I0** *By results in Remark 1, it follows that*
  $I_{S,T,N}(0, 0) = S(1, T(0, 0)) = S(1, 0) = 1$;
  $I_{S,T,N}(0, 1) = S(1, T(0, 1)) = S(1, 0) = 1$;
  $I_{S,T,N}(1, 1) = S(0, T(1, 1)) = S(1, 1) = 1$;
- **I2** *Since S, T are monotonic functions, if $y_1 \le y_2$ then $T(x, y_1) \le T(x, y_2)$ and consequently, $I_{S,N,T}(x, y_1) = S(N(x), T(x, y_1)) \le S(N(x), T(x, y_2)) = I_{S,N,T}(x, y_2)$.*
- **I4** $I_{S,N,T}(0, y) = S(1, T(0, y)) = 1$.
- **I9** $I_{S,N,T}(x, 1) = S(N(x), T(x, 1)) \le S(N(x), x) = 1$.
- **I10**a *When $x_1 \ge x_2$ then $N(x_1) \le N(x_2)$. Therefore, $I_{S,N,T}(x_1, 0) = S(N(x_1), T(x_1, 0)) = S(N(x_1), 0) \le S(N(x_2), 0) = S(N(x_2), T(x_2, 0)) = I_{S,N,T}(x_2, 0)$.*
- **I10**b *When $y_1 \ge y_2$ then $I_{S,N,T}(1, y_1) = S(0, T(1, y_1)) \ge S(0, T(1, y_2)) = I_{S,N,T}(1, y_2)$.*

Corollary 1. *The operator $I_{S_i,N_S,T_P} \in \mathcal{J}$ verifies **Ik** for **k** $\in \{0, 2, 4, 9, 10a, 10b\}$.*

Proof. *Straightforward from Proposition 7.*

Remark 3. *Let $I : U^2 \to U$ be a function given by Eq.(7). By taking a t-subconorm S, a fuzzy negation N and a t-subnorm T, the function I does not verify neither **I0** nor **I1**:*

(i) $I(1, 1) = S(N(1), T(1, 1)) = S(0, T(1, 1)) \ge T(1, 1)$;
(ii) $I(1, 0) = S(N(1), T(1, 0)) = S(0, 0) \ge 0$.

Therefore, I is not necessarily a subimplicator.

Clearly, a QL-implication is always a QL-subimplication. See in Figure 3 other instances J_1, J_2, J_3 of such class $\mathcal{J}$. In particular, $J_1 \in \mathcal{J}$ is a QL-implication [38].

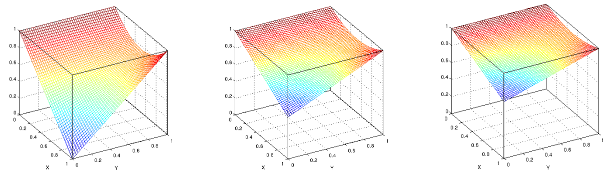

Fig. 2. Fuzzy QL-subimplications of family $\mathcal{J}_{QL} = \{J_1, J_2, J_3\}$.

C. Fuzzy D-(sub)implication class

This section reviews properties of fuzzy D-(sub)implication class.

Definition 2. *A function $I_{S,T,N}: U^2 \to U$ is called a **D-(sub)implicator** if, for all $x, y \in U$, there exist a t-(sub)conorm S, a t-norm T and a fuzzy negation N such that:*

$$I_{S,T,N}(x, y) = S(T(N(x), N(y)), y). \quad (9)$$

In this case, a D-subimplication $I_{S,T,N}$ indicates the underlying t-(sub)conorm, t-norm and fuzzy negation as S, T and N, respectively. The family of all fuzzy D-subimplicators is referred as $\mathcal{I}_D$.

Proposition 8. *A function $I_i : U^2 \to U$, given by*

$$I_i(x, y) = 1 - \frac{1}{i}(1 - y) + \frac{1}{i}(y - 1)^2(1 - x), \forall x, y \in U, \quad (10)$$

is a fuzzy D-subimplication.

Proof. *Consider the functions* $T_P(x,y) = xy$, $S_i(x,y) = 1 - \frac{1}{i}(1 - x - y + xy)$ *and* $N_S(x) = 1 - x$. *It holds that*

$$\begin{aligned} I_{S_i,N_S,T_P}(x,y) &= S_i(1-x, 1-y), y) \\ &= 1 - \frac{1}{i}(1-y)(1-(1-x)(1-y)) \\ &= 1 - \frac{1}{i}(1-y) + \frac{1}{i}(1-y)^2(1-x) = I_i(x,y). \end{aligned}$$

So, $J_i \in \mathcal{I}_D$ *which means it is a fuzzy D-subimplication.*

Proposition 9. *For all* $x, y \in U$, *there exist t-(sub)conorm* S, *a t-norm* T *and a strong fuzzy negation* N *such that:*

$$I_{S,T,N}(x,y) \quad = \quad I_{S,N,T}(N(y), N(x))). \tag{11}$$

The following proposition considers the main algebraic properties which characterize the $\mathcal{I}_D$ class.

Proposition 10. *A D-subimplicator* $I_{S,T,N} \in \mathcal{J}$ *verifies* ***Ik*** *for* $\mathbf{k} \in \{0, 2, 10a, 10b\}$ *together with the additional property:*

I11 : *If* $S(N(y), y) = 1$ *then* $I(0, y) \leq y$, *for all* $y \in U$.

Proof. *For* $x_1, x_2, x, y_1, y_2, y \in U$, *the following is verified.*

I0 *By results in Remark 1, it follows that*
$I_{S,T,N}(0,0) = S(T(1,1),0) = S(1,0) = 1$;
$I_{S,T,N}(0,1) = S(T(1,0),1) = S(0,1) = 1$;
$I_{S,T,N}(1,1) = S(T(0,0),1) = S(0,1) = 1$;

I2 *Since* S, T *are monotonic functions,* $x_1 \leq x_2$ *implies that* $N(x_1) \geq N(x_2)$. *Therefore* $T(N(x_1), N(y)) \geq T(N(x_2), N(y))$ *and consequently, it holds that* $I_{S,T,N}(x, y_1) = S(T(N(x_1), N(y)), y) \geq S(T(N(x_2), N(y)), y) = I_{S,N,T}(x_2, y)$.

I10a *When* $x_1 \geq x_2$, $I_{S,T,N}(x_1, 0) = S(T(N(x_1), 1), 0) \leq S(T(N(x_2), 1), 0) = I_{S,T,N}(x_2, 0)$.

I10b *When* $y_1 \geq y_2$, $I_{S,T,N}(1, y_1) = S(T(0, N(y_1)), y_1) = S(0, y_1) \geq S(0, y_2) = S(T(0, N(y_2)), y_2) = I_{S,T,N}(1, y_2)$.

I11 $I_{S,T,N}(0,y) = S(T(1, N(y)), y) \leq S(N(y), y) = 1$.

Corollary 2. *The operator* $I_{S_i,N_S,T_P} \in \mathcal{I}_D$ *verifies* ***Ik*** *for* $\mathbf{k} \in \{0, 2, 10a, 10b, 11\}$.

Proof. *Straightforward from Proposition 10.*

Remark 4. *Let* $I : U^2 \to U$ *be a function given by Eq.(9). By taking a t-subconorm* S, *a fuzzy negation* N *and a t-subnorm* T, *the function* I *does not verify neither* **I0**, **I1** *nor* **I4**:

(i) $I_{S,T,N}(0,0) = S(T(1,1),0) \leq S(1,0) = 1$;
(ii) $I_{S,T,N}(1,0) = S(T(0,1),0) = S(0,0) \geq 0$;
(iii) $I_{S,T,N}(1,y) = S(T(0,N(y)),y) = S(0,y) \geq y, \ \forall y \in U$.

Therefore, $I_{S,T,N}$ *is not necessarily a subimplicator.*

Clearly, a D-implication is always a D-subimplication.

See in Figure 3 other instances I_1, I_2, I_3 of such class $\mathcal{I}_D$. In particular, $I_1 \in \mathcal{I}_D$ is a D-implication [38].

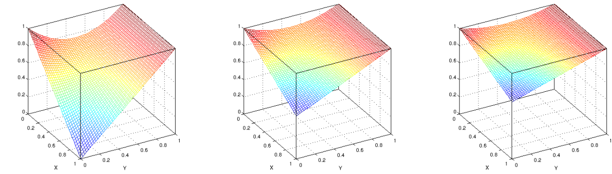

Fig. 3. Fuzzy D-subimplications of family $\mathcal{I}_D = \{I_1, I_2, I_3\}$.

V. Aggregating fuzzy connectives from the OWA operator

Consider $A\colon U^n \to U$ as an n-ary aggregation function and $\mathcal{F} = \{F_i\colon U^k \to U\}$, with $i \in \{1, 2, \ldots, n\}$ as a family of binary functions in the following results of this section.

Definition 3. *[26, Prop. 5.1] An k-ary function* $\mathcal{F}_A\colon U^k \to U$ *is called as* $(A, \mathcal{F})$***-operator on*** U *and given by:*

$$\mathcal{F}_A(x_1, \ldots, x_k) = A(F_1(x_1, \ldots, x_k), \ldots, F_n(x_1, \ldots, x_k)). \tag{12}$$

Proposition 11. *[26, Proposition 6.1] Let* $A\colon U^n \to U$ *be an aggregation function and* $(\mathcal{S})\mathcal{T} = \{(S_i)T_i\colon U^2 \to U\}$, *with* $i \in \{1, 2, \ldots, n\}$ *be a family of t-sub(co)norms. Then the function* $(\mathcal{S}_A : U^2 \to U)$ $\mathcal{T}_A : U^2 \to U$, *called* $((A, \mathcal{S})$*-operator)* $(A, \mathcal{T})$*-operator, is a t-sub(co)norm whenever the following two conditions are verified:*

(i) *A satisfies property* **A8***; and*
(ii) *for all* i, j *such that* $0 \leq i, j \leq n$ *and* $x, y, z \in U$, *each t-sub(co)norm* (S_i) T_i *satisfies the* ***generalized associativity***[1]*:*

$$\begin{aligned} S_i(x, S_j(y,z)) &= S_i(S_j(x,y), z); \\ T_i(x, T_j(y,z)) &= T_i(T_j(x,y), z), \end{aligned} \tag{13}$$

Proposition 12. *Let* $\mathcal{T}$ *and* $\mathcal{S}$ *be the families of t-subnorms and t-subconorms described in Proposition 2. For all* $i, j \geq 1$, *each pair* $T_i, T_j \in \mathcal{T}$ *and* $S_i, S_j \in \mathcal{S}$ *verifies Eqs. (13)a and (13)b, respectively.*

Proof. *For all* x, y, $z \in U$, $T_i(x, T_j(y,z)) = T_i(x, \frac{1}{j}yz) = \frac{1}{ij}(xyz) = \frac{1}{i}(T_j(x,y) \cdot z) = T_i(T_j(x,y), z)$ *Then,* $\mathcal{T}$ *satisfies the Eq.(13)a. The proof for* $\mathcal{S}$ *and related to Eq.(13) can be analogously obtained.*

Proposition 13. *Let* $\mathcal{T}_P$ *and* $\mathcal{S}_P$ *be the corresponding families:*

$$\begin{aligned} \mathcal{T}_P &= \{T_i = \frac{1}{i}xy\colon i \geq 1\}; \\ \mathcal{S}_P &= \{S_i(x) = 1 - \frac{1}{i}(1 - x - y + xy)\colon i \geq 1\}. \end{aligned}$$

Based on the OWA operator, according to Eq. (12), both operators $\mathcal{S}_{OWA}, \mathcal{T}_{OWA}\colon U^2 \to U$, *respectively given as follows*

$$(\mathcal{T}_P)_{OWA}(x, \vec{y}) \quad = \quad \sum_{i=0}^{n} w_i T_{\sigma(i)}(x,y) \tag{14}$$

$$(\mathcal{S}_P)_{OWA}(x, \vec{y}) \quad = \quad \sum_{i=0}^{n} w_i S_{\sigma(i)}(x,y) \tag{15}$$

verify Property **A8**.

Proof. *For all* $x \in U$ *and* $\vec{y} = y_1, \ldots, y_n \in U^n$,

[1]Eq. (13) are particular cases of Eq. (GA) in [40].

(i) *For a t-subnorm $T_i \in \mathcal{T}$, we have that:*

$$\begin{aligned}(\mathcal{T}_P)_{OWA}(x,\vec{y}) &= OWA(T_i(x,y_1),\ldots,T_i(x,y_n))\\ &= OWA(\frac{1}{i}xy_1,\ldots,\frac{1}{i}xy_n)\\ &= \frac{1}{i}x(w_1y_{\sigma(1)},\ldots,w_ny_{\sigma(n)}))\\ &= \frac{1}{i}x\,OWA(\vec{y}) = T_i(x,OWA(\vec{y})).\end{aligned}$$

(ii) *Otherwise, it holds that:*

$$\begin{aligned}(\mathcal{S}_P)_{OWA}(x,y) &= OWA(S_i(x,y_1),\ldots,S_i(x,y_n))\\ &= w_1(1-\frac{1}{i}(1-x-y_{\sigma(1)}+xy_{\sigma(1)}))+\ldots\\ &\quad + w_{\sigma(n)}(1-\frac{1}{i}(1-x-y_{\sigma(n)}+xy_{\sigma(n)}))\\ &= 1-\frac{1}{i}(1-x-OWA(\vec{y})+xOWA(\vec{y}))\\ &= S_i(x,OWA(\vec{y})).\end{aligned}$$

Therefore $\mathcal{T}_M$ and $\mathcal{S}_M$ satisfy **A8**.

Corollary 3. *The operator $((\mathcal{S}_P)_M)$ $(\mathcal{T}_P)_M$ is a t-sub(co)norm.*

Proof. *Straightforward from Propositions 11, 12 and 13.*

The following proposition, reported from [26], states the conditions under which a fuzzy subimplication I_M verifies the generalized exchange principle.

Proposition 14. *[26, Proposition 5.5] Let $A\colon U^n \to U$ be an n-ary aggregation function and $\mathcal{I} = \{I_i\colon U^k \to U\}$, for $i \in \{1,2,\ldots,n\}$ be a family of fuzzy subimplication functions. $\mathcal{I}_A$ verifies* **I5** *when the aggregation A verifies* **A8** *and the following properties is verified:*

I10: ***Generalized Exchange Principle**:*

$$I_i(x, I_j(y,z)) = I_i(y, I_j(x,z)). \tag{16}$$

$\forall x,y,z \in U$ *and* $I_i, I_j \in \mathcal{I}$*, such that* $0 \le i,j \le n$.[2]

A. Aggregating fuzzy (S,N)-subimplications by OWA operator

This section describes the class of aggregating fuzzy (S,N)-subimplications obtained by considering the OWA operator.

Proposition 15. *Let $M\colon U^n \to U$ be an n-ary idempotent aggregation and $\mathcal{I}$ is the family of fuzzy (S,N)-subimplications. $\mathcal{I}_M$ verifies* **I0**, **I2**, **I3**, **I4** *and* **I6** *when all the member-function of $I_i \in \mathcal{I}$ verifies* **I0**, **I2**, **I3**, **I4** *and* **I6**, *respectively.*

Proof. *From Prop. 5, each I_{ij} with $1 \le j \le n$ satisfies* **I0**, **I2**, **I3**, **I4** *and* **I6**. *So, for all $x,y,z \in U$, the following holds.*

I0: *Since the median M verifies* **A0**, *we have that:*

$$\mathcal{I}_M(1,1) = M(I_1(1,1),\ldots,I_n(1,1)) = M(1,\ldots,1) = 1$$
$$\mathcal{I}_M(0,0) = M(I_1(0,0),\ldots,I_n(0,0)) = M(1,\ldots,1) = 1$$
$$\mathcal{I}_M(0,1) = M(I_1(0,1),\ldots,I_n(0,1)) = M(1,\ldots,1) = 1$$

So, $\mathcal{I}_M$ verifies **I0**.

[2] This property also can be considered as a generalization of the extended migrative property, see [41, Definition 2].

I2: *By the monotonicity of M, if $x \le z$, it follows that*

$$\begin{aligned}\mathcal{I}_M(x,y) &= M(I_1(x,y),I_2(x,y),\ldots,I_n(x,y))\\ &\ge M(I_1(z,y),I_2(z,y),\ldots,I_n(z,y)) = \mathcal{I}_M(z,y).\end{aligned}$$

I3: *By the monotonicity of M, if $y \le z$, it follows that*

$$\begin{aligned}\mathcal{I}_M(x,y) &= M(I_1(x,y),I_2(x,y),\ldots,I_n(x,y))\\ &\le M(I_1(x,z),I_2(x,z),\ldots,I_n(x,z)) = \mathcal{I}_M(x,z).\end{aligned}$$

I4: *By property* **A0**, *we obtain the following:*

$$\mathcal{I}_M(0,y) = M(I_1(0,y),\ldots,I_n(0,y)) \quad = M(1,\ldots,1) = 1.$$

I6: *By the contrapositive symmetry of I_n, it holds that*

$$\begin{aligned}\mathcal{I}_M(N(y),N(x)) &= M(I_1(N(y),N(x)),\ldots,I_n(N(y),N(x)))\\ &= M(I_1(x,y),\ldots,I_n(x,y)) = \mathcal{I}_M(x,y).\end{aligned}$$

I7: *When M is an idempotent function, it holds that*

$$\mathcal{I}_M(1,y) = M(I_1(1,y)\ldots,I_n(1,y)) = M(y,y,\ldots,y) = y.$$

I8: *When $x \le y$, the following is also verified: $N_{\mathcal{I}_M}(x) = \mathcal{I}_M(x,0) = M(I_1(x,0)\ldots,I_n(x,0)) \ge M(I_1(y,0)\ldots,I_n(y,0)) = N_{\mathcal{I}_M}(y)$*

Therefore, Proposition 15 is verified.

Proposition 16. *The $(A,\mathcal{I})$-operator defined by OWA aggregator and the family $\mathcal{I}_{(S,N)}$ of (S,N))-subimplications I_i, which is previously defined in Eq (6), verifies* **I5**.

Proof. *According to Prop. 14, it is enough to prove that:*

(ii) *For all $x, y_1,\ldots,y_n \in U$, the following holds:*

$$\begin{aligned}&OWA(I_i(x,y_1),\ldots,I_i(x,y_n)) =\\ &= w_1(1-\frac{1}{i}(x-xy_{\sigma(1)})),\ldots,w_n(1-\frac{1}{i}(x-xy_{\sigma(n)}))\\ &= \sum_{i=0}^{n} w_i - \frac{1}{i}(x - x\sum_{i=0}^{n} w_iy_{\sigma(i)})\\ &= 1-\frac{1}{i}(x - x\cdot OWA(\vec{y})) = I_i(x,OWA(\vec{y})).\end{aligned}$$

Therefore, the OWA operator verifies **A8**.

(ii) *Now, for $I_{i_1}, I_{i_2} \in \mathcal{I}$, we obtain the following:*

$$\begin{aligned}I_{i_1}(x, I_{i_2}(y,z)) &= I_{i_1}(x, 1-\frac{1}{i_2}(y-yz))\\ &= 1-\frac{xy}{i_1i_2}(1-z)\\ &= 1-\frac{1}{i_1}(y-y(1-\frac{1}{i_2}(x-xz)))\\ &= I_{i_1}(y, I_{i_2}(x,z)).\end{aligned}$$

So, $\mathcal{I}_M$ verifies the generalized exchange principle. Concluding, $\mathcal{I}_M$ verifies **I5**.

Corollary 4. *Let $(A,\mathcal{I})$-operator be the OWA operator and $\mathcal{I}$ be the family of (S,N)-subimplications previously defined in Eq.(6) The operator $\mathcal{I}_M$ is an $(\mathcal{S}_M, N_S)$-implication given by*

$$\mathcal{I}_M(x,y) = \mathcal{S}_M(N_S(x),y) \tag{17}$$

Proof. *Straightforward from Propositions 11, 15 and 16.*

Summarizing the main result in Proposition 17, the diagram presented in Figure 4 is showing that the $M = OWA$ operator preserves the (S,N)-subimplication class defined in Prop. 15, which means, $\mathcal{I}_{\mathcal{S}_M,N_S}$ is also an (S,N)-subimplication.

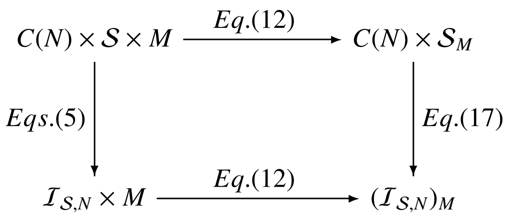

Fig. 4. $(\mathcal{S}_M, N_S)$-implication class.

Analogously, they are obtained by aggregating subimplications in the S-subimplication class.

B. Aggregating fuzzy D- and QL-subimplications

This section analyses under which conditions the class of fuzzy D- and QL-subimplications are preserved by the OWA operator, investigating related properties.

Additionally, we present the subclass of fuzzy QL-subimplication represented by a t-norm T_P, the standard negation N_S together with a t-subconorm $\mathcal{S}_P$, which is obtained by aggregating n fuzzy t-subconorms of the family $\mathcal{S}_P$.

Proposition 17. *Let $M\colon U^n \to U$ be the OWA aggregator and (i) $\mathcal{J} = \{I_i\colon U^2 \to U\colon I_i(x,y) = S_P(N_S(x), T_i(x,y)), 0 \le i \le n\}$; (ii)$\mathcal{L} = \{I_i\colon U^2 \to U\colon I_i(x,y) = S_i(N_S(x), T_P(x,y)), 0 \le i \le n\}$; be the families of fuzzy QL- and D-subimplications, respectively. Then, for all $x,y \in U$, the function $\mathcal{J}_{OWA}(\mathcal{L}_{OWA})\colon U^2 \to U$ is a QL(D)-subimplication, whose definition is given by*

$$\mathcal{J}_{OWA}(x,y) = \mathcal{I}_{\mathcal{S}_P,\mathcal{T}_{OWA},N_S}(x,y); \quad (18)$$

$$(\mathcal{L}_{OWA}(x,y) = \mathcal{I}_{\mathcal{S}_{OWA},\mathcal{T}_P,N_S}(x,y)). \quad (19)$$

Proof. *According to Prop. 3, the following holds.*

$$(i)\mathcal{J}_{OWA}(x,y) = OWA(I_1(x,y),\ldots,I_n(x,y)) = \sum_{i=1}^{n} w_i \cdot I_{\sigma(i)}(x,y)$$

$$= \sum_{i=1}^{n} w_i \cdot S_P(N(x), T_{\sigma(i)}(x,y))$$

$$= S_P(N(x), \sum_{i=1}^{n} w_i \cdot T_{\sigma(i)}(x,y)) = \mathcal{I}_{\mathcal{S}_P,\mathcal{T}_{OWA},N_S}(x,y).$$

$$(ii)\mathcal{L}_{OWA}(x,y) = OWA(I_1(x,y),\ldots,I_n(x,y)) = \sum_{i=1}^{n} w_i \cdot I_{\sigma(i)}(x,y)$$

$$= \sum_{i=1}^{n} w_i \cdot S_{\sigma(i)}(T_P(N_S(x), N_S(y)), y)$$

$$= \mathcal{I}_{\mathcal{S}_P,\mathcal{T}_{OWA},N_S}(x,y).$$

Therefore, Proposition 18 is verified.

Proposition 18. *Let $M\colon U^n \to U$ be the OWA operator and $\mathcal{J}$ be the family of all fuzzy (D-) QL-subimplicators as preseted in Prop.17. Then the $(\mathcal{J}, OWA)$-operator, referred as($\mathcal{L}_{OWA}$) $\mathcal{J}_{OWA}$, verifies the properties **Ik** for* **k** $\in$ $(\{0,2,10a,10b,11\})$ $\{0,2,4,9,10a,10b\}$.

Proof. *Straightforward from Prop. (10) 7.*

Corollary 5. *Let $M\colon U^n \to U$ be the OWA aggregator and $\mathcal{J} = \{I_i\colon U^k \to U\}$ be a family of fuzzy (D-) QL-subimplications given by (Eq. (9)) Eq. (7). Then $\mathcal{J}_M$ verifies (***I0**, **I2**, **10***a*, **10***b and* **11***)* **I0**, **I2**, **I4**, **9**, **10***a and* **10***b*.

Proof. Straightforward from Propositions 6, 17 and 18. □

Summarizing, in Fig. 5, a diagrammatic representation of the result stated in Prop. 17 is presented. In such graphical description the OWA operator preserves the fuzzy D-subimplication class.

1) Firstly, we obtain $\mathcal{S}_{OWA}$ by the OWA operator performed over n t-subconorms $\mathcal{S}_i$. And after that, we are able to define an $(\mathcal{L}, A)$-operator as a fuzzy D-subimplication represented by a t-norm T_P, the standard negation N_S together with a t-subconorm $\mathcal{S}_i$.
2) For each t-subconorm S_i, the family $\mathcal{I}_D$ of D-implications whose explicitly representable member-functions are given by I_{S_i,T_P,N_S}, are constructed. And after that, as a consequence, by aggregating n member-functions of $\mathcal{I}_D$, we obtain an $(\mathcal{J}, A)$-operator.

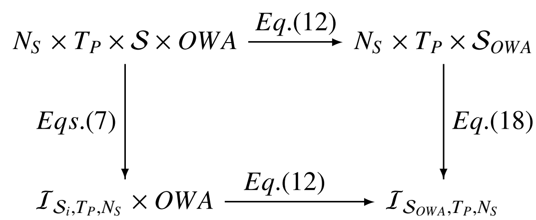

Fig. 5. $(\mathcal{S}_{OWA}, T_P, N_S)$-subimplication class obtained by the OWA operator.

VI. Conclusion and Final Remarks

We have briefly discussed some aspects of the theory of aggregation functions, including the review of some properties and classes of n-ary aggregation functions, and some construction methods.

Thus, (S,N)- QL- and D-subimplications are characterized with respect to the OWA operator.

In particular, the underlying principle of the proof related to properties preserved by the new (S,N)- QL-and D-subimplications, which are obtained by the OWA operator is similar. Since such classes of subimplication are represented by t-subconorms and t-subnorms which are characterized by generalized associativity, the corresponding (S,N)- QL- and D-subimplications are characterized by distributive n-ary aggregation together with related generalizations, as the exchange and neutrality principles.

Further investigations can be done for associative generated aggregation operators. We also consider the study in more detail of the interrelations between these subimplication clsses and their possible conjugate functions. Another interesting issue is to investigate how the method can take into account their dual constructions [42], [43].

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