QUASICLASSICAL METHOD IN THE THEORY OF SUPERCONDUCTIVITY

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It is shown that replacement of quantum-mechanical averages by the average values of the corresponding classical quantities over all trajectories with a prescribed energy is not valid in the general case. The dependence of the penetration depth on the field is found without making any assumptions about the weakness of the interaction between the electrons and the field of the impurities; the case of very dirty films is also considered.

1. Use of a quasiclassical method[1,2] has turned out to be convenient in a number of problems in the theory of superconductivity. In this method the calculation of various characteristics of a superconductor reduces to the calculation of the average value of a product of matrix elements of single-particle operators. Usually the quantum-mechanical averages are replaced by the average values of the corresponding classical quantities over all classical trajectories of a given energy. It is shown below that in certain cases such a substitution leads to erroneous results. This is associated with the fact that the momentum operators of an electron, situated in an impurity field, do not commute at different moments of time. The problem of the dependence of the penetration depth on the magnetic field and on the impurity concentration is considered in detail in the London case. The interaction between electrons and impurities is not assumed to be weak. And in this case the dependence of the penetration depth on the field is determined not only by the cross sections but by the scattering amplitudes themselves.

The case of very dirty films[3] is also considered, where the interaction with impurities is not assumed to be weak or isotropic. The result only depends on the transport time τ_{tr}. An equation is found for the critical field associated with an arbitrary concentration of impurities.

2. The current density is expressed in terms of the Green's function by the formula

$$\mathbf{j}=\frac{e}{m}T\,\mathrm{Sp}\sum_{\omega}\delta(\mathbf{r}-\hat{\mathbf{r}})\hat{\mathbf{p}}\tau_z\hat{G}(\hat{\mathbf{r}},\mathbf{r}), \tag{1}$$

where the Green's function $\hat{G}(\mathbf{r},\mathbf{r}')$ satisfies the Gor'kov equations which are conveniently written in matrix form:

$$\left[-\hat{H}+i\omega\tau_z+\begin{pmatrix}0 & \Delta\\ -\Delta^* & 0\end{pmatrix}+\frac{e}{m}(\hat{\mathbf{p}}\mathbf{A})\tau_z\right]\hat{G}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),$$
$$\hat{H}=-\frac{1}{2m}\frac{\partial^2}{\partial\mathbf{r}^2}+V(\mathbf{r})-\mu, \tag{2}$$

where $\hat{H}$ is the Hamiltonian of an electron in the field of the impurities.

In the approximation linear with respect to the field, the expression for the current in the representation of eigenfunctions of the Hamiltonian $\hat{H}$ takes the form

$$\mathbf{j}(\mathbf{r})=-\left(\frac{e}{m}\right)^2\mathrm{Sp}\,T\sum_{\omega}\langle n|\delta(\mathbf{r}-\hat{\mathbf{r}})\hat{\mathbf{p}}|m\rangle\langle m|(\hat{\mathbf{p}}\mathbf{A})|n\rangle\tau_zG_m\tau_zG_n, \tag{3}$$

where

$$G_n=\frac{1}{\varepsilon_n^2+\omega^2+\Delta^2}\begin{pmatrix}-\varepsilon_n-i\omega, & -\Delta\\ \Delta, & -\varepsilon_n+i\omega\end{pmatrix}.$$

Changing to a time representation and using the weak dependence of the diagonal matrix element on energy near the Fermi surface, we obtain

$$\mathbf{j}(\mathbf{r})=-2\pi\nu\left(\frac{e}{m}\right)^2T\sum_{\omega}\mathrm{Sp}\int_{-\infty}^{\infty}\langle\!\langle\delta(\mathbf{r}-\mathbf{r}(0))\hat{\mathbf{p}}(0)$$
$$\times(\hat{\mathbf{p}}(t_1)\mathbf{A}(t_1))\rangle\!\rangle\,\tau_zG(-t_1)\tau_zG(t_1)\,dt_1, \tag{4}$$

where

$$\hat{\mathbf{p}}(t)=\exp(i\hat{H}t)\hat{\mathbf{p}}\exp(-i\hat{H}t),\qquad \nu=mp_0/2\pi^2,$$

and $\langle\langle\ldots\rangle\rangle$ denotes the average over all states at the Fermi surface,

$$G(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G_\xi\exp(i\xi t)\,d\xi$$
$$=-\frac{i}{2E}e^{-E|t|}(E\,\mathrm{sign}\,t+\omega\tau_z+\Delta\tau_y),\qquad E^2=\omega^2+\Delta^2. \tag{5}$$

For a Fourier component Eq. (4) takes the form

$$\mathbf{j}(\mathbf{g})=-\frac{Ne^2}{m}Q(\mathbf{g})\mathbf{A}(\mathbf{g}). \tag{6}$$

Below we confine our attention to consideration of the limiting London case when the change in the potential $\mathbf{A}$ is small over distances of the order of the size of a pair. In this case it is sufficient to find $\mathbf{Q}(0)$. Taking formula (5) into account, we obtain

$$Q(0)=\frac{3\pi T\Delta^2}{p_0^2}\sum_{\omega}\int_{-\infty}^{\infty}E^{-2}\exp(-2E|t|)\langle\!\langle p_z(0)p_z(t)\rangle\!\rangle\,dt. \tag{7}$$

The average appearing in this expression may be found with the aid of the kinetic equation and is given by

$$\langle\langle p_z(0)p_z(t)\rangle\rangle=\tfrac{1}{3}p_0^2\exp(-|t|/\tau_{tr}). \tag{8}$$

Substituting this expression into formula (7), we obtain the usual expression for $\mathbf{Q}$:[4]

$$Q(0)=\pi\Delta^2T\sum_{\omega}[E^2(E+1/2\tau_{tr})]^{-1}. \tag{9}$$

It is important to note that formula (8) is obtained with the aid of the kinetic equation for the pair correlation function. The collision term in this equation is expressed in terms of the scattering cross section and

in the general case has the same form both in the Born limit and in the classical limit. Therefore formula (9) may be obtained both[4] in the Born approximation with subsequent replacement of the Born amplitude by the exact amplitude and in the method of classical trajectories with replacement of the classical cross section by the exact cross section.

However, in the next approximation cubic in the field, both of these methods are erroneous in the general case. As shown below the Born approximation leads to the correct result only in the limit of a small mean free path, and the method of classical trajectories is not applicable even in this case.

3. One can find the next terms in an expansion of the current density in powers of the field in analogy to formulas (4) and (7). In the London limit we obtain

$$j_z = -2\pi\nu\left(\frac{e}{m}\right)^4 A^3 T \sum_\omega \mathrm{Sp}\int_{-\infty}^{\infty}\int dt_1\,dt_2\,dt_3 \langle\!\langle \hat{p}_z(0)\hat{p}_z(t_1)\hat{p}_z(t_2)\hat{p}_z(t_3)\rangle\!\rangle$$
$$\times\tau_z G(-t_1)\tau_z G(t_1-t_2)\tau_z G(t_2-t_3)\tau_z G(t_3)$$
$$= -\left(\frac{e}{m}\right)^4 A^3 T \sum_\omega \mathrm{Sp}\int_{-\infty}\int\int\int \frac{d\omega_1}{2\pi}\frac{d\omega_2}{2\pi}\frac{d\omega_3}{2\pi}\frac{d\omega_4}{2\pi} P(\omega_1\omega_2\omega_3\omega_4)$$
$$\times\tau_z G_{-\omega_1}\tau_z G_{-\omega_2}\tau_z G_{-\omega_3}\tau_z G_{-\omega_4}, \qquad (10)$$

where

$$P(\omega_1\omega_2\omega_3\omega_4)$$
$$= (2\pi)^4 \mathrm{Sp}\{\hat{p}_z\delta(\hat{H}-\omega_1)\hat{p}_z\delta(\hat{H}-\omega_2)\hat{p}_z\delta(\hat{H}-\omega_3)\hat{p}_z\delta(\hat{H}-\omega_4)\}, \qquad (11)$$

$$\langle\!\langle p_z(0)p_z(t_1)p_z(t_2)p_z(t_3)\rangle\!\rangle = \frac{1}{\nu(2\pi)^4}\int_{-\infty}^{\infty}\int\int P(\omega_1\omega_2\omega_3 0)$$
$$\times\exp\{-i[\omega_1 t_1+\omega_2(t_2-t_1)+\omega_3(t_3-t_2)]\}\,d\omega_1\,d\omega_2\,d\omega_3. \qquad (12)$$

The average value of the product of four momenta at different moments of time is not determined by the pair correlation function and cannot be found from the kinetic equation. Therefore, for its evaluation we shall use a diagram technique.[5] In this connection the Born approximation for the scattering of an electron by impurities will not be used.

In formula (11) it is convenient to change to the momentum representation, having written each δ-function in the form of two terms

$$2\pi i\delta(\hat{H}-\omega) = G_\omega^A - G_\omega^R,$$
$$G_{pp'}^R = (G_{pp'}^A)^* = (-\hat{H}+\omega+i\delta)^{-1}. \qquad (13)$$

Expanding $G_{pp'}$ in powers of the interaction V with the impurities, averaging with respect to the positions of the impurities, and neglecting, as is customary, intersecting diagrams, we obtain the following expression for the averaged Green's function:

$$G_p^R = (\omega-\xi-nf_{pp})^{-1}, \quad G_p^A = (G_p^R)^*, \qquad (14)$$

where $f_{pp'}$ satisfies the equation which is graphically depicted in Fig. 1:

$$f_{pp'} = V_{pp'} + \int\frac{dp_1}{(2\pi)^3}V_{pp_1}G_{p_1}^R f_{p_1p'}. \qquad (15)$$

In this equation one can replace the function G_p^R by the function $G_p^0 = (\omega-\xi+i\delta)^{-1}$ since these functions differ from one another only in a narrow region near the pole, and the integrals over this region entering into formula (15) coincide for them. Therefore, the

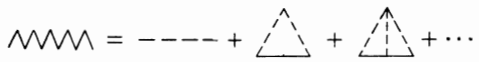

FIG. 1

function $f_{pp'}$ differs from the exact scattering amplitude only by the factor $(-2\pi/m)$. The scattering amplitude at zero degrees, whose real part gives an unimportant energy shift and whose imaginary part is expressed in terms of the total cross section σ, enters into formula (14). As a result

$$G_p^R = \left(\omega-\xi+\frac{i}{2\tau}\right)^{-1}, \quad \frac{1}{\tau} = n v\sigma, \quad \mathrm{Im}\,f_{pp} = -\frac{v\sigma}{2}. \qquad (16)$$

The diagrams shown in Fig. 2 arise upon averaging expression (11) with respect to the positions of the impurities, where each wavy line depicts a scattering amplitude, each point depicts the impurity concentration n. All solid lines on one side of a square represent either G^R or G^A. The vertex of a square corresponds to a momentum p_z if both lines coming together at this vertex represent G^R or G^A. However,

FIG. 2

if one line is G^R and the other G^A, then the vertex corresponds to an expression $\tilde{p}_z$, which satisfies the equation

$$\tilde{p}_z = p_z + \frac{n}{(2\pi)^3}\int G_{\omega_1}^A(p_1)G_{\omega_2}^R(p_1)\,|f_{pp_1}|^2\,\tilde{p}_z(p_1)\,dp_1 \qquad (17)$$

and equals

$$\tilde{p}_z = p_z\frac{\omega_2-\omega_1+i/\tau}{\omega_2-\omega_1+i/\tau_{tr}}. \qquad (18)$$

Each line from the sides of a square may represent either G^R or G^A; therefore expression (11) represents a total of sixteen terms. Three of them are substantially different:

$$P_1(\omega_1\omega_2\omega_3\omega_4) = (2\pi)^4\mathrm{Sp}\{p_z G_{\omega_1}^A p_z G_{\omega_2}^A p_z G_{\omega_3}^R p_z G_{\omega_4}^R\}$$
$$= -\frac{mp_0^5}{\pi}\frac{C(\omega_3-\omega_1)+C(\omega_4-\omega_2)}{(\omega_4-\omega_1+i/\tau_{tr})(\omega_3-\omega_2+i/\tau_{tr})},$$
$$P_2(\omega_1\omega_2\omega_3\omega_4) = (2\pi)^4\mathrm{Sp}\{p_z G_{\omega_1}^A p_z G_{\omega_2}^R p_z G_{\omega_3}^R p_z G_{\omega_4}^R\}$$
$$= \frac{mp_0^5}{\pi}\frac{C(\omega_3-\omega_1)}{(\omega_2-\omega_1+i/\tau_{tr})(\omega_4-\omega_1+i/\tau_{tr})},$$
$$P_3(\omega_1\omega_2\omega_3\omega_4) = (2\pi)^4\mathrm{Sp}\{p_z G_{\omega_1}^A p_z G_{\omega_2}^R p_z G_{\omega_3}^A p_z G_{\omega_4}^R\}$$
$$= \frac{mp_0^5}{\pi}\left\{\frac{i}{5}(\omega_1+\omega_3-\omega_2-\omega_4)+B\right\}\left\{\left(\omega_2-\omega_1+\frac{i}{\tau_{tr}}\right)\right.$$
$$\left.\times\left(\omega_4-\omega_1+\frac{i}{\tau_{tr}}\right)\left(\omega_2-\omega_3+\frac{i}{\tau_{tr}}\right)\left(\omega_4-\omega_3+\frac{i}{\tau_{tr}}\right)\right\}^{-1}, \qquad (19)$$

where

$$C(\omega) = \frac{1}{\omega+i/\tau}\left[\frac{i}{5}-\frac{1}{9\tau}\frac{1}{\omega+i\delta}-\frac{4}{45}\frac{1}{\omega\tau_2+i(\tau_2/\tau-1)}\right],$$
$$B = \frac{8}{45}\left(\frac{1}{\tau}-\frac{1}{\tau_2}\right)+\frac{nmp_0}{2(2\pi)^3}\left\{\frac{(mp_0)^2}{(2\pi)^4}\int\int\int\int f_{p_1p_2}f^*_{p_2p_3}f_{p_3p_4}f^*_{p_4p_1}\right.$$
$$\times(\cos\theta_{p_1}\cos\theta_{p_2}\cos\theta_{p_3}\cos\theta_{p_4}-\cos^2\theta_{p_1}\cos^2\theta_{p_3})\,d\Omega_{p_1}\,d\Omega_{p_2}\,d\Omega_{p_3}\,d\Omega_{p_4}$$
$$+\frac{mp_0}{\pi^2}\mathrm{Im}\int\int\int f^*_{p_1p_2}f_{p_2p_3}f_{p_3p_1}\cdot$$
$$\left.\times(\cos\theta_{p_1}\cos\theta_{p_2}-\cos^2\theta_{p_2})\cos^2\theta_{p_3}\,d\Omega_{p_1}\,d\Omega_{p_2}\,d\Omega_{p_3}\right\}. \qquad (20)$$

The remaining nonvanishing terms are expressed in terms of P_1, P_2, and P_3 with the aid of permutation of their arguments or complex conjugation. As a result we obtain:

$$P(\omega_1, \omega_2, \omega_3, \omega_4) = 2\mathrm{Re}\,\{P_1(\omega_1\omega_2\omega_3\omega_4) + P_1(\omega_4\omega_1\omega_2\omega_3) - P_2(\omega_1\omega_2\omega_3\omega_4) - P_2(\omega_4\omega_1\omega_2\omega_3) - P_2(\omega_3\omega_4\omega_1\omega_2) - P_2(\omega_2\omega_3\omega_4\omega_1) + P_3(\omega_1\omega_2\omega_3\omega_4)\}. \quad (21)$$

Thus, the average value of the product of four momenta is expressed not only in terms of the cross section but also in terms of the amplitude for the scattering of electrons by impurities. This means that the average under consideration cannot be obtained with the aid of a classical distribution function. In the Born approximation the terms containing the scattering amplitudes vanish. It is of more interest that they also vanish in the limit of large impurity concentrations when $\omega\tau \ll 1$. In this limit

$$P = {}^8/_9 m p_0^5 \tau_{tr}^2 [\delta(\omega_1 - \omega_3) + \delta(\omega_2 - \omega_4)] \quad (22)$$

and the average value of a product of four momenta decomposes into a product of pair averages:

$$\langle\langle p_z(t_1) p_z(t_2) p_z(t_3) p_z(t_4)\rangle\rangle = \langle\langle p_z(t_1) p_z(t_2)\rangle\rangle\langle\langle p_z(t_3) p_z(t_4)\rangle\rangle + \langle\langle p_z(t_1) p_z(t_4)\rangle\rangle\langle\langle p_z(t_2) p_z(t_3)\rangle\rangle. \quad (23)$$

In this case the result is expressed in terms of a transport cross section which, however, is not the classical result since it significantly depends on the order in which the momentum operators appear inside the averaging sign. This follows from the fact that the third possible product of pair averages is absent from the right-hand side of formula (23). Apparently in the limit of a large impurity concentration the product of an arbitrary number of momentum operators decomposes into a product of pair averages such that one can connect them by nonintersecting lines.

4. In order to clarify the dependence of the average on the order of the operators it is convenient to consider the special case of pairwise coincident times. From formulas (12), (19), and (20) it follows that

$$\langle p_z^2(0) p_z^2(t)\rangle = \frac{p_0^4}{9}\left[1 + \frac{4}{5}\exp\left\{-|t|\left(\frac{1}{\tau} - \frac{1}{\tau_2}\right)\right\}\right], \quad (24)$$

$$\langle p_z(0) p_z(t) p_z(0) p_z(t)\rangle = p_0^4 \exp\left\{-\frac{2|t|}{\tau_{tr}}\right\}\left[\frac{1}{5} + |t|\left(B - \frac{2}{5\tau_{tr}}\right)\right]. \quad (25)$$

Expression (24) agrees with the result which is obtained from the kinetic equation or by averaging over the classical trajectories. It is impossible to obtain expression (25) by such methods. This is associated with the fact that in formula (25) the momentum operators at different moments of time are arranged in an order which does not correspond to a classical trajectory. The commutator between momentum operators at different moments of time is not small even though the electrons move like free particles almost all the time.

Let us consider how this commutator changes in the quasiclassical limit

$$\langle [p_z(t) p_z(0)]^2\rangle = h^2 \left\langle \left(\frac{\partial \mathbf{p}_z(t)}{\partial z(0)}\right)^2 \right\rangle, \quad (26)$$

where one can calculate the average in the right-hand side of formula (26) with respect to the classical trajectories. For the calculation we introduce the more general quantities

$$X_j^i = \left\langle \left(\frac{\partial p_i(t)}{\partial r_j(0)}\right)^2 \right\rangle, \quad Y_j^i = \left\langle \left(\frac{\partial r_i(t)}{\partial r_j(0)}\right)^2 \right\rangle, \quad Z_j^i = \left\langle \frac{\partial p_i(t)}{\partial r_j(0)} \frac{\partial r_i(t)}{\partial r_j(0)} \right\rangle. \quad (27)$$

For small times t the quantity X_j^i is proportional to the number of collisions and is determined by the change of the scattering angle associated with a change of the impact parameter. Averaging with respect to the positions of the impurities, we obtain

$$X_j^i = tA_j^i, \quad A_j^i = p_0^2 n v (a + b\delta_{ij});$$

$$a = \frac{1}{15}\int_0^\infty \left[\left(\frac{\partial\chi}{\partial\rho}\right)^2 (\cos^2\chi + 2\sin^2\chi) + \frac{\sin^2\chi}{\rho^2}\right]\rho d\rho,$$

$$b = 2a - \frac{1}{3}\int_0^\infty \left(\frac{\partial\chi}{\partial\rho}\right)^2 \rho \sin^2\chi d\rho, \quad (28)$$

where χ is the scattering angle as a function of the impact parameter ρ. For a potential energy not small in comparison with the kinetic energy, the quantities a and b are of order unity.

At an arbitrary moment of time the quantities X, Y, and Z are connected by differential relationships which, for isotropic scattering, have the following form:

$$\dot{X}_j^i = A_k^i Y_j^k, \quad \dot{Y}_j^i = \frac{2}{m} Z_j^i, \quad \dot{Z}_j^i = \frac{1}{m} X_j^i \quad (29)$$

with initial conditions

$$X_j^i(0) = 0, \quad Y_j^i(0) = \delta_{ij}, \quad Z_j^i(0) = 0.$$

Solving this system we obtain

$$X_j^i = \frac{m^2}{18}\left[\frac{1}{t_0^2} f\left(\frac{t}{t_0}\right)(3\delta_{ij} - 1) + \frac{1}{t_1^2} f\left(\frac{t}{t_1}\right)\right], \quad (30)$$

where

$$f(t) = e^t + 2e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$

$$t_0^{-1} = v(2bn)^{1/3}, \quad t_1^{-1} = v[n(6a + 2b)]^{1/3}.$$

An estimate of the validity of the resulting formulas may be obtained from the condition that an initial displacement of a particle of the order of its wavelength hp_0^{-1} must lead at a moment of time t to a displacement which is smaller than the interaction radius $\sim\sigma^{1/2}$:

$$\lambda^2 Y(t) \ll \sigma \quad \text{or} \quad h^2 e^{t/t_0} \ll \sigma p_0^2. \quad (31)$$

At large times the wave packet is completely washed out. In order to evaluate the average of the square of the commutator in this region, it is necessary to use not the quasiclassical formulas (26) and (30) but the difference between expressions (25) and (24).

Thus, even for quasiclassical scattering of particles by impurities, the commutator of momentum operators at different moments of time increases exponentially with the time. For electrons in a metal, we apply the method of quasiclassical trajectories only in the pure case.

The problem considered above concerning the classical and quantum motion of an electron in a field

of impurities is a one-particle problem and does not have any direct connection to superconductivity. However, it should be kept in mind that in a theory of the normal state the momentum operators must follow in the order of increasing time. One can replace the average of such a product by an average over classical trajectories. A quantum phenomenon exists in superconductors: An electron, being created of a pair in the Bose-condensate, may change into a hole, which can be regarded as an electron moving backwards in time. Therefore, just as in the example considered above, averages of products of operators which are not ordered in time arise in the problems of superconductivity theory. The quasiclassical approximation is not applicable for the calculation of such averages.

5. In order to determine the corrections to the penetration depth, let us find the dependence of the current density on the field. To third order in the field, we can obtain one of the terms in the current density by substituting expression (21) into formula (10):

$$j_1 = -\frac{Ne^2}{m}\beta A^3 = \frac{2e}{\pi}\left(\frac{eA}{m}\right)^3 p_0^5 T \sum_\omega \frac{\Delta^2}{E^4(E+1/2\tau_{tr})^2}$$
$$\times\left\{\frac{\omega^2}{5(E+1/2\tau)} + \frac{\omega^2}{9E+18E^2\tau} + \frac{2\omega^2}{45(E+1/2\tau)[E+{}^1/_2(\tau_2/\tau-1)]}\right.$$
$$\left. - \frac{\Delta^2}{(E+1/2\tau_{tr})^2}\left(\frac{B}{16}+\frac{E}{20}\right)\right\}. \qquad (32)$$

In addition, the dependence of Δ on the magnetic field should be taken into account in the first-order expression (6). In order to take this dependence into account, in the equation for Δ

$$\Delta^*(\mathbf{r}) = |\lambda|\, T \sum_\omega \mathrm{Sp}\frac{1}{2}(\tau_x + i\tau_y)\hat{G}(\mathbf{r},\mathbf{r}), \qquad (33)$$

we expand the Green's function $\hat{G}$ to second order in the field. Changing, just as above, to a representation of eigenfunctions of the operator $\hat{H}$, we obtain

$$\Delta^* = 2\pi\nu\,|\lambda| T \sum_\omega \mathrm{Sp}\frac{1}{2}(\tau_x+i\tau_y)\left\{\hat{G}(0) + \left(\frac{e}{m}A\right)^2 \int_{-\infty}^{\infty}\int \langle\!\langle \hat{p}_z(t_1)\hat{p}_z(t_2)\rangle\!\rangle \right.$$
$$\left. \cdot G(-t_1)\tau_z G(t_1-t_2)\tau_z G(t_2)\,dt_1\,dt_2\right\}. \qquad (34)$$

Substituting here expression (8) for the average of a product of momenta and expression (5) for the Green's function G(t), we obtain the correction of second order in the field to the absolute value of Δ:

$$\Delta_2 = -\left(6\Delta_0 \sum_\omega E^{-3}\right)^{-1}\left(\frac{e}{m}p_0 A\right)^2$$
$$\times \sum_\omega \frac{1}{E^3(E+1/2\tau_{tr})^2}\left(\frac{\omega^2}{E\tau_{tr}} + 2\omega^2 - \Delta^2\right). \qquad (35)$$

Taking this correction into account in expression (6) for the current to first order in the field and the third-order correction (32), we obtain

$$j = -\frac{Ne^2}{m}Q(0)A(1-\alpha A^2), \qquad (36)$$

where

$$\alpha = \left(\beta - \frac{\Delta_2}{A^2}\frac{\partial Q}{\partial \Delta}\right)\Big/ Q(0), \qquad (37)$$

and the quantities $Q(0)$, β, and Δ_2 are determined by formulas (9), (32), and (35).

From here it follows that in the London case the field dependence of the penetration depth is determined by the expression

$$\delta = \delta_0\left[1 + \frac{\alpha H^2}{4}\delta_0^2\right]. \qquad (38)$$

Thus, the dependence of the penetration depth on the magnetic field in superconducting alloys is expressed not only in terms of a cross section but also in terms of a scattering amplitude. In the most interesting case of a large impurity concentration, when the mean free path is small in comparison with the dimensions of a pair, it is only necessary to keep the first term in formula (38); here the result agrees with that obtained by Melik-Barkhudarov[6] in the Born approximation.

6. In the general case it is convenient to carry out averaging of the Green's functions with respect to the positions of the impurities without isolating the average value of a product of matrix elements.

Let us expand the solution of the Gor'kov equations[4] in powers of the interaction $V(\mathbf{r})$ with impurities and average each term of this expansion. For a fixed difference of coordinates, the dependence of the Green's function for a pure metal in a magnetic field on the total coordinate is much weaker than the dependence on the coordinates of the interaction potential. Changing from the difference of coordinates to a momentum representation, we obtain

$$\left[-\xi - \frac{i}{m}\left(\mathbf{p}\frac{\partial}{\partial \mathbf{r}}\right) + i\omega\tau_z + \frac{e}{m}(\mathbf{pA})\tau_z + \hat{\Delta} - n\Sigma_{\mathbf{pp}}(\mathbf{r})\right] G_{\mathbf{p}}(\mathbf{r}) = 1,$$
$$\xi = \frac{\mathbf{p}^2}{2m} - \mu, \quad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0\end{pmatrix}. \qquad (39)$$

In all terms except the first one can regard the momenta $\mathbf{p}$ as lying on the Fermi surface. The matrix Σ is represented by the diagrams shown in Fig. 1, in which smooth lines correspond to the function $G_{\mathbf{p}}(\mathbf{r})$. Summing the diagrams, we obtain

$$\Sigma_{\mathbf{pp'}} = V_{\mathbf{pp'}} + \frac{1}{(2\pi)^3}\int V_{\mathbf{pp_1}} G_{\mathbf{p_1}}(\mathbf{r})\Sigma_{\mathbf{p_1p'}}(\mathbf{r})\,d\mathbf{p}_1. \qquad (40)$$

In this equation it is convenient to distinguish regions far away from the Fermi surface. For this purpose we consider a function χ satisfying the equation

$$\chi_{\mathbf{pp'}} = V_{\mathbf{pp'}} + \frac{1}{(2\pi)^3}\int V_{\mathbf{pp_1}}\frac{1}{-\varepsilon_{\mathbf{p_1}}+\mu}\chi_{\mathbf{p_1p'}}\,d\mathbf{p}_1. \qquad (41)$$

This function differs from the scattering amplitude (15) by the fact that the integral is to be understood in the sense of a principal value. The spherical harmonic functions χ are equal to the tangents of the scattering phase shifts. The amplitude (15) is related to the function χ by the relationship

$$f_{\mathbf{pp'}} = \chi_{\mathbf{pp'}} - \frac{i\vartheta}{4}\int \chi_{\mathbf{pp_1}} f_{\mathbf{p_1p'}}\,d\Omega_{\mathbf{p_1}}, \quad \vartheta = \frac{mp_0}{2\pi^2}. \qquad (42)$$

The integral in this equation is taken over the angles of the vector $\mathbf{p}_1$, whose magnitude is equal to the momentum p_0 at the Fermi surface. Similarly Eq. (40) takes the form

$$\Sigma_{\mathbf{pp'}} = \chi_{\mathbf{pp'}} - \frac{i\vartheta}{4}\int \chi_{\mathbf{pp_1}}\overline{G}_{\mathbf{p_1}}(\mathbf{r})\Sigma_{\mathbf{p_1p'}}(\mathbf{r})\,d\Omega_{\mathbf{p_1}}, \qquad (43)$$

where

$$\overline{G}_{\mathbf{p}}(\mathbf{r}) = \frac{i}{\pi}\int G_{\mathbf{p}}(\mathbf{r})\,d\xi. \qquad (44)$$

For large values of ξ it is necessary to understand

the integral (44) in the sense of a principal value, and also $\overline{G}_{\mathbf{p}}(\mathbf{r})$ only depends on the angles of the vector $\mathbf{p}$, which is situated on the Fermi surface.

The expression for Σ (formula (43)), and also all physical quantities, for example, the current density, only contain $\overline{G}_{\mathbf{p}}(\mathbf{r})$. In order to find the equation for this quantity, we write Eq. (39) in the form

$$\left[-\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)+\hat{\omega}\right]G_{\mathbf{p}}(\mathbf{r}\mathbf{r}')=2p\nu\delta(\mathbf{p},\mathbf{r}-\mathbf{r}'), \tag{45}$$

where

$$G_{\mathbf{p}}(\mathbf{r}\mathbf{r}')=\frac{i}{\pi}\int G_{\mathbf{p}}(\mathbf{r})\exp\left[-i\xi\frac{(\mathbf{p},\mathbf{r}-\mathbf{r}')}{p\nu}\right]d\xi,$$

$$\overline{G}_{\mathbf{p}}(\mathbf{r})=G_{\mathbf{p}}(\mathbf{r}\mathbf{r}),\quad \hat{\omega}=\omega\tau_z-ie(\mathbf{vA})\tau_z-i\hat{\Delta}+in\Sigma_{\mathbf{pp}}(\mathbf{r}). \tag{46}$$

One can show that it is also possible to write Eq. (45) in the form

$$\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}'}\right)G_{\mathbf{p}}(\mathbf{r}\mathbf{r}')+G_{\mathbf{p}}(\mathbf{r}\mathbf{r}')\hat{\omega}(\mathbf{r}')=2p\nu\delta(\mathbf{p},\mathbf{r}-\mathbf{r}'). \tag{47}$$

Subtracting Eq. (47) from Eq. (45) and setting $\mathbf{r}=\mathbf{r}'$, we obtain

$$-\left(\mathbf{v}\frac{\partial}{\partial \mathbf{r}}\right)\overline{G}_{\mathbf{p}}(\mathbf{r})+\hat{\omega}\overline{G}_{\mathbf{p}}(\mathbf{r})-\overline{G}_{\mathbf{p}}(\mathbf{r})\hat{\omega}=0. \tag{48}$$

In the region where the field is equal to zero or $\Delta=0$, it follows from Eq. (39) that $\mathrm{Tr}\,\overline{G}_{\mathbf{p}}(\mathbf{r})=0$, and from Eq. (47) it is seen that $\mathrm{Tr}\,\overline{G}_{\mathbf{p}}(\mathbf{r})$ does not depend on $\mathbf{r}$. Therefore it is equal to zero everywhere, and one can write the Green's function $\overline{G}_{\mathbf{p}}(\mathbf{r})$ in the form $\overline{G}_{\mathbf{p}}(\mathbf{r})=(\mathbf{a}\cdot\tau)$. Similarly one can convince oneself that

$$\mathbf{a}^2=1. \tag{49}$$

The system of equations (43), (48), and (49) determines the Green's function $\overline{G}_{\mathbf{p}}(\mathbf{r})$ in an arbitrary magnetic field. In the Born approximation these equations were obtained in another way by Eilenberger.[7]

In the presence of a magnetic field, $\overline{G}_{\mathbf{p}}(\mathbf{r})$ depends on the angles of the vector $\mathbf{p}$; therefore the system of Eqs. (43), (48), and (49) may be solved only in limiting cases. As mentioned above, certain results may depend not only on the cross sections but also on the scattering amplitudes.

7. In the most interesting cases of the critical field, weak field, or large concentration of impurities, the physical results can be expressed in terms of a cross section. In all of these cases, in the zero-order approximation $\overline{G}_{\mathbf{p}}(\mathbf{r})$ does not depend on the angles of the vector $\mathbf{p}$, and therefore from Eq. (43) with (42) taken into account we obtain

$$\Sigma_0=\mathrm{Re}\,f_{\mathbf{pp}'}+i\overline{G}_0\,\mathrm{Im}\,f_{\mathbf{pp}'}. \tag{50}$$

The first-order correction $\overline{G}_1$ anticommutes with $\overline{G}_0$, which follows from Eq. (49), and therefore from Eq. (43) we obtain the first-order correction to Σ:

$$\Sigma^{(1)}_{\mathbf{pp}}(\mathbf{r})=-\frac{i\nu}{2}\int\sigma_{\mathbf{pp}_1}\overline{G}^{(1)}_{\mathbf{p}_1}(\mathbf{r})\,d\Omega_{\mathbf{p}_1}. \tag{51}$$

Substituting (51) into Eq. (48), we obtain formula (9) in the case of a weak field in the London limit.

In order to determine the critical field, we expand Eq. (48) to first order in Δ. As a result we obtain the following system of equations:

$$\left[\mathrm{sign}\,\omega\cdot\mathbf{v}\left(\frac{\partial}{\partial \mathbf{r}}-2ie\mathbf{A}\right)+2|\omega|\right]K_{\mathbf{p}}(\mathbf{r})=\Delta+n\nu\int\sigma_{\mathbf{pp}_1}(K_{\mathbf{p}_1}-K_{\mathbf{p}})\,d\Omega_{\mathbf{p}_1},$$

$$\Delta=\frac{|\lambda|mp_0}{(2\pi)^2}T\sum_{\omega}\int K_{\mathbf{p}}(\mathbf{r})\,d\Omega_{\mathbf{p}}, \tag{52}$$

where

$$K_{\mathbf{p}}(\mathbf{r})=\tfrac{1}{4}\mathrm{Sp}(\tau_y-i\tau_x)\overline{G}^{(1)}_{\mathbf{p}}(\mathbf{r}).$$

For a thin superconducting film whose thickness is large in comparison with the mean free path but small in comparison with the dimensions of a pair, the system of equations (43), (48), and (49) may be solved for an arbitrary magnetic field. We seek the zero-order approximation to the Green's function in the form

$$\overline{G}_0=\cos\varphi\tau_y+\sin\varphi\tau_z. \tag{53}$$

In the zero-order approximation with respect to the mean free path, the angle φ is arbitrary and is determined from the condition that the average value of the correction to it is equal to zero. This correction shows up in the solution of Eq. (48) to second-order in the field and to first-order in ω and Δ. From the condition that $\overline{G}^{(2)}\sim\overline{G}_0$, we discover an equation for φ:

$$\Delta\,\mathrm{tg}\,\varphi=\omega+\tfrac{2}{3}e^2v^2\tau_{tr}\langle A^2\rangle\sin\varphi. \tag{54}$$

Equations (52) and (54) agree in appearance with the equations obtained previously[3,7,8] in the Born approximation.

8. Thus, one can assert that the majority of results in the theory of superconducting alloys, obtained in the Born approximation, remain valid even for a strong interaction between electrons and impurity atoms. Only in certain cases terms appear which are not present in the Born approximation and which depend not only on the cross section for the scattering of electrons by impurity atoms, but also depend on the explicit form of the amplitude.

Application of the method of classical trajectories to the theory of superconducting alloys is apparently limited to the case of weak fields. In order to calculate the dependence of the penetration depth on the field, the quantum character of the scattering of electrons by impurities turns out to be very important. Usually the method of classical trajectories has been applied in order to describe the diffuse reflection of electrons from a boundary, which is assumed to be classical. It is possible that in this case the quantum effects are less important.

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