Nonlinear conductivity of superconductors in the mixed state

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The current-voltage characteristic of superconductors in the mixed state is N-shaped. Owing to the long energy-relaxation time, nonlinear effects appear in comparatively weak electric fields when the transport current is much smaller than the critical pair-breaking current.

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1. INTRODUCTION

If no account is taken of the vortex pinning, superconductors in the mixed state obey Ohm's law: the density of the transport current is proportional to the voltage. This linear relation ceases to hold at relatively weak currents, much smaller than the pair-breaking currents. The reason is the long energy-relaxation time of the conduction electrons.

For a rough estimate of the effect, we consider a case when the time of inelastic collisions of the electrons with one another is small in comparison with the time of their interaction with the phonons or of the phonons with the thermostat. We denote the longest time by τ^*. The electron temperature can differ noticeably from the thermostat temperature T. Therefore the conductivity, which is determined by the electron temperature, is equal to

$$\sigma(E,T)=\sigma(0,T+\sigma E^2\tau^*/c),$$

where c is the specific heat of the electrons (or of the electrons with the phonons). It is seen from this expression that at large τ^* the nonlinear effects can be large in a relatively weak electric field.

We consider below another limiting case, when the heat is efficiently carried away and the lattice is in equilibrium with the thermostat, while the times of the energy relaxation via interelectron collisions is larger than or of the same order as the electron-phonon interaction time. In this case, too, owing to the large energy relaxation time, the change of the conductivity is large and is determined by the change of the distribution function, which in this case no longer reduces to a change of the electron temperature, but is more complicated.

The effect is strongest at temperatures close to critical, when a small change in the distribution function leads to a large change in the order parameter or in its coordinate dependence. This causes in turn a decrease of the conductivity and the dependence of the current on the voltage becomes N-shaped. The parameters of current-voltage characteristic depend significantly on the magnitude of the magnetic field and on the concentration of the magnetic impurities. The strongest nonlinear effects arise in weak magnetic fields at low magnetic-impurity concentrations, when a gap exists in the excitation spectrum.

2. EQUATIONS FOR THE DISTRIBUTION FUNCTION

To describe the behavior of a superconductor in a strong magnetic field, it is convenient to use the method proposed by Keldysh.[1] The Gor'kov equations then take the form

$$\left\{i\tau_z\frac{\partial}{\partial t}+\frac{1}{2m}\partial^2+\hat{\Delta}-e\varphi+\mu-\hat{\Sigma}\right\}\hat{G}(\mathbf{r},\mathbf{r}',t,t')=\delta(t-t')\delta(\mathbf{r}-\mathbf{r}'), \quad (1)$$

where $\mathbf{A}(t)$ is the vector potential, φ is the scalar potential, and $\partial = \partial/\partial\mathbf{r} - ie\mathbf{A}\tau_z$. In this equation, the Green's function $\hat{G}$ and the self-energy part of the $\hat{\Sigma}$ matrix are given by

$$\hat{G}=\begin{pmatrix} G^R; & G \\ 0; & G^A \end{pmatrix}, \quad \hat{\Sigma}=\begin{pmatrix} \Sigma^R; & \Sigma \\ 0; & \Sigma^A \end{pmatrix} \quad (2)$$

where G and $G^{(R,A)}$ are in turn 2 × 2 matrices made up of the usual Green's functions g and of the Gor'kov functions F:

$$G=\begin{pmatrix} g_1; & F_1 \\ -F_2; & g_2 \end{pmatrix}.$$

The Pauli matrix τ_z acts on the matrices G and $G^{(R,A)}$. The self-energy part of $\hat{\Sigma}$ takes into account the interaction of the electrons with the phonons and with the impurities. Assuming the interaction of the electrons with the phonons and with the impurities to be weak, we obtain for the self-energy part the expression

$$\hat{\Sigma}_{\mathbf{p}}(t,t')=\hat{\Sigma}_{\mathbf{p}}^{imp}(t,t')+\hat{\Sigma}_{\mathbf{p}}^{ph}(t,t'),$$

$$\hat{\Sigma}_{\mathbf{p}}^{imp}(t,t')=-\frac{inu_F}{2}\int d\Omega_{\mathbf{p}_1}\sigma_{\mathbf{p}\mathbf{p}_1}\hat{G}_{\mathbf{p}_1}(t,t')-\frac{i}{2\tau_s}\int\frac{d\Omega_{\mathbf{p}_1}}{4\pi}\tau_z\hat{G}_{\mathbf{p}_1}(t,t')\tau_z,$$

$$\Sigma_{\mathbf{p}}^{ph(R,A)}(t,t')=\frac{\nu g^2}{8}\int d\Omega_{\mathbf{p}_1}\{D_{\mathbf{p}-\mathbf{p}_1}(t'-t)G_{\mathbf{p}_1}^{(R,A)}(t,t')+D_{\mathbf{p}-\mathbf{p}_1}^{(A,R)}(t'-t)G_{\mathbf{p}_1}(t,t')\},$$

$$\Sigma_{\mathbf{p}}^{ph}(t,t')=\frac{\nu g^2}{8}\int d\Omega_{\mathbf{p}_1}\{D_{\mathbf{p}-\mathbf{p}_1}(t'-t)G_{\mathbf{p}_1}(t,t')-$$

$$-(D^R_{\mathbf{p}-\mathbf{p}_1}(t'-t)-D^A_{\mathbf{p}-\mathbf{p}_1}(t'-t))(G_{\mathbf{p}_1}^R(t,t')-G_{\mathbf{p}_1}^A(t,t'))\}, \quad (3)$$

where

$$\hat{G}_{\mathbf{p}}(t,t')=\frac{i}{\pi}\int d\xi\hat{G}(t,t',\mathbf{R}+\mathbf{r}/2,\mathbf{R}-\mathbf{r}/2)e^{-i\mathbf{p}\mathbf{r}}d\mathbf{r},$$

$\xi = p^2/2m - \mu$, u_F is the velocity on the Fermi surface, $\nu = mp/2\pi^2$ is the density of states on the Fermi surface, τ_s is the free path time of the electron with spin flip, $\sigma_{pp'}$ is the electron-impurity scattering cross section, n is the impurity density, g is the electron-phonon interaction constant, and $D_{p-p'}(\mathbf{R}, t'-t)$ is the phonon Green's function.

We assume that we have separated from $\hat{\Sigma}^{ph}$ that part which leads to the renormalization of the velocity on the Fermi surface and to effective interaction λ_{eff} between electrons, which determines the ordering parameter. The remaining part describes the energy relaxation. The ordering parameter Δ, the current density j, and the charge density ρ are expressed in terms of the Green's function by the formulas

$$\Delta(t)=\begin{pmatrix} 0; & \Delta_1(t) \\ -\Delta_2(t); & 0 \end{pmatrix}, \quad \Delta_{1,2}(t)=\frac{|\lambda_{eff}|mp}{4\pi}F_{1,2}(t,t),$$

$$\mathbf{j}(t)=-\frac{ep}{4\pi}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\operatorname{Sp}\tau_z\mathbf{p}G_{\mathbf{p}}(t,t),$$

$$\rho(t)=-ev\left\{2e\varphi(t)+\frac{\pi}{2}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\operatorname{Sp}G_{\mathbf{p}}(t,t)\right\}. \tag{4}$$

Just as in the static case,[2,3] it becomes possible to obtain, with quasiclassical accuracy, equations for the Green's function integrated with respect to the energy variable ξ:

$$\frac{1}{m}\mathbf{p}\frac{\partial\hat{G}}{\partial\mathbf{R}}+\tau_z\frac{\partial\hat{G}}{\partial t}+\frac{\partial\hat{G}}{\partial t'}\tau_z+\hat{H}(t)\hat{G}-\hat{G}\hat{H}(t')$$

$$+i\int_{-\infty}^{\infty}dt_2\{\hat{\Sigma}_{tt_2}\hat{G}_{t_2t'}-\hat{G}_{tt_2}\hat{\Sigma}_{t_2t'}\}=0, \tag{5}$$

where

$$\hat{G}\equiv\hat{G}_{\mathbf{p}}(\mathbf{R},t,t'),\quad \hat{H}(t)=-\frac{ie}{m}\mathbf{p}\mathbf{A}(t)\tau_z-i\hat{\Delta}(t)+ie\varphi(t),$$
$$\hat{\Sigma}\equiv\hat{\Sigma}_{\mathbf{p}}(\mathbf{R},t,t').$$

The system (5) was derived by Éliashberg.[4] The use of the matrix form (2) for the Green's function $\hat{G}$ enables us to establish an additional normalization relation

$$\int_{-\infty}^{\infty}dt_2\,\hat{G}_{\mathbf{p}}(tt_2)\hat{G}_{\mathbf{p}}(t_2t')=\hat{\delta}(t-t'), \tag{6}$$

which turns out to be very useful when it comes to solving the system (5). Differentiating (6) with respect to the coordinate $\mathbf{R}$, we can verify that it does not contradict the system (5). The coefficient of the $\hat{\delta}$ function in the right-hand side of (6) does not depend on the coordinates and should therefore be the same as in a superconductor without an electromagnetic field. Direct calculation for this case shows that this coefficient is equal to unity. Equations (3–6) together with Maxwell's equations determine fully the behavior of the superconductor in an arbitrary electromagnetic field.

In the static case, in an arbitrary magnetic field, the Green's functions depend only on the time differences and satisfy the relation

$$\hat{G}_{\mathbf{p}}(R,\varepsilon)=\int dt\,\hat{G}_{\mathbf{p}}(R,t-t')\exp[i\varepsilon(t-t')],$$
$$G_{\mathbf{p}}(\mathbf{R},\varepsilon)=(G_{\mathbf{p}}^R(\mathbf{R},\varepsilon)-G_{\mathbf{p}}^A(\mathbf{R},\varepsilon))\operatorname{th}(\varepsilon/2T). \tag{7}$$

In the self-consistent field approximation, when no account is taken of the energy relaxation, i.e., $\hat{\Sigma}^{ph}=0$, the system (5), (6) is satisfied by a function $G_p(\mathbf{R},\epsilon)$ in the form (7), in which $\tanh(\epsilon/2T)$ is replaced by an arbitrary distribution function $f(\epsilon)$. The form of the latter function is determined by the small terms in (5), which are proportional to the square of the electric field or are inversely proportional to the energy relaxation time. Since the energy relaxation time is large, the effects nonlinear in the electric field and connected with the change of the distribution function $f(\epsilon)$ appear in relatively weak fields.

To find the equation for the distribution function, we average the system (5) over the angles of the vector p. We then take the trace of each of the three equations for the Green's functions G and $G^{(R,A)}$. As a result we obtain three equations in the form

$$\operatorname{Sp}_\tau\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\left\{\frac{1}{m}\mathbf{p}\frac{\partial\hat{G}_{\mathbf{p}}(t_1,t')}{\partial\mathbf{R}}+\tau_z\left(\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t'}\right)\cdot\right.$$
$$\left.\cdot\hat{G}_{\mathbf{p}}(t_1,t')+(\hat{H}(t_1)-\hat{H}(t'))\hat{G}_{\mathbf{p}}(t_1,t')\right\}=\mathrm{St},$$

$$\mathrm{St}=-i\operatorname{Sp}_\tau\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\int_{-\infty}^{\infty}dt_2\{\hat{\Sigma}^{ph}_{t_1t_2}\hat{G}_{t_2t'}-\hat{G}_{t_1t_2}\hat{\Sigma}^{ph}_{t_2t'}\}. \tag{8}$$

We shall be interested below in the case of an electric field of low frequency and amplitude, when the transport current is much smaller than the critical pair-breaking current. The dependence of the Green's function on the time difference t_1-t' is therefore faster than the dependence on the sum of the times $t=(t_1+t')/2$. It is therefore convenient to change over in (8) to a frequency representation in terms of the variable t_1-t'. Averaging the equations in (8) for the coordinates, we obtain

$$\operatorname{Sp}_\tau\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\left\langle\left[\tau_z\frac{\partial}{\partial t}-i\frac{\partial\hat{H}(t)}{\partial t}\frac{\partial}{\partial\varepsilon}\right]\hat{G}_{\mathbf{p}}(\varepsilon,t)\right\rangle=\langle\mathrm{St}_\varepsilon\rangle. \tag{9}$$

We seek the Green's function $G_p(\mathbf{R},t_1,t')$ in the form

$$G_{\mathbf{p}}(\mathbf{R},t_1,t')$$
$$=\int_{-\infty}^{\infty}dt_2[G_{\mathbf{p}}^R(\mathbf{R},t_1,t_2)f(t_2,t')-f(t_1,t_2)G_{\mathbf{p}}^A(\mathbf{R},t_2,t')]+G_{\mathbf{p}}^b(\mathbf{R},t_1,t'). \tag{10}$$

The distribution function f does not depend on the coordinates. As applied to the motion of the vortex structure in an electric field, this condition means that the distance between the vortices is small in comparison with the distance negotiated by the electron during the time of the energy relaxation. The last term of G^b in the right-hand side of (10) is an odd function of the electric field and is small if the transport current is small in comparison with the critical pair-breaking current.

Taking the Fourier transform with respect to the time difference in (10), we obtain

$$G_{\mathbf{p}}(\mathbf{R},\varepsilon,t)=(G_{\mathbf{p}}^R(\mathbf{R},\varepsilon,t)-G_{\mathbf{p}}^A(\mathbf{R},\varepsilon,t))f_\varepsilon(t)$$
$$-\frac{i}{2}\frac{\partial f}{\partial\varepsilon}\frac{\partial}{\partial t}(G_{\mathbf{p}}^R(\mathbf{R},\varepsilon,t)+G_{\mathbf{p}}^A(\mathbf{R},\varepsilon,t))+G_{\mathbf{p}}^b(\mathbf{R},\varepsilon,t). \tag{11}$$

We shall consider below stationary motion of a vortex lattice in a constant electric field. In this case, in a sufficiently large volume over which the averaging is carried out in (9), the vortex velocity u can be regarded as independent of the coordinates. The time dependence of the Green's functions takes in this case the form

$$G_{\mathbf{p}}^{(R,A)}(\mathbf{R},\varepsilon,t)$$
$$=\begin{pmatrix} g_1^{(R,A)}(\mathbf{R}-\mathbf{u}t,\varepsilon); & F_1^{(R,A)}(\mathbf{R}-\mathbf{u}t,\varepsilon)\exp[2ie\chi(t)] \\ -F_2^{(R,A)}(\mathbf{R}-\mathbf{u}t,\varepsilon)\exp[-2ie\chi(t)]; & g_2^{(R,A)}(\mathbf{R}-\mathbf{u}t,\varepsilon)\end{pmatrix}$$
$$\mathbf{A}(\mathbf{R},t)=\mathbf{A}(\mathbf{R}-\mathbf{u}t)+\frac{\partial\chi}{\partial\mathbf{R}},$$
$$\Delta(\mathbf{R},t)=\begin{pmatrix}0; & \Delta_1(\mathbf{R}-\mathbf{u}t)\exp[2ie\chi(t)] \\ -\Delta_2(\mathbf{R}-\mathbf{u}t)\exp[-2ie\chi(t)]; & 0\end{pmatrix} \tag{12}$$

It is convenient to choose a gauge in such a way that the phase χ satisfy the equation

$$\partial\chi/\partial t=(\mathbf{u}\mathbf{A}(\mathbf{R}-\mathbf{u}t)). \tag{13}$$

In this case

$$\partial\mathbf{A}(\mathbf{R},t)/\partial t=[\mathbf{u}\mathbf{H}(\mathbf{R}-\mathbf{u}t)],$$

and the scalar potential $\varphi(\mathbf{R},t)$ is a periodic function of the coordinates.

Equation (9) leads to two equations for the Green's functions $G^{(R,A)}$:

$$\operatorname{Sp}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\left\langle\frac{\partial\hat{H}(t)}{\partial t}G_{\mathbf{p}}^{(R,A)}(\mathbf{R},\varepsilon,t)\right\rangle=0. \tag{14}$$

Substituting expression (11) for the Green's function $G_p(\mathbf{R},\epsilon,t)$ in (9) and taking (14) into account, we obtain

$$-\frac{1}{2}\frac{\partial}{\partial\varepsilon}\left\langle\operatorname{Sp}\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{\partial\hat{H}(t)}{\partial t}\left\{\frac{\partial f}{\partial\varepsilon}\frac{\partial}{\partial t}(G^R+G^A)+2iG^b\right\}\right\rangle=\langle\mathrm{St}_\varepsilon\rangle. \tag{15}$$

The expression for the function G^b in (15) must be determined accurate to terms linear in the velocity. Substituting expression (10) for the Green's function in (5) and in the additional condition (6), we obtain

$$\frac{1}{m}\mathbf{p}\frac{\partial G_{\mathbf{p}}^{b}}{\partial \mathbf{R}}+[-i\varepsilon\tau_z+\hat{H}, G_{\mathbf{p}}^{b}]$$

$$+\frac{nu_F}{2}\int d\Omega_{\mathbf{p}_1}\sigma_{\mathbf{p}\mathbf{p}_1}\{G_{\mathbf{p}_1}^{R}G_{\mathbf{p}}^{b}+G_{\mathbf{p}_1}^{b}G_{\mathbf{p}}^{A}-G_{\mathbf{p}}^{b}G_{\mathbf{p}_1}^{A}-G_{\mathbf{p}}^{R}G_{\mathbf{p}_1}^{b}\}$$

$$+\frac{1}{2\tau_s}\int\frac{d\Omega_{\mathbf{p}_1}}{4\pi}\{\tau_z G_{\mathbf{p}_1}^{R}\tau_z G_{\mathbf{p}}^{b}+\tau_z G_{\mathbf{p}_1}^{b}\tau_z G_{\mathbf{p}}^{A}-G_{\mathbf{p}}^{b}\tau_z G_{\mathbf{p}_1}^{A}\tau_z-G_{\mathbf{p}}^{R}\tau_z G_{\mathbf{p}_1}^{b}\tau_z\}$$

$$=i\frac{\partial f}{\partial \varepsilon}\left\{G_{\mathbf{p}}^{R}\frac{\partial \hat{H}}{\partial t}-\frac{\partial \hat{H}}{\partial t}G_{\mathbf{p}}^{A}\right\}; \tag{16}$$

$$G_{\mathbf{p}}^{R}(\varepsilon)G_{\mathbf{p}}^{b}(\varepsilon)+G_{\mathbf{p}}^{b}(\varepsilon)G_{\mathbf{p}}^{A}(\varepsilon)=0. \tag{17}$$

The collision term in the right-hand side of (15) will be calculated under the assumption that the phonons are in thermal equilibrium with the temperature T. In this case the phonon Green's function takes the form

$$D_{\mathbf{k}}(\omega)=(D_{\mathbf{k}}^{R}(\omega)-D_{\mathbf{k}}^{A}(\omega))\,\mathrm{cth}\frac{\omega}{2T},$$

$$D_{\mathbf{k}}^{R}(\omega)=D_{\mathbf{k}}^{A*}(\omega)=-\frac{\omega^2(\mathbf{k})}{\omega^2(\mathbf{k})-(\omega+i\delta)^2}. \tag{18}$$

At temperatures lower than the Debye temperature, the main contribution to the energy relaxation is made by acoustic phonons, for which $\omega(\mathbf{k}) = s|\mathbf{k}|$. Using (8), (11) and (18), we obtain for the collision term in (15) the expression

$$\langle \mathrm{St}_\varepsilon\rangle=-\frac{\pi\nu g^2}{8(sp)^2}\int d\Omega_{\mathbf{p}}\int\frac{d\varepsilon_1}{4\pi}\mathrm{Sp}\Big\langle(\varepsilon_1-\varepsilon)^2\,\mathrm{sign}(\varepsilon_1-\varepsilon)\cdot$$

$$\cdot\left[(1-f(\varepsilon)f(\varepsilon_1))+\mathrm{cth}\frac{\varepsilon_1-\varepsilon}{2T}(f(\varepsilon)-f(\varepsilon_1))\right]\cdot$$

$$\cdot(G_{\mathbf{p}}^{R}(\varepsilon_1)-G_{\mathbf{p}}^{A}(\varepsilon_1))(G_{\mathbf{p}}^{R}(\varepsilon)-G_{\mathbf{p}}^{A}(\varepsilon))\Big\rangle \tag{19}$$

The system of equations (5), (15–17) and (19) describes the behavior of a superconductor in strong electric and magnetic fields. This system becomes simpler for superconductors with small electron mean free paths.

3. SUPERCONDUCTORS WITH SMALL ELECTRON MEAN FREE PATHS

In the case $l \ll \xi$, the integral equation (16) for the function $G_{\mathbf{p}}^{b}(\mathbf{R}, \epsilon)$ reduces to a differential equation. The function $G_{\mathbf{p}}^{b}$ depends here little on the angles of the vector $\mathbf{p}$, and can be written in the form

$$G_{\mathbf{p}}^{b}(R,\varepsilon)=G^{b}(\mathbf{R},\varepsilon)+(\mathbf{p}\mathbf{G}_1^{b}(\mathbf{R},\varepsilon)). \tag{20}$$

The second term in (20) is small and can be easily obtained from (16) and (17):

$$\mathbf{G}_1^{b}(\varepsilon)=-\frac{l_{tr}}{p}\left\{(G^R\,\partial G^b+G^b\,\partial G^A)-e\frac{\partial \mathbf{A}}{\partial t}\frac{\partial f}{\partial \varepsilon}(\tau_z-G^R\tau_z G^A)\right\} \tag{21}$$

where the operator ∂ is defined in (1). We obtain similarly for the functions $G_{\mathbf{p}}^{(R,A)}$

$$G_{\mathbf{p}}^{(R,A)}(\varepsilon)=G^{(R,A)}(\varepsilon)+(\mathbf{p}\mathbf{G}_1^{(R,A)}(\varepsilon)), \tag{22}$$

$$\mathbf{G}_1^{(R,A)}(\varepsilon)=-\frac{l_{tr}}{p}\{ie\mathbf{A}\tau_z+G^{(R,A)}\,\partial G^{(R,A)}\}.$$

Substituting (20), (21), and (22) in (16) and averaging over the angles of the vector $\mathbf{p}$, we obtain an equation for the function $G^b(\epsilon)$:

$$-[\varepsilon\tau_z+\Delta, G^b]+iD[\partial, G^R\,\partial G^b+G^b\,\partial G^A]$$

$$-\frac{i}{2\tau_s}[\tau_z, G^R\tau_z G^b+G^b\tau_z G^A]=i\frac{\partial f}{\partial \varepsilon}\left\{\left(\frac{\partial \Delta}{\partial t}G^A-G^R\frac{\partial \Delta}{\partial t}\right)\right.$$

$$\left.-eD[\mathbf{u}\times\mathbf{H}](\tau_z G^A\,\partial G^A-G^R\,\partial G^R\tau_z+[\partial, G^R\tau_z G^A])-4\pi e D\tau_z(\mathbf{u}\mathbf{j})\right\}, \tag{23}$$

where $D = u_F l_{tr}/3$ is the diffusion coefficient.

Similarly, substituting (20), (21), and (22) in (15), we obtain an equation for the distribution function $f(\epsilon)$. This equation takes the form

$$\frac{\partial}{\partial \varepsilon}\left\{D_\varepsilon\frac{\partial f}{\partial \varepsilon}\right\}=\mathrm{St}, \tag{24}$$

where the collision term is given by (19).

The coefficient D of diffusion along the energy axis is conveniently represented in the form of a sum of two terms:

$$D_\varepsilon=D_1(\varepsilon)+D_2(\varepsilon), \tag{25}$$

where

$$-D_1(\varepsilon)=2e^2D\left\langle[\mathbf{u}\times\mathbf{H}]^2\left[1-g_1^R g_1^A-\frac{1}{2}(F_1^R F_2^A+F_1^A F_2^R)\right]\right\rangle$$

$$+\frac{i}{2}\langle(\mathbf{u}\partial_-\Delta)\mathbf{u}\partial_+(F_2^R+F_2^A)+(\mathbf{u}\partial_+\Delta^*)\mathbf{u}\partial_-(F_1^R+F_1^A)\rangle$$

$$+\frac{ieD}{2}\left\langle\left(\left(\mathbf{u}\frac{\partial}{\partial \mathbf{R}}\right)[\mathbf{u}\times\mathbf{H}]\right)[F_2^R\partial_- F_1^R-F_1^R\partial_+ F_2^R+F_2^A\partial_- F_1^A-F_1^A\partial_+ F_2^A]\right\rangle$$

$$D_2(\varepsilon)\frac{\partial f}{\partial \varepsilon}=\left\langle\mathrm{Sp}\left\{\begin{pmatrix}0, & \mathbf{u}\partial_-\Delta\\ -\mathbf{u}\partial_+\Delta^*, & 0\end{pmatrix}G^b+eD[\mathbf{u}\times\mathbf{H}]\tau_z(G^R\,\partial G^b+G^b\,\partial G^A)\right\}\right\rangle \tag{26}$$

where

$$\partial_\pm=\partial/\partial\mathbf{R}\pm 2ie\mathbf{A}.$$

We shall consider below superconductors with large Ginzburg-Landau parameters κ. The last terms of (26) are small in comparison with the square of this term.

The effects nonlinear in the electric field manifest themselves most strongly at a temperature close to critical. In this region it becomes possible to simplify Eqs. (2) and (24) further. The most significant change in the conductivity is due to the change of the ordering parameter Δ in the electric field. The equation for the ordering parameter Δ can be obtained from the general formula (4) by using for the Green's function F the expression (11), in which we need retain only the first term. Separating the terms corresponding to the static Ginzburg-Landau equation, we obtain

$$\left\{1-\frac{T}{T_c}+\frac{\pi D}{8T}\partial_-^2-\frac{7\zeta(3)|\Delta|^2}{8\pi^2T^2}\right\}\Delta$$

$$+\frac{1}{4}\int_{-\infty}^{\infty}d\varepsilon\left(f(\varepsilon)-\mathrm{th}\frac{\varepsilon}{2T}\right)(F_1^R(\varepsilon)-F_1^A(\varepsilon))=0. \tag{27}$$

It is seen from (27) that a strong change in the ordering parameter Δ, and hence also in the conductivity $\sigma(E)$, occurs in relatively weak fields, when the distribution function $f(\epsilon)$ differs little from its equilibrium value. Therefore in the left-hand side of (24) the function $f(\epsilon)$ can be replaced by $\tanh(\epsilon/2T)$. At practically all values of the magnetic field, with the exception of a very narrow vicinity of H_{c2}, the significant changes of the distribution function take place in the region $\epsilon \sim \Delta \ll T$. Therefore only the particle departures play an important role in the collision term, and this term can be represented in the form

$$\mathrm{St}=-\frac{1}{\tau_\varepsilon}\left(f(\varepsilon)-\mathrm{th}\frac{\varepsilon}{2T}\right)\langle\mathrm{Sp}\,\tau_z(G^R-G^A)\rangle, \tag{28}$$

$$\tau_\varepsilon^{-1}=7\zeta(3)\pi\nu g^2T^3/2(sp)^2. \tag{29}$$

4. SUPERCONDUCTOR WITH MAGNETIC IMPURITIES

In this section we consider gapless superconductors, when the magnetic-impurity concentration is large enough and the condition $\Delta\tau_s \ll 1$ is satisfied. The Green's functions $G^{(R,A)}$ can then be obtained by expansion in the ordering parameter Δ. In the principal approximation we have

$$g_1^R(\varepsilon)=-g_1^A(-\varepsilon)=1-\frac{|\Delta|^2}{2(-i\varepsilon+\tau_s^{-1})^2},\quad F_1^R(\varepsilon)=F_1^A(-\varepsilon)=\frac{-i\Delta}{-i\varepsilon+\tau_s^{-1}}. \tag{30}$$

Substituting these expressions in (25), (26), and (28), we reduce (24) to the form

$$f(\varepsilon)-\operatorname{th}\frac{\varepsilon}{2T}=\frac{\tau_\varepsilon}{4}\frac{\partial}{\partial\varepsilon}\Big\{\langle(\mathbf{u}\partial_-\Delta)F_2^b+(\mathbf{u}\partial_+\Delta^*)F_1^b\rangle + \frac{\partial f}{\partial\varepsilon}\Big[4e^2D\Big\langle[\mathbf{u}\times\mathbf{H}]^2\Big(1+|\Delta|^2\Big(\frac{\varepsilon}{\varepsilon^2+\tau_s^{-2}}\Big)^2\Big)\Big\rangle + \frac{2\tau_s^{-1}}{\varepsilon^2+\tau_s^{-2}}\langle(\mathbf{u}\partial_-\Delta)(\mathbf{u}\partial_+\Delta^*)\rangle\Big]\Big\}. \tag{31}$$

To find the functions $F^b_{1,2}$ we use the normalization condition (17). Substituting (30) in this condition, we get

$$F_1^b(\varepsilon)=F_2^{b*}(\varepsilon)=\frac{i\Delta}{2}\Big(\frac{g_1^b}{i\varepsilon+\tau_s^{-1}}+\frac{g_2^b}{-i\varepsilon+\tau_s^{-1}}\Big). \tag{32}$$

The functions $g^b_{1,2}$ are obtained from the system (23) which reduces, when account is taken of (30) and (32), to the form

$$\Big(-D\frac{\partial^2}{\partial\mathbf{R}^2}\Big)(g_1^b-g_2^b)=\frac{2\varepsilon}{\varepsilon^2+\tau_s^{-2}}\frac{\partial f}{\partial\varepsilon}\Big(\mathbf{u}\frac{\partial|\Delta|^2}{\partial\mathbf{R}}\Big),$$
$$\Big\{|\Delta|^2-\frac{\varepsilon^2+\tau_s^{-2}}{2\tau_s^{-1}}D\frac{\partial^2}{\partial\mathbf{R}^2}\Big\}(g_1^b+g_2^b)=-i\frac{\partial f}{\partial\varepsilon}[\Delta(\mathbf{u}\partial_+\Delta^*)-\Delta^*(\mathbf{u}\partial_-\Delta)]. \tag{33}$$

We consider the case when the concentration of the magnetic impurities is not too high, and the magnetic fields are not close to H_{c2}, so that the following conditions are satisfied:

$$T_c\tau_s\gg 1,\quad \Delta^2\tau_s\gg T_c-T. \tag{34}$$

When these conditions are satisfied, the Laplacian in the right-hand side of the second equation of (33) can be left out. We substitute the solution of (32) and (33) in (31), and replace in the right-hand side of the latter $f(\epsilon)$ by $\tanh(\epsilon/2T)$. Using the expression thus obtained for the distribution function $f(\epsilon)$, we reduce Eq. (27) for the ordering parameter Δ to the form

$$\Big\{1-\frac{T}{T_c}+\frac{\pi D}{8T}\partial_-^2-\frac{7\zeta(3)|\Delta|^2}{8\pi^2T^2}-\frac{\pi\tau_s^2\tau_\varepsilon}{32T}\Big\langle\Big(\mathbf{u}\frac{\partial|\Delta|}{\partial\mathbf{R}}\Big)^2\Big\rangle\Big\}\Delta=0. \tag{35}$$

It follows from this equation that the coordinate dependence is preserved and that the effect of the electric field reduces to renormalization of T_c. The average current density j can be expressed by using the linear-approximation formula,[5-7], which takes the following form if the conditions (34) are satisfied:

$$\mathbf{j}=\sigma_N\mathbf{E}\Big\{1+\frac{\pi\tau_s}{64e^2B^2D^2T}\Big\langle(|\Delta|^4-\langle|\Delta|^2\rangle^2)+D\tau_s^{-1}\frac{1}{|\Delta|^2}\Big(\frac{\partial|\Delta|^2}{\partial\mathbf{R}}\Big)^2\Big\rangle\Big\} \tag{36}$$

where σ_N is the conductivity of the metal in the normal state. In magnetic fields close to the critical field H_{c2}, we obtain from (35) and (36)

$$\mathbf{j}=\sigma_N\mathbf{E}\Big\{1+\frac{\pi^2\langle|\Delta|^2\rangle}{64T(T_c-T)}\Big[1+\frac{\pi\tau_s}{16(T_c-T)}(\beta_A-1)\langle|\Delta|^2\rangle\Big]\Big\},$$
$$\langle|\Delta|^2\rangle=\Big(1-\frac{T}{T_c}\Big)\Big(1-\frac{H}{H_{c2}}\Big)\Big[\frac{7\zeta(3)\beta_A}{8\pi^2T^2}+\frac{\pi^2\tau_s^2\tau_\varepsilon e^2E^2D}{256T(T_c-T)}\Big]^{-1}, \tag{37}$$

where $\beta_A = 1.16$ and E is the average electric field. As follows from this formula, with increasing electric field the conductivity of the superconductor decreases and approaches the conductivity of the normal metal.

Of greater interest is the case of a weak magnetic field $H \ll H_{c2}$, when the conductivity of the superconductor in a weak electric field greatly exceeds the conductivity of the normal metal. Using the results of a numerical calculation[8] to determine the mean value in (35), we obtain for the current density the expression

$$\mathbf{j}=\sigma\mathbf{E}/[1+(E/E^*)^2], \tag{38}$$

where

$$(E^*)^2=5\frac{B}{H_{c2}}\Big(1-\frac{T}{T_c}\Big)[e^2D\tau_s^2\tau_\varepsilon]^{-1},\quad \sigma=\sigma_N\Big(\frac{\pi^3}{28\zeta(3)}\Big)^2\frac{H_{c2}}{B}(\pi T_c\tau_s).$$

As the foltage increases, the current reaches its maximum value at $E = E^*$. In electric fields $E > E^*$, the current decreases with increasing voltage, to its value in the normal metal. In this voltage region, the ordering parameter drops to values such that formulas (37) become valid. The current-voltage characteristic thus has the typical N-shaped form.

5. FIELDS CLOSE TO THE CRITICAL FIELD H_{c2}

In the region of fields close to the critical value H_{c2}, the Green's functions can be expanded in powers of the ordering parameter Δ at arbitrary paramagnetic-impurity concentration. In formulas (30) and (32) for the Green's function and for the distribution function, it is necessary to make the substitution

$$\tau_s^{-1}\to\tau_s^{-1}+\lambda,\quad \lambda=4\pi^{-1}(T_c(E)-T), \tag{39}$$

where $T_c(E)$ is the electric-field-dependent transition temperature. If the condition

$$\Delta^2<(T_c-T)(T_c-T+\tau_s^{-1}) \tag{40}$$

is satisfied, we can neglect in (31) the first terms, which contain $F^b_{1,2}$, since $F^b_{1,2}\sim\Delta^3$. Substituting formula (31) corrected in this manner in (27), we obtain

$$\Big\{T_c(E)-T+\frac{\pi D}{8}\partial_-^2-\frac{7\zeta(3)|\Delta|^2}{8\pi^2T}-\frac{\pi\hat{\lambda}\tau_\varepsilon u^2\langle|\Delta|^2\rangle}{32D(\lambda+\tau_s^{-1})^2}\Big\}\Delta=0, \tag{41}$$

where

$$T_c(E)=T_c-\frac{7\zeta(3)\tau_\varepsilon e^2E^2D}{2\pi^2T}. \tag{42}$$

In the calculation of the shift of the transition temperature (formula (42)), the important integration region is $\epsilon\sim T$. The use of a collision term in the form (28) is incorrect in this case. Therefore formula (42) for the shift of the transition temperature is valid accurate to the numerical coefficient of E^2.

The current density is expressed in terms of the ordering parameter Δ in accordance with the linear-approximation formula.[9] Solving (41), we obtain

$$\mathbf{j}=\sigma_N\mathbf{E}\Big\{1+\frac{\pi\langle|\Delta|^2\rangle}{8T}\Big(\frac{1}{\lambda}+\frac{1}{\lambda+\tau_s^{-1}}\Big)\Big\}$$
$$\langle|\Delta|^2\rangle=eD(H_{c2}(E)-H)\frac{2\pi^3T}{7\zeta(3)\beta_A}\Big[1+\Big(\frac{E}{E^*}\Big)^2\Big]^{-1}, \tag{43}$$

where

$$H_{c2}(E)=\frac{4}{\pi eD}(T_c(E)-T),$$
$$(E^*)^2=\frac{1792\zeta(3)\beta_A(T_c-T)^3}{\pi^6T\tau_\varepsilon e^2D}\Big(1+\frac{\pi}{4\tau_s(T_c-T)}\Big)^2.$$

From formula (43) for $\langle|\Delta|^2\rangle$ it follows that the magnetic-field region defined by the inequality (40) breaks up into two. In the magnetic field range $(1-T/T_c)^2 < 1-H/H_{c2} < 1-T/T_c$ the dependence of H_{c2} on the electric field E is inessential. In the region $1-H/H_{c2} \ll (1-T/T_c)^2$, the entire dependence of the conductivity on the electric field E is determined by the shift of the transition temperature.

6. SUPERCONDUCTORS WITHOUT MAGNETIC IMPURITIES

The effects nonlinear in the electric field turn out to be strongest in superconductors without magnetic impurities, in magnetic fields that are weak in comparison with the critical field H_{c2}. At a temperature close to critical, when $\Delta \gg T_c - T$, the gradient terms in the equations for the Green's functions $G^{(R,A)}$ are small. In this case the Green's functions are expressed in terms of the value of the ordering parameter at the same point

$$G^{(R,A)} = -\frac{i}{[|\Delta|^2-(\varepsilon \pm i\delta)^2]^{1/2}} \begin{pmatrix} \varepsilon, & \Delta \\ -\Delta^*, & -\varepsilon \end{pmatrix}. \tag{44}$$

Solving the system (23), with allowance for expression (44) for the Green's functions $G^{(R,A)}$ and for the normalization condition (17), we obtain for $|\Sigma| < |\Delta|$

$$\Delta^* F_1^b = \Delta F_2^b = |\Delta|^2 g_1^b/\varepsilon, \quad g_2^b = -g_1^b;$$
$$\left(D\frac{\partial^2}{\partial \mathbf{R}^2}\right)\{(\varepsilon^2-|\Delta|^2)^{1/2} g_1^b\} = 2\varepsilon \frac{\partial f}{\partial \varepsilon}\left(\mathbf{u}\frac{\partial(\varepsilon^2-|\Delta|^2)^{1/2}}{\partial \mathbf{R}}\right) \tag{45}$$

In the region $|\epsilon| < |\Delta|$ the functions $F^b_{1,2}$ are relatively small. Substituting expressions (44) and (45) in (26) and (28), we reduce Eq. (24) for the distribution function to the form

$$\left(f(\varepsilon) - \text{th}\frac{\varepsilon}{2T}\right)\int_{V_\varepsilon} d^2\mathbf{r}\frac{|\varepsilon|}{(\varepsilon^2-|\Delta|^2)^{1/2}}$$
$$= \tau_\varepsilon \frac{\partial}{\partial \varepsilon}\left\{\frac{\partial f}{\partial \varepsilon}\int_{V_\varepsilon} d^2\mathbf{r}\left(\mathbf{u}\frac{\partial(\varepsilon^2-|\Delta|^2)^{1/2}}{\partial \mathbf{R}}\right)\left(-D\frac{\partial^2}{\partial \mathbf{R}^2}\right)^{-1}\left(\mathbf{u}\frac{\partial(\varepsilon^2-|\Delta|^2)^{1/2}}{\partial \mathbf{R}}\right)\right\}, \tag{46}$$

where V_ϵ is that region inside the cell for which $|\epsilon| > |\Delta|$.

Near the transition temperature, strong changes in the conductivity and in the ordering parameter Δ occur in weak fields, when the distribution function still varies weakly. In this case the distribution function $f(\epsilon)$ in the right-hand side of (46) can be replaced by $\tanh(\epsilon/2T)$. In the case of weak magnetic fields $H \ll H_{c2}$ the vortices are located far from each other. As will be shown below, the dependence of the ordering parameter Δ on the coordinates remains qualitatively the same also in a strong electric field, namely Δ vanishes at the center of the vortex and assumes a constant value at large distances from its center. This circumstance allows us to solve Eq. (46) in various limiting cases:

$$\frac{f(\varepsilon)-\text{th}(\varepsilon/2T)}{\tau_\varepsilon u^2/DT} = \begin{cases} \varepsilon/36, & |\varepsilon| \ll \Delta_\infty \\ \dfrac{\sqrt{2}}{\pi}\Delta_\infty(1-\varepsilon/\Delta_\infty)^{1/2}, & 0 < \Delta_\infty - \varepsilon \ll \Delta_\infty. \\ -\dfrac{1}{8\varepsilon^3}\langle(|\Delta|^2 - \langle|\Delta|^2\rangle)^2\rangle, & \varepsilon \gg \Delta_\infty \end{cases} \tag{47}$$

It follows from this formula that the number of normal excitations decreases inside the vortices ($\epsilon < \Delta$) and increases on the outside.

As seen from (27), the angular dependence of the ordering parameter Δ is preserved in an electric field. A change in the distribution function causes only a change in the dependence of the absolute value of Δ on the distance to the center of the vortex:

$$\left\{1-\frac{T}{T_c}+\frac{\pi D}{8T}\left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho}-\frac{1}{\rho^2}\right)-\frac{7\zeta(3)\Delta^2}{8\pi^2 T^2}\right.$$
$$\left.+\int_\Delta^\infty \frac{d\varepsilon}{(\varepsilon^2-\Delta^2)^{1/2}}\left(f(\varepsilon)-\text{th}\frac{\varepsilon}{2T}\right)\right\}\Delta = 0. \tag{48}$$

Since $f(\epsilon)$ changes little at $\epsilon > \Delta_\infty$ in the case of loosely-spaced vortices, Δ assumes at large distances the same constant value as in the absence of an electric field. In a region far from the center of the vortex, the ordering parameter takes the form

$$\Delta^2(\rho) = \Delta_\infty^2(1-\xi^2(u)/\rho^2).$$

From (47) and (48) we obtain for $\xi^2(u)$

$$\xi^2(u) = \xi^2(0)\left[1+\left(\frac{u}{u^*}\right)^2\right]^{-1}, \quad \xi^2(0) = \frac{\pi D}{8(T_c-T)},$$
$$u^{*2} = \frac{D(14\zeta(3))^{1/2}(1-T/T_c)^{1/2}}{\pi\tau_\varepsilon}. \tag{49}$$

It follows from (48) and (49) that the dimension of the vortex $\xi(u)$ can change strongly, but both in the case of small and in the case of large velocities u the ordering parameter Δ depends only on the ratio ρ/ξ. It follows from (49) that the vortex dimension (10) decreases rapidly with increasing velocity u. This decrease of the dimension is due to the decrease in the number of normal excitation s inside the vortex. The viscosity coefficient η and the conductivity σ are expressed in terms of the distribution function and the ordering parameter Δ by means of the linear-approximation formulas (see[6, 10])

$$\eta = \frac{\sigma_N}{2\pi^2 D^2}\int_0^\infty d\varepsilon\frac{\partial f}{\partial \varepsilon}\int_{V_\varepsilon} d^2\mathbf{r}\left((\varepsilon^2-\Delta^2)^{1/2}-\frac{1}{V_\varepsilon}\int_{V_\varepsilon}(\varepsilon^2-\Delta^2)^{1/2}d^2\mathbf{r}\right)^2, \quad \sigma = \frac{\pi\eta}{eB}, \tag{50}$$

To calculate the integral in (50) it is necessary to know the dependence of the ordering parameter Δ on the coordinates. At large distances, this dependence is given by (48). At short distances we have $\Delta \sim \rho$. Just as in[10], we assume that

$$\Delta^2(\rho) = \Delta_\infty^2\frac{\rho^2}{\rho^2+\xi^2}. \tag{51}$$

Substituting this expression in (50), we obtain

$$\eta(u) = \eta(0)\frac{\xi^2(u)}{\xi^2(0)}, \quad \eta(0) = 0{,}45\frac{\sigma_N T_c}{D}\left(1-\frac{T}{T_c}\right)^{1/2}, \tag{52}$$

where $\xi(u)$ is given by (49). Thus, the viscous-friction force has a maximum at a vortex velocity $u = u^*$, determined by formula (49). At $u^2 \sim u^{*2}(1 - T/T_c)^{-1/2}$ the viscosity coefficient ceases to decrease, since an important role is assumed in it by terms that are not very sensitive to the distribution function and to the coordinate dependence of the ordering parameter. The dependence of the current on the voltage is determined by the formula

$$\mathbf{j} = \sigma\mathbf{E}\left\{\frac{1}{1+(E/E^*)^2}+c\left(1-\frac{T}{T_c}\right)^{1/2}\right\}, \tag{53}$$

where c is a number on the order of unity,

$$\sigma = \sigma_N\frac{H_{c2}}{B}\frac{1{,}1}{\sqrt{1-T/T_c}}, \quad E^{*2} = \frac{B^2 D}{\pi\tau_\varepsilon}\sqrt{14\zeta(3)}\sqrt{1-T/T_c}.$$

The numerical coefficients in the expressions for $\eta(0)$, σ, and E^* in (52) and (53) are approximate, since expression (51) was used for $\Delta(\rho)$.

Thus, just as in the case of a superconductor with magnetic impurities, the current-voltage characteristic is N-shaped. We note that the nonlinear effects in superconductors without magnetic impurities in weak magnetic fields set in much earlier than in superconductors with magnetic impurities. The cause of the difference is that superconductors with sufficiently high concentration of magnetic impurities are gapless, and the normal excitations are distributed almost uniformly over the entire volume of the superconductor. Therefore their distribution function is altered by the vortex motion much less than in superconductors with gaps.

CONCLUSION

Thus, the current voltage characteristic of a superconductor in the mixed state has a characteristic N-shape. The nonlinear effects appear in relatively weak electric fields, when the transport current is small in comparison with the critical pair-breaking current. When the vortices move, the energy of the normal excitations increases. Owing to the large energy-relaxation time, the excitation-energy distribution function can change quite strongly at a relatively low vortex velocity. The number of excitations inside the vortex decreases; this increases the vortex dimensions, causing in turn a decrease in the viscosity coefficient. The viscous-friction force is not a monotonic function of the velocity, but has a maximum at $u = u^*$, where u^* is defined in (49).

If there is no gap in the excitation spectrum, owing to magnetic impurities, then the nonlinear effects appear in stronger electric fields. The reason is that the distribution of the excitations changes throughout the volume, and not only in the region inside the vortex. The vortex motion results in this case in effective heating, which is equivalent to a lowering of the superconducting-transition temperature. A similar picture arises in magnetic fields close to the critical field H_{c2}.

It was assumed above that the heat is effectively carried away and that the phonon temperature is independent of the electric field. If the heat dissipation is poor, then the temperature T in all the formulas derived above must be replaced by the local phonon temperature, which depends on the critical field and can be obtained from the heat-conduction equations.

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