CS E6204 Lecture 6
Cayley Graphs

Abstract

There are frequent occasions for which graphs with a lot of symmetry are required. One such family of graphs is constructed using groups. These graphs are called Cayley graphs and they are the subject of this chapter. Cayley graphs generalize circulant graphs. There are variations in how different authors define Cayley graphs. This is typical of mathematical literature.

1. Construction and Recognition
2. Prevalence
3. Isomorphism
4. Subgraphs
5. Factorization
6. Further Reading

Adapted from §6.2 of HBGT, by Brian Alspach, University of Regina, Canada.

GENERATING SETS IN A GROUP

FROM APPENDIX A.4 OF GTAIA:

A ***group*** $\mathcal{B} = \langle B, \cdot \rangle$ comprises

- a nonempty set B and
- an associative binary operation $\cdot$

such that

- $\mathcal{B}$ has an identity element, and
- each $g \in G$ has an inverse.

A subset $X \subseteq B$ is a ***generating set*** of $\mathcal{B}$ if every element of B is obtainable as the product (or sum) of elements of X.

Example 0.1 For the group $\mathbb{Z}_n$, a nonempty set of integers mod n is a generating set iff its gcd is 1. This observation employs this standard fact*:*

> The gcd of a set of positive integers equals the smallest positive integer that can be formed by taking sums and differences of numbers in the set.

For instance, the set $\{4, 7\}$ generates $\mathbb{Z}_{24}$, since

$$gcd(4, 7) \ = \ 1$$

but $\{6, 9\}$ does not generate $\mathbb{Z}_{24}$, since

$$gcd(6, 9) \ = \ 3$$

0.1 Review of Cayley Graphs

Let $\mathcal{B} = \langle B, \cdot \rangle$ be a group with generating set X. Then the ***Cayley digraph*** $\vec{C}(\mathcal{B}, X)$ has as its vertex-set and arc-set, respectively,

$$V_{\vec{C}(\mathcal{B},X)} = B \qquad \text{and} \qquad E_{\vec{C}(\mathcal{B},X)} = \{x_b \mid x \in X, b \in B\}$$

Arc x_b joins vertex b to vertex bx. (Bidirected arcs are sometimes used for generators of order 2.)

Let $\mathcal{B} = \langle B, \cdot \rangle$ be a group with generating set X. The ***Cayley graph*** $C(\mathcal{B}, X)$ is the *underlying graph* of the ***Cayley digraph*** $\vec{C}(\mathcal{B}, X)$.

Example 0.2 Figure 1 shows the Cayley digraph for $\mathcal{Z}_5$ with generating set $\{1\}$ and the corresponding Cayley graph, which is clearly isomorphic to the circulant graph $circ(5 : 1)$.

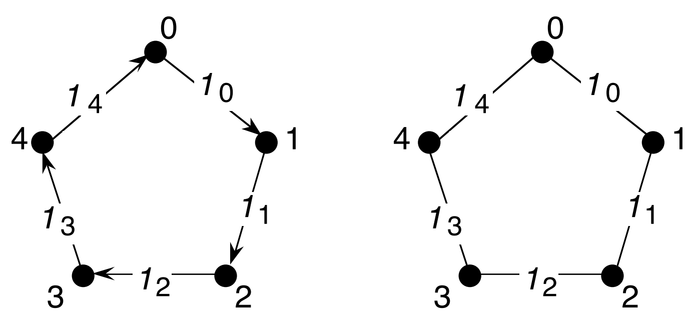

Figure 1: The Cayley digraph $\vec{C}(\mathbb{Z}_5, \{1\})$ and the Cayley graph $C(\mathbb{Z}_5, \{1\})$.

Example 0.3 We observe in Figure 2 that the circulant graph $circ(24 : 6, 9)$ has three components, which are mutually isomorphic.

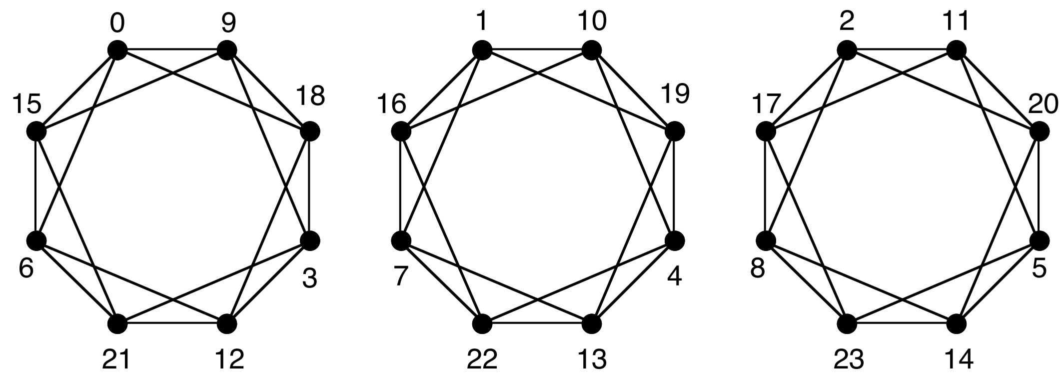

Figure 2: The circulant graph $circ(24 : 6, 9)$ has three components.

- Every Cayley graph is connected, because the edges are defined by a generating set.

- A circulant graph is a Cayley graph if and only if it is connected.

Let $\mathcal{B} = \langle B, \cdot \rangle$ be a group with generating set X. The ***fiber over a generator*** $x \in X$ in the Cayley digraph $\vec{C}(\mathcal{B}, X)$ or Cayley graph $C(\mathcal{B}, X)$ is the set

$$\tilde{x} = \{x_b \mid b \in B\}$$

TERMINOLOGY NOTE The traditional way to draw a Cayley digraph $\vec{C}(\mathcal{B}, X)$ labels the vertices by group elements. Edges were not given distinct names. Instead, a different color or graphic feature was used for each edge fiber $\tilde{x}$, which led to the terminology *Cayley color graph*.

Example 0.4 Figure 3 shows the Cayley digraph for the cyclic group $\mathbb{Z}_8$ with generating set $\{2, 3\}$. The legend at the right identifies the edge fibers, according to their graphic features.

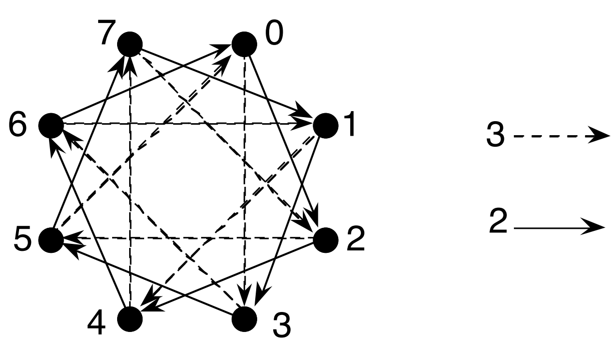

Figure 3: A traditional drawing of the Cayley digraph $\vec{C}(\mathcal{Z}_8, \{2, 3\})$.

An arbitrary graph G is said to be a ***Cayley graph*** if there exists a group $\mathcal{B}$ and a generating set X such that G is isomorphic to the Cayley graph for $\mathcal{B}$ and X.

REMARK Figure 3 illustrates that a non-minimal generating set for a group can be used in a Cayley-graph specification of a graph. A minimum generating set for $\mathbb{Z}_8$ would have only one generator, and the corresponding graph would specify an 8-cycle, not the graph shown above.

Example 0.5 The Cayley digraph $\vec{C}(\mathbb{Z}_8, \{2, 3\})$ can also be specified by the $\mathbb{Z}_8$-voltage graph in Figure 4.

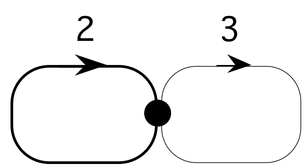

Figure 4: A $\mathbb{Z}_8$-voltage graph that specifies $\vec{C}(\mathbb{Z}_8, \{2, 3\})$.

REMARK In fact, every Cayley graph can be specified by assigning voltages to a bouquet.

MESHES: DIRECT SUMS OF CYCLIC GROUPS

NOTATION The notation used here for an element of a direct sum of k small cyclic groups is a string of k digits, rather than a k-tuple. For instance, 20 stands for the element $(2, 0)$. This convention avoids cluttering the drawings with parentheses and commas.

Example 0.6 Figure 5 illustrates that the 3×4 wraparound mesh is a Cayley graph for the group $\mathbb{Z}_3 \times \mathbb{Z}_4$.

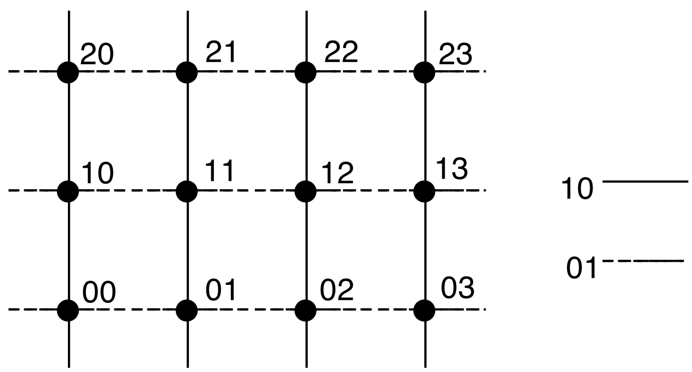

Figure 5: The Cayley graph $\vec{C}(\mathbb{Z}_3 \times \mathbb{Z}_4, \{01, 10\})$.

Example 0.7 The complete graph K_{2n+1} is a Cayley graph for the group $\mathbb{Z}_{2n+1}$, with generating set

$$X = \{1, 2, \ldots, n\}$$

Figure 6 illustrates this for K_7.

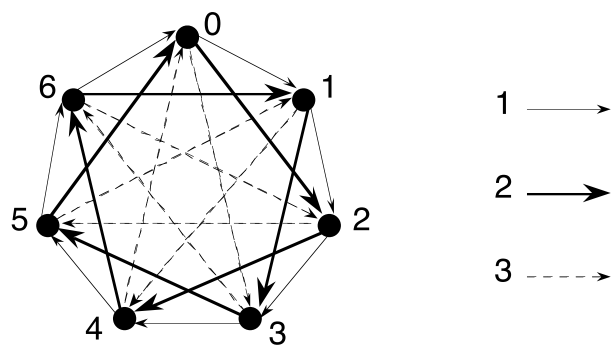

Figure 6: The Cayley digraph $\vec{C}(\mathbb{Z}_7, \{1, 2, 3\})$.

An element of a generating set X of order 2 would cause doubled edges to appear in the Cayley graph. Collapsing such doubled edges to a single edge enhances the usefulness of Cayley graphs in algebraic specification.

Bidirected-arc convention: Let y be a generator of order 2 in a group $\mathcal{B}$. Then the pair of arcs that would otherwise join vertex b to vertex by and vertex by to vertex b in a Cayley digraph for $\mathcal{B}$ is replaced by a single bidirected arc between b and by. The ***underlying graph of a digraph with bidirected arcs*** has a single edge for each bidirected arc.

NOTATION ε denotes the identity element of a group.

Example 0.8 The dihedral group $\mathcal{D}_4$ is the group of rigid-body motions on the unit square. Let r denote a 90° clockwise rotation and let s denote a reflection through a vertical axis. Then the elements of $\mathcal{D}_4$ are

$$\{\varepsilon, r, r^2, r^3, s, rs, r^2s, r^3s\}$$

Figure 7 shows a Cayley digraph for $\mathcal{D}_4$ with generating set $\{r, s\}$.

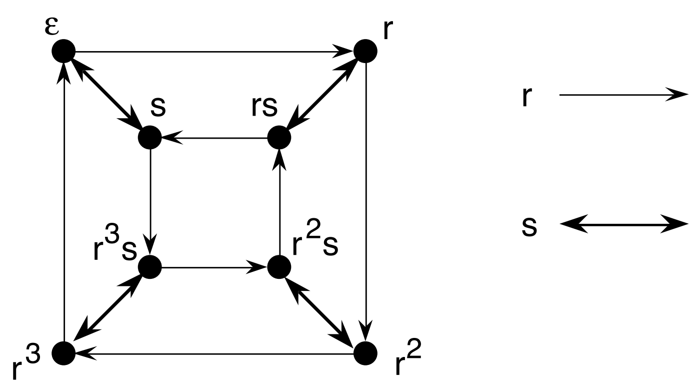

Figure 7: A Cayley digraph for the dihedral group $\mathcal{D}_4$.

JG REVIEW ENDS HERE

1 Construction and Recognition

We restrict ourselves to finite graphs, which means we use finite groups, but the basic construction is the same for infinite groups.

Let $\mathcal{G}$ be a finite group with identity 1. Let $\mathcal{S}$ be a subset of $\mathcal{G}$ such that

- $1 \notin \mathcal{S}$
- $\mathcal{S} = \mathcal{S}^{-1}$, that is, $s \in \mathcal{S}$ if and only if $s^{-1} \in \mathcal{S}$.

The ***Cayley graph*** on $\mathcal{G}$ with ***connection set*** $\mathcal{S}$, denoted $\mathrm{Cay}(\mathcal{G}; \mathcal{S})$, satisfies:

- the vertices of $\mathrm{Cay}(\mathcal{G}; \mathcal{S})$ are the elements of $\mathcal{G}$;
- there is an edge joining $g, h \in \mathrm{Cay}(\mathcal{G}; \mathcal{S})$ if and only if $h = gs$ for some $s \in \mathcal{S}$.

NOTATION The set of all Cayley graphs on $\mathcal{G}$ is denoted $\mathrm{Cay}(\mathcal{G})$.

Remark We do **not** require that the connection set $\mathcal{S}$ generate the group $\mathcal{G}$. It is standard to use additive notation when $\mathcal{G}$ is an abelian group and multiplicative notation for nonabelian groups.

A Cayley graph on the cyclic group $\mathbb{Z}_n$ is always a ***circulant graph***. We use the special notation $\mathrm{Circ}(n; \mathcal{S})$ for a circulant graph on $\mathbb{Z}_n$ with connection set $\mathcal{S}$.

Remark We recall that the connections for a circulant graph on $\mathbb{Z}_n$ generate $\mathbb{Z}_n$ if and only if their gcd is equal to 1. Thus, a circulant graph is not a Cayley graph unless it satisfies this condition.

(JG) Cayley noted that group can be regarded as a set of permutations on its own elements, acting by left or right multiplication.

Notation g_L denotes the permutation on the group $\mathcal{G}$ given by the rule $g_L(h) = gh$.

Notation $\mathcal{G}_L$ denotes the permutation group

$$\{g_L : g \in \mathcal{G}\}$$

which is called the ***left-regular representation*** of $\mathcal{G}$.

An ***automorphism of a simple graph*** G is a bijection f on the vertex set $V(G)$ such that

$$\langle u, v\rangle \text{ is an edge if and only if } \langle f(u), f(v)\rangle \text{ is an edge.}$$

Let $\mathcal{G}$ be a transitive permutation group acting on a finite set Ω. If $\mathcal{G}$ satisfies any one of the following three equivalent conditions, then it is said to be a ***regular*** action:

- The only element of $\mathcal{G}$ fixing an element of Ω is the identity permutation;
- $|\mathcal{G}| = |\Omega|$;
- for any $\omega_1, \omega_2 \in \Omega$, there is a unique element $g \in \mathcal{G}$ satisfying the equation $\omega_1 g = \omega_2$.

Fact-JG: **** The left and right actions of a group on its own elements are regular. Moreover, the action of a group $\mathcal{G}$ on the vertices of any Cayley graph for $\mathcal{G}$ is a regular group of automorphisms on that Cayley graph.

Examples

Example 1.1 The hypercube Q_n may be represented as a Cayley graph on the elementary abelian 2-group $\mathbb{Z}_2^n$ using the standard generators

$$X : e_1, e_2, \ldots, e_n$$

for the connection set, where e_i has a 1 in the i-th coordinate and zeroes elsewhere.

Example 1.2 The complete graph K_n is representable as a Cayley graph on any group $\mathcal{G}$ of order n, where the connection set is the set of non-identity elements of the group. We get the complement of K_n by using the empty set as the connection set.

Example 1.3 he complete symmetric multipartite graph $K_{m;n}$, with m parts, each of cardinality n, is realizable as a circulant graph on $\mathbb{Z}_{mn}$, with the connection set

$$X \;=\; \{j : j \not\equiv 0 \mod m\}$$

EXERCISE Draw the complete symmetric multipartite graph $K_{3;4}$ as a circulant graph.

The graph formed on the additive group of the finite field $GF(q^r)$, with $q \equiv 1 \pmod 4$, where the connection set is the set of quadratic residues in $GF(q^r)$, is called a ***Paley graph***.

(JG) For $q = 5$, we calculate squares:

$$1^2 = 1 \quad 2^2 = 4$$

thus, the Paley graph is $Circ\,(5:1)$, the 5-cycle.

(JG) For $q = 13$, we calculate squares:

$$1^2 = 1 \quad 2^2 = 4 \quad 3^2 = -4 \quad 4^2 = 3 \quad 5^2 = -1 \quad 6^2 = -3$$

thus, the Paley graph is $Circ\,(13:1,3,4)$.

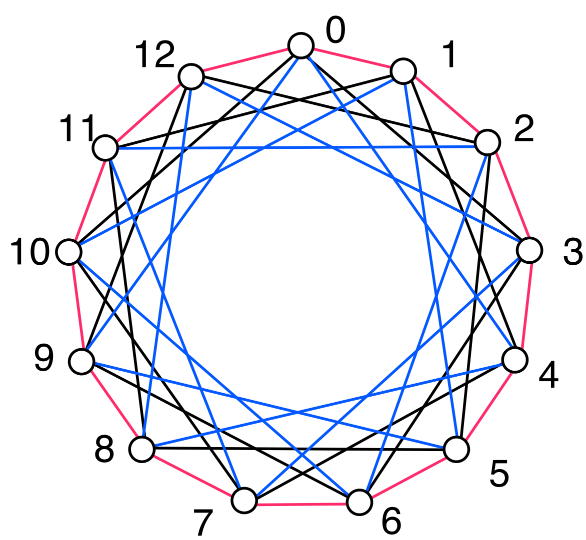

Figure 8: Paley graph of order 13.

(JG) All Paley graphs are self-complementary.

Exercise Prove that the Paley graph of order 13 is self-complementary.

The circulant graph of even order n with connection set $S = \{\pm 1, n/2\}$ is known as the ***Möbius ladder*** of order n.

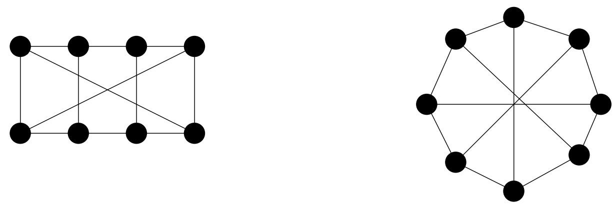

Figure 9: Two drawings of the Möbius ladder of order 8.

Facts

FACT Every Cayley graph is vertex-transitive.

FACT The Cayley graph $\mathrm{Cay}(\mathcal{G}; \mathcal{S})$ is connected if and only if $\mathcal{S}$ generates $\mathcal{G}$.

(JG) However, counting the number of automorphisms of a graph is at least as hard as the graph isomorphism problem.

Sabidussi's Theorem is the basis for all work on recognizing whether or not an arbitrary graph is a Cayley graph. It is an absolutely fundamental result.

FACT [Sa58] **Sabidussi's Theorem**: A graph G is a Cayley graph if and only if $\mathrm{Aut}(G)$ contains a subgroup that acts regularly on G.

Example 1.4 $\mathrm{Aut}(C_4) = \mathbb{D}_4$, and $\mathbb{D}_4$ does not act regularly on C_4 because the diagonal reflections have fixed points. However, The subgroup of rotations acts regularly, so C_4 is a Cayley graph.

(JG) The forward direction of Sabidussi's theorem follows from the topological theorem that a graph is a regular covering space if and only if there is a regular action, from which it follows that graph is a Cayley graph if and only if it is a regular covering graph of a bouquet.

REMARK The Cayley graphs on the group $\mathcal{Z}_\ell^n$ with the standard generators of the group as connection set are used as theoretical models of interconnection networks of homogeneous processors in computer science.

2 Prevalence

Since all Cayley graphs are vertex-transitive, a natural question is whether or not the family of Cayley graphs encompasses all finite vertex-transitive graphs. The Petersen graph (which Alspach denotes JP_5) is the smallest vertex-transitive graph that is not a Cayley graph, and it suggests the topic of this section.

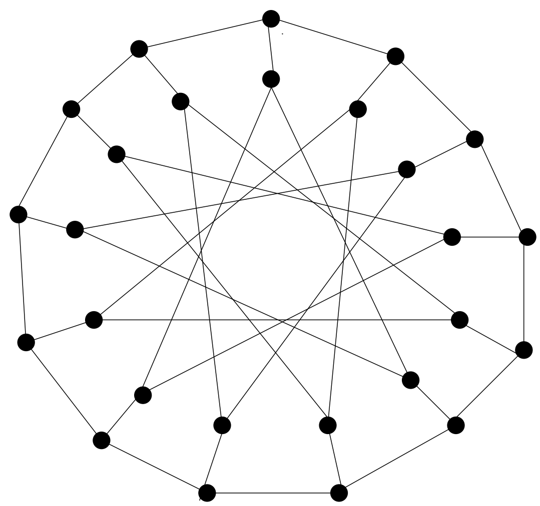

Figure 10: A non-Cayley vertex-transitive graph of order 26.

(JG) Proof that JP_5 is non-Cayley follows from the fact that there are only two groups of order 10, namely, $\mathbb{Z}_{10}$ and $\mathbb{D}_5$. Similarly, there are only two groups of order 26, namely, $\mathbb{Z}_{26}$ and $\mathbb{D}_{13}$. It is provable from this that JP_{13} is non-Cayley.

(⇒) The next four pages are for **REFERENCE ONLY**.

NOTATION $\mathbb{NC}$ denotes the set of integers n for which there exists a vertex-transitive graph of order n that is non-Cayley.

Example 2.1 If $n \in \mathbb{NC}$, then any multiple of n belongs to $\mathbb{NC}$. This follows by taking the appropriate number of vertex-disjoint copies of a non-Cayley, vertex-transitive graph of order n. Thus, in order to determine $\mathbb{NC}$, it suffices to find the minimal elements belonging to $\mathbb{NC}$.

Facts

The first two facts reduce the problem of trying to characterize membership in $\mathbb{NC}$ to the consideration of square-free integers.

FACT A prime power $p^e \in \mathbb{NC}$ whenever $e \geqq 4$.

FACT Any positive integer, other than 12, divisible by a square is in $\mathbb{NC}$.

FACT Let p and q be distinct primes with $p < q$. Then $pq \in \mathbb{NC}$ if and only if one of the following holds:

- p^2 divides $q - 1$;
- $q = 2p - 1 > 3$ or $q = \frac{p^2+1}{2}$;
- $q = 2^t + 1$ and either p divides $2^t - 1$ or $p = 2^{t-1} - 1$;
- $q = 2^t - 1$ and $p = 2^{t-1} + 1$; and
- $p = 7$, $q = 11$;

FACT Let p and q be odd primes satisfying $p < q$. Then $2pq \in \mathbb{NC}$ if and only if one of the following holds:

- p^2 divides $q - 1$;
- $p \equiv 1 (\text{mod } 4)$ or $q \equiv 1 (\text{mod } 4)$;
- $p = 7$, $q = 11$;
- $p \equiv q \equiv 3 (\text{mod } 4)$, p divides $q - 1$, and p^2 does not divide $q - 1$;
- $p \equiv q \equiv 3 (\text{mod } 4)$, and $p = \frac{q+1}{4}$; and
- $p = 7$, $q = 19$.

FACT Let p, q, r be distinct odd primes satisfying $p < q < r$. Then $pqr \in \mathbb{NC}$ if and only if at least one of pq, pr or qr is a member of $\mathbb{NC}$, or none of pq, pr and qr is a member of $\mathbb{NC}$ but one of the following holds:

- $pqr = (2^{2^t} + 1)(2^{2^{t+1}} + 1)$, for some t;
- $pqr = (2^{d\pm 1} + 1)(2^d - 1)$, for some prime d;
- $pq = 2r \pm 1$ or $pq = (r + 1)/2$;
- $pq = (r^2 + 1)/2$ or $pr = (q^2 + 1)/2$;
- $pq = (r^2 - 1)/24x$ or $pr = (q^2 - 1)/24x$, where $x \in \{1, 2, 5\}$;
- $ab = 2^t + 1$ and c divides $2^t - 1$, where $\{a, b, c\} = \{p, q, r\}$;
- the largest power of p dividing $q - 1$ is p^p and the largest power of q dividing $r - 1$ is q^q;
- $q = (3p+1)/2$ and $r = 3p+2$, or $q = 6p-1$ and $r = 6p+1$;
- $q = (r - 1)/2$ and p divides $r + 1$, where $p > q$ when $p = (r + 1)/2$;

- $p = (k^{d/2}+1)/(k+1)$, $q = (k^{d/2}-1)/(k-1)$, $r = (k^{d-1}-1)/(k-1)$, where $k, d-1, d/2$ are primes and $p > q$ may be the case;
- $p = (k^{(d-1)/2}+1)/(k+1)$, $(k^{(k-1)/2}-1)/(k-1)$, $r = (k^d-1)/(k-1)$, where $k, d, (d-1)/2$ are primes and $p > q$ may be the case;
- $p = k^2-k+1$, $q = (k^5-1)/(k-1)$, $r = (k^7-1)/(k-1)$, where k is prime;
- $p = 3$, $q = (2^d+1)/3$, $r = 2^d-1$, where d is a prime;
- $p = (2^d+1)/3$, $q = 2^d-1$, $r = 2^{2d\pm 2}+1$, where $d = 2^t \pm 1$ is prime;
- $p = 5$, $q = 11$ and $r = 19$; and
- $p = 7$, $q = 73$ and $r = 257$.

Research Problem

P1: Is there a number $k > 0$ such that every product of k distinct primes is in $\mathbb{NC}$? There is no known characterization of the members of $\mathbb{NC}$ that are products of four distinct primes.

3 Isomorphism

Some of the most interesting and deepest work on Cayley graphs has revolved around the question of trying to determine when two Cayley graphs are isomorphic.

A Cayley graph $\mathrm{Cay}(\mathcal{G};\mathcal{S})$ is a ***CI-graph*** if whenever

$$\mathrm{Cay}(\mathcal{G};\mathcal{S}) \cong \mathrm{Cay}(\mathcal{G};\mathcal{S}')$$

there exists an automorphism $\alpha \in \mathcal{A}\mathrm{ut}(\mathcal{G})$ such that $\mathcal{S}' = \alpha(\mathcal{S})$.

A group $\mathcal{G}$ is a ***CI-group*** if every Cayley graph on $\mathcal{G}$ is a CI-graph.

Example 3.1 The two circulant graphs

$$\mathrm{Circ}(7;1,2) \quad \text{and} \quad \mathrm{Circ}(7;1,3)$$

are isomorphic via the mapping that takes g to $3g$ for all elements of $\mathbb{Z}_7$. This mapping is an automorphism of $\mathbb{Z}_7$.

Example 3.2 For $n = 25$, let

$$[\mathcal{S} = \{1, 4, 5, 6, 9, 11\}] \quad \text{and} \quad [\mathcal{S}' = \{1, 4, 6, 9, 10, 11\}]$$

The two circulant graphs Circ$(25; \mathcal{S})$ and Circ$(25; \mathcal{S}')$ are isomorphic since both are wreath products of a 5-cycle with a 5-cycle. On the other hand, it is easy to see there is no $a \in \mathbb{Z}_{25}^*$ for which $\mathcal{S}' = a\mathcal{S}$ is satisfied. Thus, $\mathbb{Z}_{25}$ is not a CI-group.

FACT [Mu97] The cyclic group $\mathbb{Z}_n$ is a CI-group if and only $n = 2^e m$, where m is odd and square-free and $e \in \{0, 1, 2\}$ or $n \in \{8, 9, 18\}$.

(JG) For $e = 0$, this yields $\mathbb{Z}_n$ where n is a product of one or more distinct odd primes.

EXERCISE Show that Z_8 is a CI-group. This is case-by-case analysis.

The Cayley graphs on a CI-group can be enumerated in a straightforward way using Pőlya enumeration [Br64]. The next two theorems illustrate this for circulant graphs.

FACT [Tu67] If p is an odd prime, then the # of isomorphism classes of vertex-transitive graphs of order p is

$$\left[\frac{2}{p-1}\sum_{d}\Phi(d)2^{(p-1)/2d}\right]$$

where the summation runs over all divisors d of $(p-1)/2$ and Φ denotes the Euler totient function.

Example 3.3 For $p = 7$, Tutte's theorem gives the number of vertex-transitive 7-vertex graphs as

$$\begin{aligned}\frac{2}{7-1} & \left[\Phi(1)2^{(7-1)/2\cdot 1} + \Phi(3)2^{(7-1)/2\cdot 3}\right] \\ & = \tfrac{1}{3}\left[1\cdot 2^3 + 2\cdot 2^1\right] = \tfrac{12}{3} = 4\end{aligned}$$

The four graphs are $7K_1$, C_7, $Circ(7:1,2)$, and K_7.

EXERCISE Evaluate Tutte's theorem for $p = 11$ and list the different graphs.

Research Problem

P2: For an odd prime p, determine the values of e for which $\mathbb{Z}_p^e$ is a CI-group.

4 Subgraphs

There are interesting results and questions regarding subgraphs of Cayley graphs. Some of the results we mention hold for all vertex-transitive graphs, and we state them accordingly.

A graph G is ***Hamilton-connected*** if for any two vertices u, v of G, there is a Hamilton path whose terminal vertices are u and v.

A bipartite graph with parts A and B is ***Hamilton-laceable*** if for any $u \in A$ and $v \in B$, there is a Hamilton path whose terminal vertices are u and v.

FACT [ChQi81] Let G be a connected Cayley graph on a finite abelian group. If G is bipartite and has degree at least 3, then G is Hamilton-laceable. If G is not bipartite and has degree at least 3, then G is Hamilton-connected.

Facts

FACT Let G be a connected vertex-transitive graph. If G has even order, then G has a 1-factor. If G has odd order, then $G-v$ has a 1-factor for every vertex $v \in G$.

FACT If a d-regular graph G is connected and vertex-transitive, then G is d-edge-connected.

FACT For every positive integer m, there exists a Paley graph containing all graphs of order m as induced subgraphs.

FACT [Wi84] Every connected Cayley graph on a group of order p^e, where p is a prime and $e \geqq 1$, has a Hamilton cycle.

5 Factorization

Definitions

A 1-***factorization*** of a graph is a partition of the edge set into 1-factors.

The connection set $\mathcal{S}$ is a ***minimal generating Cayley set*** for the group $\mathcal{G}$ if $\mathcal{S}$ generates $\mathcal{G}$, but

$$\mathcal{S} - \{s, s^{-1}\}$$

generates a proper subgroup for every $s \in \mathcal{S}$.

A ***Hamilton decomposition*** of a graph G is a partition of the edge set into Hamilton cycles when the degree is even, or a partition into Hamilton cycles and a 1-factor when the degree is odd.

An ***isomorphic factorization*** of a graph G is a partition of the edge set of G so that the subgraphs induced by the edges in each part are pairwise isomorphic.

Facts

FACT [St85] A given connected Cayley graph on the group $\mathcal{G}$ has a 1-factorization if one of the following holds:

- $|\mathcal{G}| = 2^k$ for an integer k;
- $\mathcal{G}$ is an even order abelian group; or
- $\mathcal{G}$ is dihedral or dicyclic.

FACT [Li96, Li03ta] If $G = \mathrm{Cay}(\mathcal{G}, \mathcal{S})$ is a connected Cayley graph on an abelian group $\mathcal{G}$ and $\mathcal{S}$ is a minimal generating Cayley set, then G has a Hamilton decomposition.

FACT [Fi90] If T is any tree with n edges, then the n-dimensional cube Q_n has an isomorphic factorization by T. Furthermore, there is an isomorphic factorization so that each copy of T is an induced subgraph.

Research Problem

P3: Let $\mathcal{C}$ be one of the classes of circulant graphs, or Cayley graphs, or vertex-transitive graphs. Is it the case that for every graph $G \in \mathcal{C}$, whenever d divides $|E(G)|$, then there is an isomorphic factorization of G into d subgraphs?

6 Further Reading

Remarks

Remark There is a long history and an extensive literature about imbedding Cayley graphs on orientable and non-orientable surfaces. See Chapter 7 in this volume. The books [GrTu87], [Ri74], and [Wh01] and a recent excellent survey [RiJaTuWa03ta] provide a good starting point for this topic.

Remark There are a variety of meaningful applications of Cayley graphs.

- Circulant graphs appear in the study of circular chromatic number. For a recent survey see [Zh01].
- Cayley graphs occur frequently in the literature on networks. A recent book on this topic is [Xu01] and a fundamental paper is [AkKr89].
- Cayley graphs play a central role in some work on expanders. Two excellent references are [Al95] and [Lu95].

Remark A survey on Cayley graph isomorphism is provided in [Li02].

Remark A good general discussion about vertex-transitive graphs and Cayley graphs is [Ba95]. A good starting point for reading about $\mathbb{NC}$ is [IrPr01].

References

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