KYBERNETIKA — VOLUME 9 (1973), NUMBER 4

Algebraic Theory of Discrete Optimal Control for Single–Variable Systems III

Closed-Loop Control

VLADIMÍR KUČERA

This part completes Part I: Preliminaries [3] and Part II: Open-Loop Control [4] to form a comprehensive and unified treatment of the algebraic theory of discrete optimal control for single-input single-output systems.

The object to be discussed here is the closed-loop optimal control theory. To recall, given a system s we are to find such a controller r feeded by the error signal e that the output y of the system follows a given reference signal w in a prescribed manner. This configuration, shown in Fig. 1, is of feedback type, i.e., it counteracts possible disturbances in the control loop.

First the pole assignment problem is mentioned. Then we consider the time optimal controls and the least squares control in the absence of disturbances. At the end the effect of disturbances, closed-loop stability and related topics are discussed.

One of the most interesting features of the closed-loop synthesis technique presented is that the optimal controller is synthesized directly without predetermining the closed-loop transfer function as is usual in the literature [1], [2], [6], [7] to garantee stability.

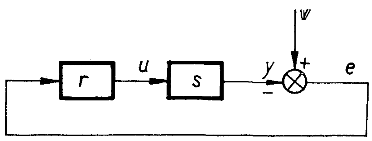

Fig. 1.

The theorems, equations, examples, etc. are numbered separately in each part of the tripaper. The usual system of references is used within this paper whereas cross-references are followed by a slash and the respective part number. The notation introduced in Part I and Part II is consistently adhered to throughout.

INTRODUCTION

By precascading the system to be controlled with a controller and closing the feedback loop we form another system. An interesting point is that this closed-loop system may not be canonical [3] even if the original components are. In fact, we shall see later that the optimum system synthesis always calls for certain procedures that produce a non-canonical closed-loop system. As a result, the transfer function does not fully describe such a system any more and we have to take into account the system response to initial conditions.

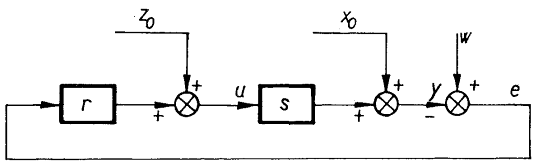

Fig. 2.

Given the system shown in Fig. 2, where

$$\boldsymbol{s} = \frac{\zeta^d b}{a}, \quad \boldsymbol{r} = \frac{s}{r},$$

the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta, b) = 1$, $(b, \zeta) = 1$, $(r, \zeta s) = 1$. Then

$$y = \frac{\zeta^d b}{a} u + \frac{g}{a},$$

$$u = \frac{s}{r} e + \frac{h}{r},$$

where $g, h \in \mathfrak{F}[\zeta]$ are arbitrary polynomials of degree one less than dimensions of $\boldsymbol{s}$ and $\boldsymbol{r}$ respectively. They characterize the effect of $\mathbf{x}_0$ and $\mathbf{z}_0$, the initial states of the $\boldsymbol{s}$ and $\boldsymbol{r}$ respectively.

The effect of the initial states upon the error $\boldsymbol{e}$ and the control $\boldsymbol{u}$ is then obtained as

$$e = \frac{ar}{ar + \zeta^d bs} w - \frac{\zeta^d b}{ar + \zeta^d bs} h - \frac{r}{ar + \zeta^d bs} g,$$

$$u = \frac{as}{ar + \zeta^d bs} w + \frac{a}{ar + \zeta^d bs} h - \frac{s}{ar + \zeta^d bs} g.$$

Now if $(a, s) \neq 1$ and/or $(b, r) \neq 1$, these factors disappear in the transfer functions relating $\boldsymbol{e}$ and $\boldsymbol{u}$ to $\boldsymbol{w}$. However, they do remain in the response of $\boldsymbol{e}$ or $\boldsymbol{u}$ to both initial states. Thus the annihilating polynomial of the closed-loop system is no less than

$$c = ar + \zeta^a bs.$$

Since it cannot be obtained from the input-output properties, a care must be exercised in stability investigations.

POLE ASSIGNMENT PROBLEM

One of the most powerful synthesis techniques is that of pole assignment. Given the configuration shown in Fig. 1, where

$$s = \frac{\zeta^d b}{a},$$

the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$. Find a controller

$$r = \frac{s}{r}$$

so as to achieve a desired annihilating polynomial $c \in \mathfrak{F}[\zeta]$, $(c, \zeta) = 1$, of the closed-loop system.

Theorem 1. *The pole assignment problem has always a solution for any c. The solution is not unique and all solutions are given by*

$$s = y\,,$$

$$r = x\,,$$

where x and y is any solution of the Diophantine equation

$$(1) \qquad ax + \zeta^d by = c$$

for which $(x, y) = 1$.

Even the controller of minimal dimension, further specified by $\partial x = \min$, *is not unique in general.*

Proof. In the Introduction the polynomial $ar + \zeta^d bs$ was shown to be the annihilating polynomial of the closed-loop system provided $(r, s) = 1$. Otherwise the annihilating polynomial would become

$$\frac{ar + \zeta^d bs}{(r, s)}.$$

Thus (1) follows and the existence of a solution is implied by $(a, \zeta^d b) = 1$. □

Example 1. Consider $\mathfrak{F} = \mathfrak{R}$,

$$s = \frac{\zeta}{1 - \zeta},$$

 and assign to the closed-loop system the annihilating polynomial

$$c = 1 .$$

The equation

$$(1 - \zeta)\, x + \zeta y = 1$$

gives

$$x = 1 + \zeta t ,$$
$$y = 1 - (1 - \zeta)\, t ,$$

$t \in \Re[\zeta]$ arbitrary. Hence any

$$r = \frac{1 - (1 - \zeta)\, t}{1 + \zeta t}$$

solves the problem. Among all controllers,

$$r = 1$$

is the only one which is of minimal dimension.

Example 2. Consider the system

$$s = \frac{\zeta(1 - 2\zeta)}{2 - 0{\cdot}5\zeta}$$

over the field $\Re$. Find such a controller that

$$c = 2 - 1{\cdot}5\zeta - 2{\cdot}5\zeta^2 + 2\zeta^3 .$$

By Theorem 1, solve

$$(2 - 0{\cdot}5\zeta)\, x + \zeta(1 - 2\zeta)\, y = 2 - 1{\cdot}5\zeta - 2{\cdot}5\zeta^2 + 2\zeta^3$$

to get

$$x = 1 - \zeta + (\zeta - 2\zeta^2)\, t ,$$
$$y = 1 - \zeta - (2 - 0{\cdot}5\zeta)\, t .$$

Thus

$$r = \frac{1 - \zeta - (2 - 0{\cdot}5\zeta)\, t}{1 - \zeta + (\zeta - 2\zeta^2)\, t} ,$$

where $t \in \Re[\zeta]$ arbitrary but such that $(x, y) = 1$.

For instance $t = 0$ is prohibited. Otherwise $(x, y) = 1 - \zeta$ and

$$r = 1$$

would yield the annihilating polynomial $2 + 0.5\zeta - 2\zeta^2$ instead of that required.

The minimal controller is not unique, either, and all such controllers are of the form

$$r = \frac{1 - \zeta - (2 - 0{\cdot}5\zeta)\,\tau}{1 - \zeta + (\zeta - 2\zeta^2)\,\tau}$$

where $\tau \in \mathfrak{R}$, $\tau \neq 0$.

In particular, feedback stabilization falls within this scope. All we need is to choose a stable c. If, on the other hand, an optimality criterion is specified, we are to find the appropriate c using the methods to follow.

We can attack the closed-loop optimal control problems in such a way as to strictly minimize the optimality criterion whatever the resulting annihilating polynomial c may be. This yields results equivalent to the open-loop control. What makes the closed-loop solution attractive, however, is the possibility of counteracting disturbances by synthesizing a stable or otherwise presprecified c.

CLOSED-LOOP OUTPUT TIME OPTIMAL CONTROL

In view of the preceding discussion the problem can be posed as follows. Given the configuration of Fig. 1, where

$$s = \frac{\zeta^d b}{a}, \quad w = \frac{q}{p},$$

the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$, $(p, \zeta q) = 1$. Fing such a controller

$$r = \frac{s}{r}$$

that the control u is stable, the error e is zero in a minimum time $k_{\min}$, and the annihilating polynomial c is stable.

The solution is contained in

Theorem 2. *The closed-loop output time optimal control problem has a solution if and only if* $p^- \mid a$. *The solution is unique and is given by*

$$s = a_0^+ \mathfrak{y},$$
$$r = p_0 b^+ \mathfrak{x}$$

where $\mathfrak{x}$ *and* $\mathfrak{y}$ *is such solution of the Diophantine equation*

$$a_0^- p x + \zeta^d b^- y = q^+$$

that $\partial \mathfrak{x} = \min$.

Moreover

$$e = a_0^- q^- \hat{x} ,$$

$$u = \frac{a_0 q^- \hat{y}}{p_0 b^+} ,$$

and

$$k_{\min} = 1 + \partial a_0^- + \partial q^- + \partial \hat{x} .$$

Finally,

$$c = \frac{a_0^+ b^+ q^+}{(r, s)} .$$

Proof. On inspecting Fig. 1 we get

$$e = \frac{1}{1 + sr} w = \frac{ar}{ar + \zeta^d bs} \frac{q}{p} .$$

Using (14/I),

$$e = \frac{a_0 r}{ar + \zeta^d bs} \frac{q}{p_0} .$$

The error being zero in a finite time, it calls for a polynomial e. Therefore $p_0 \mid r$ and in order to reduce the expression for e as much as possible while preserving u and $ar + \zeta^d bs$ stable, we write

$$s = a_0^+ y ,$$

$$r = p_0 b^+ x .$$

Then

$$e = \frac{a_0^- qx}{a_0^- px + \zeta^d b^- y} \quad (2)$$

and the utmost reduction yields

$$a_0^- px + \zeta^d b^- y = q^+ \quad (3)$$

since the denominator in (2) must be stable.

It follows that

$$e = a_0^- q^- x .$$

But e is to be zero in a minimum time a hence we have to extract that solution $\hat{x}$ of (3) for which $\partial \hat{x} = \min$. Then

$$k_{\min} = 1 + \partial a_0^- + \partial q^- + \partial \hat{x} .$$

We infer from Fig. 1 that

$$u = re = \frac{a_0 q^- \hat{y}}{p_0 b^+} .$$

The control is stable if and only if p_0 is stable, that is, if $p^- \mid a$. This implies $(a_0^- p, \zeta^d b^-) = 1$ and, in turn, the existence of a solution to (3).

Inasmuch as r and s are required to be relatively prime, we obtain

$$c = \frac{ar + \zeta^d bs}{(r, s)} = \frac{a_0^+ b^+ q^+}{(r, s)}$$

which is indeed stable. □

Remark 1. In case **s** originated from a continuous system by the process of sampling the error need not vanish in between sampling points. However, it is stable.

Remark 2. If we drop the requirement that c be stable, we obtain

$$s = a_0 \hat{y} ,$$
$$r = p_0 b^+ \hat{x}$$

where

$$px + \zeta^d b^- y = q$$

and $\partial \hat{x} = \min$.

Then

$$e = \hat{x} ,$$
$$u = \frac{a_0 \hat{y}}{p_0 b^+}$$

and

$$k_{\min} = 1 + \partial \hat{x} .$$

Comparing the results with Theorem 1/II we see that the solutions are identical. Thus the $k_{\min}$ is as small as for the open-loop control, but

$$c = \frac{a_0 b^+ q}{(r, s)}$$

is not generally stable.

Example 3. Let $\mathfrak{F} = \mathfrak{R}$ and

$$s = \frac{\zeta(1 - 2{\cdot}5\zeta + \zeta^2)}{1 - 5\zeta + 4\zeta^2} , \quad w = \frac{1}{1 - \zeta} .$$

 Obtain the closed-loop output time optimal control. We have

$$a_0 = 1 - 4\zeta\,, \quad p_0 = 1\,,$$
$$b^+ = 1 - 0{\cdot}5\zeta\,, \quad b^- = 1 - 2\zeta\,.$$

The Diophantine equation to be solved reads

$$(1 - 5\zeta + 4\zeta^2)\,x + \zeta(1 - 2\zeta)\,y = 1\,.$$

We arrange the computations into the table below, see (9/I):

$1 - 5\zeta + 4\zeta^2$	$\zeta - 2\zeta^2$	$1 - 3\zeta$	$\frac{1}{9}$
	-2	$-\frac{1}{9} + \frac{6}{9}\zeta$	$9 - 27\zeta$
1	-2	$\frac{11}{9} - \frac{12}{9}\zeta$	$9 - 45\zeta + 36\zeta^2$
0	1	$-\frac{1}{9} + \frac{6}{9}\zeta$	$9\zeta - 18\zeta^2$

Using (5/I) and (6/I), the general solution becomes

$$x = 1 - 6\zeta + (9\zeta - 18\zeta^2)\,t\,,$$
$$y = 11 - 12\zeta - (9 - 45\zeta + 36\zeta^2)\,t\,.$$

The condition $\partial\hat{x} = \min$ gives

$$\hat{x} = 1 - 6\zeta\,,$$
$$\hat{y} = 11 - 12\zeta\,.$$

Therefore the optimal controller is

$$r = \frac{11 - 12\zeta}{(1 - 0{\cdot}5\zeta)(1 - 6\zeta)}\,.$$

Further we compute

$$e = 1 - 10\zeta + 24\zeta^2$$
$$u = \frac{(1 - 4\zeta)(11 - 12\zeta)}{1 - 0{\cdot}5\zeta}\,,$$
$$k_{\min} = 3\,.$$

The annihilating polynomial of the closed-loop system becomes

$$c = 1 - 0{\cdot}5\zeta\,.$$

Example 4. Consider $\mathfrak{F} = \mathfrak{R}$,

$$s = \zeta\,, \quad w = \frac{2 - \zeta + \zeta^2}{2 - \zeta}$$

and find the closed-loop output time optimal control.

Since

$$a = 1, \qquad b^- = 1,$$
$$p_0 = 2 - \zeta, \quad q^+ = 2 - \zeta + \zeta^2,$$

we obtain

$$(2 - \zeta)\, x + \zeta y = 2 - \zeta + \zeta^2.$$

The solution satisfying $\partial \hat{x} = \min$ reads

$$\hat{x} = 1,$$
$$\hat{y} = \zeta,$$

and hence

$$s = \zeta,$$
$$r = 2 - \zeta.$$

Therefore

$$\boldsymbol{r} = \frac{\zeta}{2 - \zeta},$$
$$\boldsymbol{e} = 1, \quad k_{\min} = 1,$$
$$\boldsymbol{u} = \frac{\zeta}{2 - \zeta}.$$

Since $(r, s) = 1$,

$$c = 2 - \zeta + \zeta^2.$$

Note that s may be divisible by, i.e., the optimal control strategy may require a delay!

CLOSED-LOOP STATE TIME OPTIMAL CONTROL

This is a modification of the preceding problem. We are to zero $\boldsymbol{e}$ in a minimum time by application of a finite control. This is equivalent to reaching equilibrium state in a finite time.

More formally, given the configuration shown in Fig. 1, where

$$\boldsymbol{s} = \frac{\zeta^d b}{a}, \quad \boldsymbol{w} = \frac{q}{p},$$

the polynomials being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$, $(p, \zeta q) = 1$; find such a controller

$$\boldsymbol{r} = \frac{s}{r}$$

that the $\boldsymbol{u}$ is zero in a finite time, the $\boldsymbol{e}$ is zero in a minimum time $k_{\min}$, and the c is stable.

 Then we claim the following

Theorem 3. *The closed-loop state time optimal control problem has a solution if and only if $p \mid a$. The solution is unique and is given as*

$$s = a_0^+ \hat{y}\,,$$
$$r = \hat{x}\,,$$

where $\hat{x}$ and $\hat{y}$ is such solution of the Diophantine equation

$$a_0^- px + \zeta^d by = q^+$$

that $\partial \hat{x} =$ min.

Moreover

$$e = a_0^- q^- \hat{x}\,,$$
$$u = a_0 q^- \hat{y}\,,$$
$$k_{\min} = 1 + \partial a_0^- + \partial q^- + \partial \hat{x}$$

and

$$c = \frac{a_0^+ q^+}{(r, s)}\,.$$

Proof. On inspecting Fig. 1 we obtain

$$e = \frac{ar}{ar + \zeta^d bs} \frac{q}{p} = \frac{a_0 r}{ar + \zeta^d bs} \frac{q}{p_0}\,.$$

To make e polynomial we first have to require $p_0 = 1$ or $p \mid a$ by (14/I). Further we set

$$s = a_0^+ y\,,$$
$$r = x\,,$$

to reduce the expression for e as much as possible,

(4) $$e = \frac{a_0^- qx}{a_0^- px + \zeta^d by}\,,$$

and to keep the $ar + \zeta^d bs$ stable.

Since the denominator in (4) must be stable, the best we can do to minimize the degree of e while preserving a polynomial u is to set

$$a_0^- px + \zeta^d by = q^+\,.$$

This Diophantine equation has indeed a solution since $p \mid a$ implies

$$(a_0^- p, \zeta^d b) = 1\,.$$

It follows that

$$e = a_0^- q^- \hat{x}$$

and

$$k_{\min} = 1 + \partial a_0^- + \partial q^- + \partial \hat{x} .$$

Further,

$$u = re = a_0 q^- \hat{y}$$

and the annihilating polynomial c is given as

$$c = \frac{ar + \zeta^d bs}{(r, s)} = \frac{a_0^+ q^+}{(r, s)}$$

since r and s are required to be relatively prime. □

Remark 3. Similarly to the open-loop control, the control time may exceed the follow-up time and hence the system need not reach equilibrium within $k_{\min}$ time units. See Example 6.

Remark 4. There is a common fallacy in the literature that

(5) $$k = \frac{ar}{ar + \zeta^d bs},$$

the overall transfer function relating e to w, must be a polynomial in order for e to vanish in a finite time. It is easily seen and also illustrated in Example 6 that a polynomial k is not the optimizing choice for the output or state time optimal control whenever $q^+ \neq 1$.

Remark 5. Had we strictly minimized $k_{\min}$ without any restrictions on c, we would have obtained

$$s = a_0 \hat{y} ,$$
$$r = \hat{x}$$

where

$$px + \zeta^d by = q$$

and $\partial \hat{x} = \min$.

Then

$$e = \hat{x} ,$$
$$u = a_0 \hat{y} ,$$

and

$$k_{\min} = 1 + \partial \hat{x} .$$

The results are identical to those in Theorem 2/II and

$$c = \frac{a_0 q}{(r, s)}$$

is not stable in general. Therefore, the closed-loop control can do no better than the open-loop control except for stabilizing the annihilating polynomial c.

Example 5. Obtain the closed-loop state time optimal control for the system

$$s = \frac{\zeta(1 - 2{\cdot}5\zeta + \zeta^2)}{1 - 5\zeta + 4\zeta^2}, \quad w = \frac{1}{1 - \zeta}$$

over the field $\mathfrak{R}$.

The respective equation to be solved becomes

$$(6) \qquad (1 - 5\zeta + 4\zeta^2)\, x + \zeta(1 - 2{\cdot}5\zeta + \zeta^2)\, y = 1$$

and its solution

$$\hat{x} = \tfrac{1}{7}(7 - 55\zeta + 26\zeta^2),$$
$$\hat{y} = \tfrac{1}{7}(90 - 104\zeta)$$

has the property that $\partial\hat{x} = \min$.

Thus we see that

$$r = \frac{90 - 104\zeta}{7 - 55\zeta + 26\zeta^2}$$

and

$$e = \tfrac{1}{7}(7 - 83\zeta + 246\zeta^2 - 104\zeta^3),$$
$$u = \tfrac{1}{7}(90 - 464\zeta + 416\zeta^2), \quad k_{\min} = 4$$

by invoking Theorem 3. Also

$$c = 1\,.$$

However, imposing no restrictions on c, we have

$$(1 - \zeta)\, x + \zeta(1 - 2{\cdot}5\zeta + \zeta^2)\, y = 1$$

instead of (6). The solution modifies to

$$\hat{x} = 1 + 3\zeta - 2\zeta^2,$$
$$\hat{y} = -2$$

and hence

$$r = \frac{-2 + 8\zeta}{1 + 3\zeta - 2\zeta^2},$$
$$e = 1 + 3\zeta - 2\zeta^2, \quad k_{\min} = 3,$$
$$u = -2 + 8\zeta,$$
$$c = 1 - 4\zeta$$

conformably with Example 3/II. As expected the c is not stable.

Example 6. Let $\mathfrak{F} = \mathfrak{R}$ and consider

$$s = \frac{2\zeta}{(1-\zeta)^2}, \quad w = \frac{1 - 0{\cdot}5\zeta}{1-\zeta}.$$

The task is to find the closed-loop state time optimal control.

By Theorem 3, we are to solve the equation

$$(1-\zeta)^2 x + 2\zeta y = 1 - 0{\cdot}5\zeta$$

under the condition that $\partial \hat{x} = \min$. The solution reads

$$\hat{x} = 1,$$
$$\hat{y} = 0{\cdot}75 - 0{\cdot}5\zeta.$$

The optimal controller results as

$$r = 0{\cdot}75 - 0{\cdot}5\zeta$$

and

$$e = 1 - \zeta, \quad u = 0{\cdot}75 - 1{\cdot}25\zeta + 0{\cdot}5\zeta^2$$
$$k_{\min} = 2, \quad c = 1 - 0{\cdot}5\zeta.$$

Notice that the equilibrium state is not attained within $k_{\min}$ time units and also that the k in (5) is not a polynomial,

$$k = \frac{(1-\zeta)^2}{1 - 0{\cdot}5\zeta}.$$

CLOSED-LOOP LEAST SQUARES CONTROL

This problem involves minimization of a quadratic functional. In fact, the basic idea of this section mimics that in the pioneering work [5]. However, our setting is different and much more general.

The problem can be formally defined as follows. Given the configuration shown in Fig. 1, where

$$s = \frac{\zeta^d b}{a}, \quad w = \frac{q}{p},$$

the polynomial being arbitrary elements of $\mathfrak{F}[\zeta]$ but $d > 0$, $(a, \zeta b) = 1$, $(b, \zeta) = 1$, $(p, \zeta q) = 1$. Synthesize such a controller

$$r = \frac{\mathbf{s}}{\mathbf{r}}$$

 that the $\boldsymbol{u}$ is stable, the cost functional

$$\varphi = \sum_{k=0}^{\infty} \varepsilon_k^2$$

is minimized, where $\boldsymbol{e} = \varepsilon_0 + \varepsilon_1 \zeta + \varepsilon_2 \zeta^2 + \ldots$, and the c is stable.

Theorem 4. *The closed-loop least squares control problem has a solution if and only if $p^- \mid a$ and $\tilde{a}_0^- \tilde{b}^- \tilde{q}^- / (\hat{x}, \hat{y})$ is stable. The solution is unique and is given by*

$$s = a_0^+ \hat{y},$$
$$r = p_0 b^+ \hat{x},$$

where $\hat{x}$ and $\hat{y}$ is such solution of the Diophantine equation

$$a_0^- p x + \zeta^d b^- y = \tilde{a}_0^- \tilde{b}^- q^*$$

that $\partial \hat{x} = \min$.

Moreover,

$$e = \frac{a_0^- q^- \hat{x}}{\tilde{a}_0^- \tilde{q}^- \tilde{b}^-},$$

$$\boldsymbol{u} = \frac{a_0 q^- \hat{y}}{p_0 \tilde{a}_0^- \tilde{q}^- b^*}$$

and

$$\varphi_{\min} = \left\langle \left(\frac{\hat{x}}{b^-} \right)^{\sim} \left(\frac{\hat{x}}{b^-} \right) \right\rangle .$$

Finally,

$$c = \frac{a_0^* b^* q^*}{(r, s)} .$$

Proof. Analogously to the proof of Theorem 3/II we have

$$\varphi = \langle \tilde{e} e \rangle$$

if and only if $\boldsymbol{e}$ is stable.

We will manipulate the expression for φ in such a way as to make the minimizing choice of $\boldsymbol{r}$ obvious. Rewrite

$$\varphi = \left\langle \left(\frac{\tilde{b}^- \tilde{a}_0^- \tilde{q}^-}{\zeta^d b^- a_0^- q^-} e \right)^{\sim} \left(\frac{\tilde{b}^- \tilde{a}_0^- \tilde{q}^-}{\zeta^d b^- a_0^- q^-} e \right) \right\rangle \tag{7}$$

and

$$e = \frac{ar}{ar + \zeta^d bs} \frac{q}{p} = \frac{q}{p} - \frac{\zeta^d bs}{ar + \zeta^d bs} \frac{q}{p} .$$

Therefore

$$(8) \qquad \frac{\tilde{b}^- \tilde{a}_0^- \tilde{q}^-}{\zeta^d b^- a_0^- q^-} e = \frac{\tilde{a}_0^- \tilde{b}^- q^*}{\zeta^d b^- a_0^- p} - \frac{\tilde{a}_0^- b^* q^* s}{(ar + \zeta^d bs) a_0^- p}$$

and consider the decomposition

$$\frac{\tilde{a}_0^- \tilde{b}^- q^*}{\zeta^d b^- a_0^- p} = \frac{x}{\zeta^d b^-} + \frac{y}{a_0^- p}$$

of the first term in (8), so that x and y are coupled via the Diophantine equation

$$(9) \qquad a_0^- px + \zeta^d b^- y = \tilde{a}_0^- \tilde{b}^- q^* .$$

Collecting the terms gives us

$$(10) \qquad \frac{\tilde{a}_0^- \tilde{b}^- \tilde{q}^-}{\zeta^d b^- a_0^- q^-} e = \frac{x}{\zeta^d b^-} + \boldsymbol{a} ,$$

where

$$(11) \qquad \boldsymbol{a} = \frac{y}{a_0^- p} - \frac{\tilde{a}_0^- b^* q^* s}{(ar + \zeta^d bs) a_0^- p} =$$

$$= \frac{ayr + (\zeta^d b^- y - \tilde{a}_0^- \tilde{b}^- q^*) b^+ s}{(ar + \zeta^d bs) a_0^- p} = \frac{a_0^+ yr - p_0 b^+ xs}{(ar + \zeta^d bs) p_0}$$

on making use of (9) and (14/I).

Substitution of (10) into (7) results in

$$\varphi = \left\langle \left(\frac{x}{b^-} \right)^{\sim} \left(\frac{x}{b^-} \right) \right\rangle + \tilde{\boldsymbol{a}}\boldsymbol{a} + 2 \left\langle \left(\frac{x}{\zeta^d b^-} \right)^{\sim} \boldsymbol{a} \right\rangle .$$

Further we refine the above by setting

$$(12) \qquad x = \hat{x} + \zeta^d b^- t , \quad \partial \hat{x} < \partial \zeta^d b^- .$$

Then the same reasoning as in the proof of Theorem 3/II justifies that

$$\left\langle \left(\frac{\hat{x}}{\zeta^d b^-} \right)^{\sim} t \right\rangle = 0 , \quad \left\langle \left(\frac{\hat{x}}{\zeta^d b^-} \right)^{\sim} \boldsymbol{a} \right\rangle = 0 .$$

Therefore

$$(13) \qquad \varphi = \left\langle \left(\frac{\hat{x}}{b^-} \right)^{\sim} \left(\frac{\hat{x}}{b^-} \right) \right\rangle + \langle \tilde{\boldsymbol{a}}\boldsymbol{a} \rangle + 2 \langle \tilde{\boldsymbol{a}} t \rangle + \langle \tilde{t} t \rangle .$$

The first term in (13) cannot be further reduced. To minimize φ, we can do no better than to set $\boldsymbol{a} = 0$, $t = 0$. Thus by virtue of (11) and (12)

$$s = a_0^+ \hat{y}\,,$$
$$r = p_0 b^+ \hat{x}\,,$$

where $\hat{x}$ and $\hat{y}$ is such solution of (9) that $\partial \hat{x} = \min$.

It follows from (9) and (14/I) that

$$\boldsymbol{e} = \frac{ar}{ar + \zeta^d bs}\,\frac{q}{p} = \frac{a_0 p b^+ q \hat{x}}{a_0^+ p b^+ (a_0^- p \hat{x} + \zeta^d b^- y)} = \frac{a_0^- q^- \hat{x}}{\tilde{a}_0^- \tilde{q}^- \tilde{b}^-}\,,$$

which is stable if and only if $\tilde{b}^- / (\hat{x}, \tilde{b}^-)$ is stable.

Also

$$\boldsymbol{u} = r\boldsymbol{e} = \frac{a_0 q^- \hat{y}}{p_0 \tilde{a}_0^- \tilde{q}^- b^*}\,,$$

which is stable if and only if $\tilde{b}^- / (\hat{y}, \tilde{b}^-)$ is stable and p_0 is stable, that is, $p^- \mid a$. This last condition implies that $(a_0^- p, \zeta^d b^-) = 1$ and hence (9) has a solution.

Finally, since r and s are required to be relatively prime, the annihilating polynomial c is given as

$$c = \frac{ar + \zeta^d bs}{(r, s)} = \frac{a_0^* b^* q^*}{(r, s)}$$

on employing (9). Note that c is stable, as required, if and only if $\tilde{a}_0^- \tilde{b}^- \tilde{q}^- / (\hat{x}, \hat{y})$ is stable. □

Remark 6. For a minimum-phase system (i.e. b stable) the closed-loop output time optimal control and the least squares control are not the same in contradistinction to the open-loop control. Compare Example 5/II and Example 8 to follow. They may equal, however, in some cases.

Remark 7. As before, not restricting c to the class of stable polynomials would yield results identical to the open-loop case (Theorem 3/II). We just need to write

$$\varphi = \left\langle \left(\frac{\tilde{b}^-}{\zeta^d b^-}\right)^{\sim} \left(\frac{\tilde{b}^-}{\zeta^d b^-}\right) \right\rangle$$

instead of (7) and follow the proof of Theorem 4 to obtain

$$s = a_0 \hat{y}\,,$$
$$r = p_0 b^+ \hat{x}\,,$$

where

$$px + \zeta^d b^- y = \tilde{b}^- q$$

and $\partial\hat{x}$ = min. As expected,

$$c = \frac{a_0 b^* q}{(r, s)}.$$

Example 7. Let $\mathfrak{F} = \mathfrak{R}$ and consider the running example

$$s = \frac{\zeta(1 - 2{\cdot}5\zeta + \zeta^2)}{1 - 5\zeta + 4\zeta^2}, \quad w = \frac{1}{1 - \zeta}.$$

The closed-loop least squares control is obtained by solving the equation

$$(1 - \zeta)(1 - 4\zeta)\, x + \zeta(1 - 2\zeta)\, y = (\zeta - 2)(\zeta - 4)$$

for such $\hat{x}$ and $\hat{y}$ that $\partial\hat{x}$ = min.

The solution is seen to be

$$\hat{x} = 8 - 37\zeta,$$
$$\hat{y} = 71 - 74\zeta.$$

It follows that

$$r = \frac{71 - 74\xi}{(1 - 0{\cdot}5\zeta)(8 - 37\zeta)}$$

and further

$$e = \frac{(1 - 4\zeta)(8 - 37\zeta)}{(\zeta - 4)(\zeta - 2)},$$

$$u = \frac{(1 - 4\zeta)(71 - 74\zeta)}{(\zeta - 4)(1 - 0{\cdot}5\zeta)(\zeta - 2)}.$$

Using the algorithm (11/II) we get

$$\varphi_{\min} = \frac{377{,}621}{315}.$$

The polynomial c follows as

$$c = -0{\cdot}5(\zeta - 4)(\zeta - 2)^2.$$

Example 8. Consider again

$$s = \frac{\zeta}{1 - \zeta}, \quad w = \frac{1 - 2\zeta}{1 - \zeta}$$

over the field $\mathfrak{R}$. We seek for the closed-loop least squares control. Bearing in mind that b is stable, we set up the equation

$$(1 - \zeta)\, x + \zeta y = \zeta - 2.$$

The solution

$$\hat{x} = -2\,,$$
$$\hat{y} = -1$$

meets the requirement $\partial\hat{x} = \min$.

Thus the optimal controller becomes

$$r = \tfrac{1}{2}$$

and

$$e = \frac{2 + 4\zeta}{2 - \zeta}\,, \quad u = \frac{1 - 2\zeta}{2 - \zeta}\,,$$

$$c = \zeta - 2\,, \quad \varphi_{\min} = 4\,.$$

Example 9. Let

$$s = \zeta^2\,, \quad w = \frac{2 + 2\zeta}{2 - \zeta^2}$$

be given over $\mathfrak{R}$. We are to find the closed-loop least squares control.

The Diophantine equation

$$(2 - \zeta^2)\,x + \zeta^2 y = 2 + 2\zeta$$

yields

$$\hat{x} = 1 + \zeta\,,$$
$$\hat{y} = 1 + \zeta\,.$$

Therefore

$$s = 1 + \zeta\,,$$
$$r = (2 - \zeta^2)(1 + \zeta)$$

and hence

$$r = \frac{1}{2 - \zeta^2}\,, \quad u = \frac{1 + \zeta}{2 - \zeta^2}\,,$$

$$e = 1 + \zeta\,, \quad \varphi_{\min} = 2\,.$$

Since $(r, s) = 1 + \zeta$, the annihilating polynomial c becomes

$$c = \frac{2 + 2\zeta}{1 + \zeta} = 2\,.$$

This example is intended to illustrate that it is the $\tilde{a}_0^- \tilde{b}^- \tilde{q}^- / (\hat{x}, \hat{y})$, not $\tilde{a}_0^- \tilde{b}^- \tilde{q}^-$, that is essential for stability. Otherwise speaking, nothing can be concluded about the optimal system stability until the Diophantine equation is solved for $\hat{x}$ and $\hat{y}$.

The most important condition imposed on control systems is that of stability. Specifically, we require both $\boldsymbol{u}$ and $\boldsymbol{e}$ to be stable, that is, we require external stability. However, this is not enough as the closed-loop system is not canonical. We have to ensure internal stability so that the system may remain stable even if disturbances occur. This is materialized by synthesizing the closed-loop system so as to make its annihilating polynomial c stable. At the same time all remaining degrees of freedom are exploited to minimize an optimality criterion, as it has been done above. Alternatively, we may choose c so as to obtain a particular performance, e.g., to make the disturbances vanish in a finite time.

Consider a nonzero initial state $\mathbf{x}_0$ of the system $\boldsymbol{s}$ and a nonzero initial state $\mathbf{z}_0$ of the controller $\boldsymbol{r}$ as typical disturbances (Fig. 2).

Then

$$y = \frac{\zeta^d b}{a} u + \frac{g}{a},$$

$$u = \frac{s}{r} e + \frac{h}{r},$$

where $g, h \in \mathfrak{F}[\zeta]$ are arbitrary polynomials of degree one less than dimensions of $\boldsymbol{s}$ and $\boldsymbol{r}$ respectively. They characterize respectively x_0, the system output due to $\mathbf{x}_0$, and $\mathbf{z}_0$, the controller output due to z_0.

The effect of g and k upon the error e is then obtained as

$$e = \frac{ar}{c} w - \frac{\zeta^d b}{c} h - \frac{r}{c} g, \tag{14}$$

where

$$c = ar + \zeta^d bs.$$

Thus the disturbances are stabilized or otherwise influenced through c.

Let us try to eliminate disturbances in a finite time. If an infinite but stable control is allowed, the problem is solved when the constraint

$$c = b^+$$

is imposed on the closed-loop output time-optimal control. If a finite control is required, we have to set

$$c = 1$$

in the closed-loop state time-optimal control problem. This results from (14) and from the form of $\boldsymbol{r}$ in Theorems 2 and 3.

Quite analogically, we can demand

$$c = b^*$$

 to obtain a sort of least squares control the physical interpretation of which is left to the reader.

As expected, these strict constraints necessitate the following modifications in Theorem 2

$$ap_0x + \zeta^d b^- y = 1\,, \quad \partial\hat{x} = \min\,,$$
$$s = \hat{y}\,, \quad r = p_0 b^+ \hat{x}\,,$$
$$c = b^+\,,$$
$$e = a_0 q\hat{x} - \zeta^d b^- h - p_0\hat{x}g\,,$$
$$u = \frac{a_0 q\hat{y}}{p_0 b^+} + \frac{a}{b^+}h - \frac{\hat{y}}{b^+}g\,.$$

Similarly in Theorem 3

(15)
$$\begin{aligned} &ax + \zeta^d by = 1\,, \quad \partial\hat{x} = \min\,,\\ &s = \hat{y}, \quad r = \hat{x}\,,\\ &c = 1\,,\\ &e = a_0 q\hat{x} - \zeta^d bh - \hat{x}g\,,\\ &u = a_0 q\hat{y} + ah - \hat{y}g\,. \end{aligned}$$

Theorem 4 will be altered to yield

$$\begin{aligned} &ap_0x + \zeta^d b^- y = \tilde{b}^-\,, \quad \partial\hat{x} = \min\,,\\ &s = \hat{y}\,, \quad r = p_0 b^+ \hat{x}\,,\\ &c = b^*\,,\\ &e = \frac{a_0 q\hat{x}}{\tilde{b}^-} - \frac{\zeta^d b^-}{\tilde{b}^-}h - \frac{p_0\hat{x}}{\tilde{b}^-}g\,,\\ &u = \frac{a_0 q\hat{y}}{p_0 b^*} + \frac{a}{b^*}h - \frac{\hat{y}}{b^*}g\,. \end{aligned}$$

Example 10. Let $\mathfrak{F} = \mathfrak{R}$ and

$$s = \frac{\zeta^2}{(1-\zeta)(\zeta-2)}\,, \quad w = \frac{1}{1-\zeta}\,.$$

We are to obtain the closed-loop state time-optimal control while

(i) stabilizing disturbances, i.e., c stable.

Theorem 3 yields the equation

$$(1-\zeta)\,x + \zeta^2 y = 1$$

which gives

$$\hat{x} = 1 + \zeta ,$$
$$\hat{y} = 1 .$$

Then

$$r = \frac{\zeta - 2}{1 + \zeta}, \quad c = \zeta - 2$$

and by (14)

$$e = (1 + \zeta) - \frac{\zeta^2}{\zeta - 2} h - \frac{1 + \zeta}{\zeta - 2} g .$$

(ii) eliminating disturbances in a finite time, i.e., $c = 1$.
Now the equation

$$(1 - \zeta)(\zeta - 2) x + \zeta^2 y = 1$$

is to be solved by (15).
It is seen that

$$\hat{x} = -0{\cdot}5 - 0{\cdot}75\zeta ,$$
$$\hat{y} = 1{\cdot}75 - 0{\cdot}75\zeta ;$$

hence

$$r = \frac{1{\cdot}75 - 0{\cdot}75\zeta}{-0{\cdot}5 - 0{\cdot}75\zeta}, \quad c = 1$$

and

(16) $$e = (1 + \zeta - 0{\cdot}75\zeta^2) - \zeta^2 h - (0{\cdot}5 + 0{\cdot}75\zeta) g .$$

Thus a faster response to w has been traded for a polynomial error in the presence of disturbances.

Remark 8. It is commonly believed that prespecifying $c = 1$ in the state time-optimal control problem always leads to $k_{\min} = n$, the dimension of the system s, whenever $w = 1/(1 - \zeta)$. This is false, however, as (16) indicates.

CONCLUSIONS

This paper has completed the discussion of the algebraic theory of discrete optimal control for single-variable systems. In Part I (Preliminaries) essential mathematical concepts have been established. Part II (Open-Loop Control) has been devoted to the basic open-loop control problems and some computational aspects whereas this part has been concerned with the closed-loop controls.

One might think of closing the loop by simply feeding back the error of the open-loop control to get the closed-loop system. However, this is not acceptable. The resulting controller need not exist (e.g. $s = \zeta$, $w = \zeta$ yields $r = 0$, $s = 1$ for any criterion) or need not be causal (e.g. $s = \zeta^2$, $w = \zeta + \zeta^2$ yields $r = \zeta$, $s = 1$ for any criterion). To make the matters worse, the closed-loop system created in this way will not be stable whenever $a_0^- q_0^- \neq 1$. Thus special synthesis procedures have been developed to produce the appropriate closed-loop system directly.

The reader will have noticed that the closed-loop control problems considered here are not completely general in that they are a priori endowed with the structure of Fig. 1. However, this configuration is reasonably general and well-established.

An interesting observation is that the closed-loop control is always inferior to the corresponding open-loop control in minimizing the optimality criterion. This is a penalty for making the annihilating polynomial c stable.

Each problem has been stated with great care because seemingly identical problems can have different solutions. For example, compare minimization of all optimality criteria under three types of constraints, viz. c unrestricted, c stable, and a fixed c prespecified.

All the problems included are, in fact, classical. The setting and the method of attack is new, however. It provides a deep and inconventional insight into the problems of discrete optimal control. With this machinery at hand it is possible to treat stable as well as unstable systems, continuum-state systems as well as finite automata, closed-loop as well as open-loop controls, etc. within a unified framework of the general theory. We have arrived at many extensions of classical results, to mention the control of unstable systems, the pole assignment problem, new existence and unicity condictions, etc. As a by-product some clasical fallacies have been elucidated.

The synthesis technique proposed in this paper is not only conceptually simple, unified and transparent, but it also yields effective and uniform computational algorithms. Last but not least, the author believes that the closed-loop *stability* problem is posed and rigorously solved for the first time here.

The theory as developed in the tripaper applies to single-variable systems only. A natural generalization to multivariable systems will be considered in a future paper.

(Received June 6, 1972.)

REFERENCES

[1] S. S. L. Chang: Synthesis of Optimum Systems Control. McGraw-Hill, New York 1961.
[2] S. Kubík, Z. Kotek, M. Šalamon: Teorie regulace II: Nelineární regulace, SNTL, Praha 1969.
[3] V. Kučera: Algebraic theory of discrete optimal control for single-variable systems, Part I: Preliminaries. Kybernetika *9* (1973), 2, 94—107.
[4] V. Kučera: Algebraic theory of discrete optimal control for single-variable systems, Part II: Open-Loop Control. Kybernetika *9* (1973), 3, 206—221.
[5] V. Peterka: On steady state minimum variance control strategy. Kybernetika *8* (1972), 3, 219—232.
[6] V. Strejc et al.: Syntéza regulačních obvodů s číslicovým počítačem. NČSAV, Praha 1965.
[7] J. T. Tou: Digital and Sampled-data Control Systems. McGraw-Hill, New York 1959.

Ing. Vladimír Kučera, CSc.; Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Vyšehradská 49, 128 48 Praha 2, Czechoslovakia.