KYBERNETIKA — VOLUME 16 (1980), NUMBER 2

Optimal Control of Stabilizable Time-Varying Linear Systems with Time Delay

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The linear-quadratic problem on the infinite time interval is considered. Optimal control is derived from the smallest nonnegative continuous bounded solution of the known system of three Riccati-type equations.

In this paper we show that the optimal control of stabilizable time-varying linear-quadratic systems with time delay on the infinite time interval is given by the formula similar to the known formula for the optimal feedback control of systems on a finite time interval. The main results are contained in Theorem 1 and Theorem 2.

Theorem 1 describes the asymptotic behavior (in T) of the solution W^T of the Riccati-type system of equations in three variables (cf. [1], [2]) subject to the initial conditions $W^T(T) = 0$. The limit is the solution of the above system on the infinite interval. Theorem 2 contains the formulas for optimal control and minimal cost and a discussion of some properties of solutions of the mentioned Riccati-type system on infinite time interval. The functional of minimal cost corresponds to the smallest nonnegative bounded continuous solution. A sufficient condition for uniqueness of this solution is presented.

Consider the system described by the equation

$$\dot{x}(t) = A_0(t) \,.\, x(t) + \int_{-h}^{0} A_1(t, \tau) \,.\, x(t + \tau) \,\mathrm{d}\tau + A_2(t) \,.\, x(t - h) + B(t)\, u(t) \tag{1}$$
$$\text{for} \quad t \in \langle t_0, \infty)$$

with the initial condition

$$x(t_0 + \tau) = \varphi(\tau) \,; \quad \tau \in \langle -h, 0 \rangle$$

where

$x(t)$ is the n-dimensional state vector
$u(t)$ is the p-dimensional control function
$A_0(t)$, $A_1(t, \tau)$, $A_2(t)$, $B(t)$ are matrix coefficients of appropriate types which are bounded and continuous on their domains.

Let $Q_1(t)$ and $Q_2(t)$ are bounded continuous matrix functions with nonnegative definite and positive definite values, respectively. Our aim is to minimize the loss function

$$(2) \qquad C_{t_0}^{\infty}(u, \varphi) = \int_{t_0}^{\infty} c(t, x(t), u(t))\, \mathrm{d}t$$

where

$$(2a) \qquad c(t, x(t), u(t)) = x'(t) \,.\, Q_1(t) \,.\, x(t) + u'(t) \,.\, Q_2(t) \,.\, u(t) \,.$$

It is well known (see [1], [2]) that for any $T > s \geqq t_0$ the optimal control of the system (1) with respect to the cost function

$$(2b) \quad C_s^T(u, x) = \int_s^T c(t, x(t), u(t))\, \mathrm{d}t\,, \quad x(s + \tau) = \varphi(\tau) \quad \text{for} \quad \tau \in \langle -h, 0\rangle$$

can be written in the form

$$(3) \qquad u^T(t) = -Q_2^{-1}(t) \,.\, B'(t) \,.\, [W_0^T(t) \,.\, x^T(t) + \int_{-h}^{0} W_1^T(t, \tau) \,.\, x^T(t + \tau)\, \mathrm{d}\tau]$$

and the corresponding minimal cost can be written in the form

$$(4) \qquad C_s^T(u^T, \varphi) = \varphi'(0) \,.\, W_0^T(s) \,.\, \varphi(0) + 2\varphi'(0) \,.\, \int_{-h}^{0} W_1^T(s, \tau)\, \varphi(\tau)\, \mathrm{d}\tau +$$
$$+ \int_{-h}^{0} \int_{-h}^{0} \varphi'(\tau) \,.\, W_2^T(s, \tau, \varrho)\, \varphi(\varrho)\, \mathrm{d}\varrho\, \mathrm{d}\tau = W^T(s)\,(\varphi)$$

where the triple $W_0^T(t)$, $W_1^T(t, \tau)$, $W_2^T(t, \tau, \varrho)$ of bounded continuous matrix functions of type $n \times n$ defined for $t \in \langle t_0, T\rangle$; $\tau, \varrho \in \langle -h, 0\rangle$ is the unique solution of the Riccati-type system of equations:

$$(5.1) \qquad \frac{\mathrm{d}W_0(t)}{\mathrm{d}t} + A_0'(t) \,.\, W_0(t) + W_0(t) \,.\, A_0(t) + W_1(t, 0) + W_1'(t, 0) +$$
$$+ Q_1(t) - W_0(t) \,.\, B_1(t) \,.\, W_0(t) = 0$$

$$(5.2) \qquad \frac{\mathrm{d}W_1(t, s - t)}{\mathrm{d}t} + A_0'(t) \,.\, W_1(t, s - t) + W_0(t) \,.\, A_1(t, s - t) +$$
$$+ W_2(t, 0, s - t) - W_0(t)\, B_1(t) \,.\, W_1(t, s - t) = 0$$

$$\text{(5.3)} \qquad \frac{\mathrm{d}W_2(t, s - t, r - t)}{\mathrm{d}t} + A_1'(t, s - t) \cdot W_1(t, r - t) +$$

$$+ W_1'(t, s - t) \cdot A_1(t, r - t) - W_1'(t, s - t) \cdot B_1(t) \cdot W_1(t, r - t) = 0$$

where $s, r \in \langle t - h; t \rangle$; $B_1 = B'Q_2^{-1} \cdot B$

$$\text{(5.4)} \qquad W_1(t, -h) = W_0(t) \cdot A_2(t)$$

$$\text{(5.5)} \qquad W_2(t, -h, \tau) = A_2'(t) \cdot W_1(t, \tau)$$

$$\text{(5.6)} \qquad W_2(t, \tau, \varrho) = W_2'(t, \varrho, \tau)$$

with the initial conditions

$$\text{(6)} \qquad W_0^T(\tau) = W_1^T(T, \tau) = W_2^T(T, \tau, \varrho) = 0 .$$

We show that all the functions $W_0^T(t)$, $W_1^T(t, \tau)$ and $W_2^T(t, \tau, \varrho)$ converge (under the condition of stabilizability of the system (1)) in T to a triple of continuous functions $W_0(t)$, $W_1(t, \tau)$ and $W_2(t, \tau, \varrho)$ which is a solution of system (5) on $\langle t_0, \infty)$. Moreover, the optimal control and minimal cost are given by (3) and (4) (with T omitted).

First we introduce some formalism. For any matrix A of type $m \times n$ we consider the Euclidean norm in $R^{m.n}$.

Definition 1. For any Lebesgue measurable subset $\mathbf{M}$ of the interval $\langle -h, 0 \rangle$ we put

$$m(\mathbf{M}) = \lambda(\mathbf{M}) + \operatorname{card}(\mathbf{M} \cap \{-h, 0\})$$

and

$$m_0(\mathbf{M}) = \lambda(\mathbf{M}) + \operatorname{card}(\mathbf{M} \cap \{0\})$$

where λ is a standard Lebesgue measure on $\langle -h, 0 \rangle$.

Definition 2. a) We denote by $\mathbf{L}_1^n(m_0)$ the system of all finite n-dimensional measurable functions on $\langle -h, 0 \rangle$ satisfying the condition

$$\|\varphi\|_1 = \|\varphi(0)\| + \int_{-h}^{0} \|\varphi(\tau)\| \, \mathrm{d}\tau = \int \|\varphi(\tau)\| \, \mathrm{d}m_0(\tau) < \infty .$$

b) Let $\mathbf{QF}(m_0)$ be the system of all matrix functions of type $n \times n$ defined on the product set $\langle -h, 0 \rangle \times \langle -h, 0 \rangle$ and having the following properties:

i) $$W(\tau, \varrho) = W'(\varrho, \tau) \quad \text{for} \quad \tau, \varrho \in \langle -h, 0 \rangle .$$

ii) If we put

$$\text{(7a)} \quad W_0 = W(0, 0), \quad W_1(\tau) = W(0, \tau), \quad W_2(\tau, \varrho) = W(\tau, \varrho) \quad \text{for} \quad \tau, \varrho \in \langle -h, 0)$$

then the functions W_1 and W_2 are continuous and continuously prolongable on the sets $\langle -h, 0\rangle$ and $\langle -h, 0\rangle \times \langle -h, 0\rangle$, respectively. Hence we can put

$$W_1(0) = \lim_{\tau \to 0} W_1(\tau), \quad W_2(0, \varrho) = \lim_{\tau \to 0} W_2(\tau, \varrho). \tag{7b}$$

Definition 3. a) We say that the function

$$W : \langle t_0, t_1 \rangle \to \mathbf{QF}(m_0)$$

is continuous if all the functions

$$W_0(t), \quad W_1(t, \tau), \quad W_2(t, \tau, \varrho)$$

are continuous on their domains.

b) For $W \in \mathbf{QF}(m_0)$ and $\varphi \in \mathbf{L}_1^n(m_0)$ we define

$$W(\varphi) = \iint \varphi'(\tau) \,.\, W(\tau, \varrho) \,.\, \varphi(\varrho) \,\mathrm{d}m_0(\varrho) \,\mathrm{d}m_0(\tau)$$

c) We introduce a partial order on $\mathbf{QF}(m_0)$ by

$$W \leqq V \Leftrightarrow \forall \varphi \in \mathbf{L}_1^n(m_0) : W(\varphi) \leqq V(\varphi)$$

$W \in \mathbf{QF}(m_0)$ is said nonnegative if $0 \leqq W$.

Now we return to study the system (1) more closely. We can rewrite it in the form

$$\dot{x}(t) = \int A(t, \tau) \,.\, x_t(\tau) \,\mathrm{d}m(\tau) + B(t)\, u(t) \tag{1a}$$

where

$$A(t, \tau) = \begin{cases} A_0(t) & \text{for} \quad \tau = 0 \\ A_1(t, \tau) & \text{for} \quad \tau \in (-h, 0) \\ A_2(t) & \text{for} \quad \tau = -h \end{cases} \tag{8a}$$

and

$$x_t(\tau) = x(t + \tau). \tag{8b}$$

Lemma 1. (cf. [1], [4]). Consider the equation

$$\dot{x}(t) = \int A(t, \tau) \,.\, x_t(\tau) \,\mathrm{d}m(\tau) \tag{9}$$

with the initial condition $x_s = \varphi \in \mathbf{L}_1^n(m_0)$

Let $X(t, s)$ be the matrix solution of the equation

$$\text{(9a)} \qquad \frac{\partial X(t, s)}{\partial t} = \int_{-h}^{0} A(t, \tau)\, X(t + \tau, s)\, dm(s)$$

subject to the initial condition $X(t, t) = I$; $X(t, s) = 0$ for $t < s$.

The solution $x(t)$ of (9) can be written in the form:

$$\text{(9b)} \qquad x(t) = \int Y(t, s, \tau)\, \varphi(\tau)\, dm_0(\tau)$$

where

$$\text{(9c)} \qquad Y(t, s, \tau) = \begin{cases} X(t, s) & \text{for} \quad \tau = 0 \\ X(t, s + \tau + h) \,.\, A_2(s + \tau + h) + & \\ \quad + \int_0^{\tau + h} X(t, s + \varrho)\, A_1(s + \varrho, \tau - \varrho)\, d\varrho & \text{for} \quad \tau \in \langle -h, 0 \rangle\,. \end{cases}$$

The following quite simple result will be very useful.

Proposition 1. Consider the solution $x(t)$ of (9) with the initial condition $x_s = \varphi \in \boldsymbol{L}_1^n(m_0)$. There exists a real function $K(a, d)$ nondecreasing in both the real variables a and d such that for $t - s \leqq d$ and

$$\sup \{\|A(r, \tau)\| : r \in \langle s, t \rangle, \tau \in \langle s - h, 0 \rangle\} \leqq a$$

the inequality

$$\text{(10)} \qquad \|x(t)\| \leqq K(a, d) \,.\, \|\varphi\|_1$$

holds.

Proof. Let the matrix function N be defined by

$$N(t, s) = -A_0(t) \,.\, \theta(t - s) - \int_{t-h}^{t} A_1(t, \tau) \,.\, \theta(\tau - s)\, d\tau - A_2(t) \,.\, \theta(t - h - s)$$

where θ is the step function

$$\theta(t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leqq 0\,. \end{cases}$$

The function $X(t, s)$ is the solution of the integral equation (cf. [4])

$$X(t, s) + \int_s^t X(t, \tau) \,.\, N(\tau, s)\, d\tau = I\,.$$

We have $\|I\| = \sqrt{n}$. Using the inequality

$$\|N(\tau, s)\| \leqq (h + 2) . a = a_1$$

and the Gronwal's lemma we get

(10a) $$\|X(t, s)\| \leqq n^{1/2} . e^{a_1(t-s)} \leqq n^{1/2} . e^{a_1 d} = K_0(a, d) .$$

Substituting into (9c) and (9b) we get

(10b) $$\|Y(t, s, \tau)\| \leqq \max(1, a_1) . \max\{\|X(t, \tau)\| : \tau \in \langle s, t\rangle\} \leqq \leqq \max(1, a_1) . K_0(a, d) = K(a, d)$$

hence

$$\|x(t)\| \leqq K(a, d) . \|\varphi\|_1 .$$

Further we concern with stable or stabilizable systems.

Definition 4. a) We say that the system (9) is stable if there exists a constant K_0 such that for any $s \in \langle t_0, \infty)$ the inequality

(11) $$\int_s^\infty \|X(t, s)\|^2 \, dt = K_0$$

holds.

b) We say that the system (1) is stabilizable if there exists a pair of continuous bounded functions $L_0(t)$, $L_1(t, \tau)$; for $t \in \langle t_0, \infty)$ $\tau \in \langle -h, 0\rangle$ such that the system

(1b) $$\dot{x}(t) = \int A(t, \tau)\, x_t(\tau)\, dm(\tau) + B(t) \int L(t, \tau)\, x_t(\tau)\, dm_0(\tau)$$

is stable.

The feedback control

(12) $$u(t) = \int L(t, \tau)\, x_t(\tau)\, dm(\tau)$$

where

$$L(t, \tau) = \begin{cases} L_0(t) & \text{for } \tau = 0 \\ L_1(t, \tau) & \text{for } \tau \in \langle -h, 0\rangle \end{cases}$$

is called stabilizing.

Proposition 2. Suppose that the function $A(t, \tau)$ is bounded on $\langle t_0, \infty) \times \langle -h, 0\rangle$. The system (9) is stable if and only if there exists a constant K_1 such that for any $s \in \langle t_0, \infty)$ and any solution $x(t)$ with the initial condition $x_s = \varphi \in L_1^n(m_0)$ the inequality

$$(13) \qquad \int_s^\infty \|x(t)\|^2 \, \mathrm{d}t \leqq K_1 \|\varphi\|_1^2$$

holds.

Proof. Put

$$(8c) \qquad a = \sup \{\|A(t, \tau)\| : t \in \langle t_0, \infty); \tau \in \langle -h, 0 \rangle\}$$

$$(8d) \qquad a_1 = (h + 2)\, a\,.$$

From (10) we get

$$\int_s^{s+h} \|x(t)\|^2 \, \mathrm{d}t \leqq h \,.\, K^2(a, h) \,.\, \|\varphi\|_1^2 = K_1' \,.\, \|\varphi\|_1^2\,.$$

For $t \in \langle s + h, \infty)$ we get from (9b) and (9c)

$$x(t) = X(t, s + h) \,.\, x(s + h) + \int_s^{s+h} [X(t, \tau + h) \,.\, A_2(\tau + h) + \\ + \int_{s+h}^{\tau+h} X(t, \varrho) \,.\, A_1(\varrho, \tau - \varrho) \, \mathrm{d}\varrho] \,.\, x(\tau) \, \mathrm{d}\tau$$

hence

$$\|x\| \leqq \|\varphi\|_1 \,.\, K(a, h) \,.\, \|X(t, s + h)\| + \int_{s+h}^{s+2h} \|X(t, \varrho)\| \,.\, \left[\|A_2(\varrho)\| + \right. \\ \left. + \int_{\varrho-h}^{s+h} \|A_1(\varrho, \tau - \varrho)\| \, \mathrm{d}\tau \right] \mathrm{d}\varrho \leqq \|\varphi\|_1 \,.\, K(a, h) \,. \\ .\left[\|X(t, s + h)\| + a_1 \int_{s+h}^{s+2h} \|X(t, \varrho)\| \, \mathrm{d}\varrho \right].$$

Therefore

$$\int_{s+h}^\infty \|x(t)\|^2 \, \mathrm{d}t \leqq \|\varphi\|_1^2 \,.\, K^2(a, h) \,.\, \int_{s+h}^\infty \left[2\|X(t; s + h)\|^2 + 2a_1^2 h \,. \right. \\ \left. .\int_{s+h}^{s+2h} \|X(t, \varrho)\|^2 \, \mathrm{d}\varrho \right] \mathrm{d}t \leqq 2K^2(a, h)\,(1 + a_1^2 h^2) \,.\, K_0 \,.\, \|\varphi\|_1^2 = K_1'' \,.\, \|\varphi\|_1^2\,.$$

Hence (13) is fulfilled for

$$K_1 = K_1' + K_1''\,.$$

Proposition 3. Let the system (9) be stable and let the function $A(t, \tau)$ be bounded on $\langle t_0, \infty) \times \langle -h, 0 \rangle$. Then for any solution $x(t)$ of (9) we have

$$\lim_{t \to \infty} x(t) = 0 . \tag{13a}$$

Proof. Let $x(t)$ be a solution of (9) with the initial condition $x_s = \varphi \in \mathbf{L}_1^n(m_0)$. From (9) and (13) we deduce that there exists a positive constant D^2 such that

$$\int_{s+h}^{\infty} \|\dot{x}(t)\|^2 \, \mathrm{d}t < D^2 .$$

Suppose that (13a) is not valid. There exists $\varepsilon > 0$ and an increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $\|x(t_n)\| > \varepsilon$. Put $\Delta = \varepsilon^2/4D^2$. The sequence $\{t_n\}_{n=1}^{\infty}$ can be choosen in such a way that $s + h \leqq t_1$, $t_{n+1} > t_n + \Delta$. For $t \in \langle t_n; t_n + \Delta \rangle$ we have

$$\|x(t)\| \geqq \|x(t_n)\| - \int_{t_n}^{t} \|\dot{x}(\tau)\| \, \mathrm{d}\tau \geqq \varepsilon \text{-} \Delta^{1/2} \left[\int_{t_n}^{t} \|\dot{x}(\tau)\|^2 \, \mathrm{d}\tau \right]^{1/2} \geqq$$

$$\geqq \varepsilon - \varepsilon/2D \, . \, D = \varepsilon/2 .$$

Hence

$$\int_{t_n}^{t_n + \Delta} \|x(t)\|^2 \, \mathrm{d}t \geqq \varepsilon^2/4 \, . \, \Delta = \varepsilon^4/16D^2 .$$

Therefore

$$\int_{t_0}^{\infty} \|x(t)\|^2 \, \mathrm{d}t \geqq \sum_{n=1}^{\infty} \int_{t_n}^{t_n + \Delta} \|x(t)\|^2 \, \mathrm{d}t = \infty$$

which contradicts to (13).

Theorem 1. Suppose that the system (1) is stabilizable. The system of functions $W^T(t, \tau, \varrho)$ converges in T to the function $W(t, \tau, \varrho)$ which has the following properties:

a) $$W(t, \tau, \varrho) \in \mathbf{QF}(m_0) \quad \text{for} \quad t \in \langle t_0, \infty)$$

b) The mapping

$$W : \langle t_0, \infty) \to \mathbf{QF}(m_0)$$

is continuous.

c) The triple $W_0(t)$, $W_1(t, \tau)$, $W_2(t, \tau, \varrho)$ given by (7a), (7b) is the solution of (5) on $\langle t_0, \infty)$.

Proof. Choose a stabilizing control

$$u_0(t) = \int L^0(t, \tau) \, x(\tau) \, \mathrm{d}m_0(\tau) .$$

Suppose that

$$x_s^0 = \varphi \in \mathbf{L}_1^n(m_0) .$$

From the fact that the functions $Q_1(t)$, $Q_2(t)$ and $L^0(t, \tau)$ are bounded we obtain that there exists a constant K_2 such that

(14) $$c(t, x^0(t), u^0(t)) \leqq K_2 \,.\, \|x_t^0\|_1^2 \,.$$

Denote

$$A^0(t, \tau) = A(t, \tau) + B(t) \,.\, L^0(t, \tau) \,.$$

Let

$$a = \sup \{\|A^0(t, \tau)\| : t \in \langle t_0, \infty); \tau \in \langle -h, 0 \rangle\} \,.$$

For $t \in \langle s, s + h \rangle$ we have

$$\|x^0(t)\| \leqq K(a, d) \,.\, \|\varphi\|_1$$

and

$$\|x_t^0\|_1 = \int_{t-h}^{0} \|\varphi(\tau)\| \, d\tau + \int_0^t \|x^0(\tau)\| \, d\tau + \|x^0(t)\| \leqq$$
$$\leqq \|\varphi\|_1 + (1 + h) \,.\, K(a, h) \,.\, \|\varphi\|_1$$

when

(15a) $$\int_s^{s+h} \|x_t^0\|_1^2 \, dt \leqq K_3' \,.\, \|\varphi\|_1^2 .$$

For $t \geqq s + h$ we have

$$\|x_t\|_1^2 \leqq (1 + h) \left[\|x(t)\|^2 + \int_{-h}^{0} \|x(\tau)\|^2 \, d\tau\right].$$

Therefore

(15b) $$\int_{s+h}^{\infty} \|x_t^0\|_1^2 \, dt \leqq (1 + h)^2 \,.\, \int_s^{\infty} \|x^0(t)\|^2 \, dt \leqq$$
$$\leqq (1 + h)^2 \,.\, K_1 \,.\, \|\varphi\|_1^2 = K_3'' \,.\, \|\varphi\|_1^2 \,.$$

Combining (14) with (15a) and (15b) we get

(15) $$\int_s^{\infty} c(t, x^0(t), u^0(t)) \, dt \leqq K_3 \,.\, \|\varphi\|_1^2 .$$

For $T \geqq t_0$ we put

(16a) $$L^T(t, \tau) = -Q_2^{-1}(t) \,.\, B'(t) \,.\, W^T(t, 0, \tau)$$

and

(16b) $$A^T(t, \tau) = A(t, \tau) + B(t) \,.\, L^T(t, \tau)$$

where W^T is the solution of (5) and (6).

Let $x^T(t)$ be the solution of (1) for the control function

(16c) $$u^T(t) = \int_{-h}^{0} {}'L^T(t, \tau)\, x_t(\tau)\, \mathrm{d}m_0(\tau)$$

and the initial condition

(16d) $$x_s^T = \varphi \in \mathbf{L}_1^n(m_0)\,.$$

Then we have

$$W_{(s)}^T(\varphi) = \int_s^T c(t, x^T(t), u^T(t))\, \mathrm{d}t \leqq \int_s^T c(t, x^0(t), u^0(t))\, \mathrm{d}t \leqq K_3 \,.\, \|\varphi\|_1^2 \,.$$

Let $T_1 \leqq T_2$. We denote by x^i and u^i the functions x^{T_i}, u^{T_i}, $i = 1; 2$. We have

$$W_{(s)}^{T_1}(\varphi) = \int_s^{T_1} c(t, x^1(t), u^1(t))\, \mathrm{d}t \leqq \int_s^{T_1} c(t, x^2(t); u^2(t))\, \mathrm{d}t \leqq$$

$$\leqq \int_s^{T_2} c(t, x^2(t), u^2(t))\, \mathrm{d}t = W_{(s)}^{T_2}(\varphi)\,.$$

Thus for $t_0 \leqq s \leqq T_1 \leqq T_2$ we have the following inequalities in $\mathbf{QF}(m_0)$:

(17) $$W_{(s)}^{T_1} \leqq W_{(s)}^{T_2} \leqq K_3 \,.\, I$$

where $K_3 \,.\, I$ is the constant matrix function on $\langle -h, 0\rangle \times \langle -h, 0\rangle$.

We denote by ***In*** the class of initial functions of the types

(18a) $$\varphi_i = e_i \,.\, \chi_{\{0\}} \quad \text{for} \quad i = 1, \ldots, n$$

(18b) $$\psi_{\tau,j}^m = m \,.\, e_j \,.\, \chi_{\langle \tau - 1/m, \tau + 1/m\rangle} \quad \text{for} \quad j = 1, \ldots, n;\ \tau \in \langle -h, 0\rangle;\ m = 1, 2, \ldots$$

(18c) $$\varphi = \varphi' \pm \varphi'';\ \varphi' \text{ and } \varphi'' \text{ are of the type (18a) or (18b)}$$

where e_i is the i-th member of the standard orthonormal base in R^n and $\chi_{\mathbf{M}}$ is the characteristic function of the set $\mathbf{M}$.

Choosing suitable initial functions from the class ***In*** we derive the inequality

(17a) $$\|W^T(s, \tau, \varrho)\| \leqq K_3 \,.\, n \quad \text{for} \quad s \leqq T;\ \tau, \varrho \in \langle -h, 0\rangle\,.$$

Substituting it into (16a) and (16b) we conclude that there exists a constant α such that for any T

(17b) $$\sup\{\|A^T(t,\tau)\| : (t,\tau) \in \langle t_0, T\rangle \times \langle -h, 0\rangle\} \leqq \alpha .$$

Therefore for the solution x^T of (1) determined by (16c) and (16b) we have

(17c) $$\|x^T(t)\| \leqq K(\alpha, (t-s)) \,.\, \|\varphi\|_1 \quad \text{for} \quad t \leqq T .$$

Considering once more suitable functions of the class ***In*** (cf. [1], [2]) we get that for any given $(s, \tau, \varrho) \in \langle t_0, \infty) \times \langle -h, 0\rangle \times \langle -h, 0\rangle$ there exists a limit

(19a) $$W(s,\tau,\varrho) = \lim_{T\to\infty} W^T(s,\tau,\varrho) .$$

We show that this convergence is uniform on $\langle t_0, t_1\rangle \times \langle -h, 0\rangle \times \langle -h, 0\rangle$ for any $t_1 \in \langle t_0, \infty)$. Put $t_2 = t_1 + h$. For any $\varepsilon > 0$ there exists $T_0 > t_2$ such that for $T_0 < T_1 < T_2$ the inequality

(19b) $$\iint \|W^{T_2}(t,\tau,\varrho) - W^{T_1}(t,\tau,\varrho)\| \, \mathrm{d}m_0(\tau)\, \mathrm{d}m_0(\varrho) < \varepsilon/n \,.\, K^2(\alpha, d)$$

holds.

Put $d = t_2 - t_0$. For $i = 1, 2$ we consider the solution $x^i = x^{T_i}$ determined by (16c) and (16d). For $s \in \langle t_0, t_1\rangle$ we have

$$W^{T_i}_{(s)}(\varphi) = \min_u \left\{\int_s^{t_2} c(t, x(t), u(t))\, \mathrm{d}t + W^{T_i}_{(t_2)}(x_{t_2})\right\} =$$
$$= \int_s^{t_2} c(t, x^i(t), u^i(t))\, \mathrm{d}t + W^{T_i}_{(t_2)}(x^i_{t_2})$$

where $u^i = u^{T_i}$ is given by (16a) and (16c).
Therefore

$$0 \leqq W^{T_2}_{(s)}(\varphi) - W^{T_1}_{(s)}(\varphi) = \int_s^{t_2} c(t, x^2(t); u^2(t))\, \mathrm{d}t + W^{T_2}_{(t_2)}(x^2_{t_2}) -$$
$$- \int_s^{t_2} c(t, x^1(t), u^1(t))\, \mathrm{d}t - W^{T_1}_{(t_2)}(x^1_{t_2}) \leqq W^{T_2}_{(t_2)}(x^1_{t_2}) - W^{T_1}_{(t_2)}(x^1_{t_2}) \leqq$$
$$\leqq \iint \|x^1(t_2+\tau)\| \,.\, \|W^{T_2}(t_2,\tau,\varrho) - W^{T_1}(t_2,\tau,\varrho)\| \,.\, \|x^1(t_2+\varrho)\|\, \mathrm{d}m_0(\tau)\, \mathrm{d}m_0(\varrho) \leqq$$
$$\leqq K^2(\alpha, d) \,.\, \|\varphi\|_1^2 \,.\, \varepsilon/(n \,.\, K^2(\alpha, d)) = \varepsilon/n \,.\, \|\varphi\|_1^2 .$$

Choosing suitable initial functions from ***In*** we derive

$$\|W^{T_2}(s,\tau,\varrho) - W^{T_1}(s,\tau,\varrho)\| \leqq \varepsilon$$

uniformly on

$$\langle t_0, t_1 \rangle \times \langle -h, 0 \rangle \times \langle -h, 0 \rangle .$$

Thus the components $W_0(t)$, $W_1(t, \tau)$, $W_2(t, \tau, \varrho)$ of the limit function are continuous on their domains.
For a given t and $T \geqq t + h$ the triple $W_0^T(t)$, $W_1^T(t, \tau)$, $W_2^T(t, \tau, \varrho)$ is the solution of the system of integral equations which we obtain by the integration of the system (5):

$$\text{(20a)} \quad W_0(t) = W_0(t + h) + \int_t^{t+h} [A_0'(s) \,.\, W_0(s) + W_0(s)\, A_0(s) + W_1(s, 0) + \\ + W_1'(s, 0) - W_0'(s) \,.\, B_1 \,.\, W_0(s) + Q_1(s)]\, \mathrm{d}s$$

$$\text{(20b)} \quad W_1(t, \tau) = W_0(t + \tau + h) \,.\, A_2(t + \tau + h) + \\ + \int_{-h}^{\tau} [W_0(t + \tau - \varrho) \,.\, A_1(t + \tau - \varrho; \varrho) + A_0'(t + \tau - \varrho) \,.\, W_1(t + \tau - \varrho; \varrho) + \\ + W_2(t + \tau - \varrho; 0; \varrho) - W_0(t + \tau - \varrho) \,.\, B_1(t + \tau - \varrho) \,.\, W_1(t + \tau - \varrho; \varrho)]\, \mathrm{d}\varrho$$

$$\text{(20c)} \quad W_2(t, \tau, \varrho) = A_2'(t + \tau + h) \,.\, W_1'(t + \tau + h; \varrho - \tau + h) + \\ + \int_{-h}^{\tau} [A_1'(t + \tau - \xi, \xi) \,.\, W_1(t + \tau - \xi, \varrho - \tau + \xi) + \\ + W_1'(t + \tau - \xi, \xi) \,.\, A_1(t + \tau - \xi, \varrho - \tau + \xi) - \\ - W_1'(t + \tau - \xi, \xi) \,.\, B_1(t + \tau - \xi) \,.\, W_1(t + \tau - \xi, \varrho - \tau + \xi)]\, \mathrm{d}\xi \\ \text{for} \quad -h \leqq \tau \leqq \varrho \leqq 0$$

and

$$\text{(20d)} \quad W_2(t, \tau, \varrho) = W_2'(t, \varrho, \tau) \quad \text{for} \quad -h \leqq \varrho \leqq \tau \leqq 0 .$$

Taking the limits with respect to T we obtain that the triple $W_0(t)$, $W_1(t, \tau)$, $W_2(t, \tau, \varrho)$ is the solution of (20) and of (5) as well.

Theorem 2. Assume that the system (1) is stabilizable and W is the function constructed above. Then the following statements hold.

a) The control

$$\text{(21)} \quad u^*(t) = \int_{-h}^{0} L^*(t, \tau)\, x_t^*(\tau)\, \mathrm{d}m_0(\tau)$$

where

(21a) $$L^*(t, \tau) = -B'(t) \,.\, Q_2^{-1}(t) \,.\, W(t, 0, \tau)$$

is the optimal control for (1) and the value of minimal cost is given by

(22) $$C_s^\infty(u, \varphi) = W_s(\varphi)\,.$$

b) The function W is the smallest nonnegative bounded continuous solution of (5) on $\langle t_0, \infty)$ (in view of Definition 3).

c) Suppose that V is any nonnegative continuous solution of (5) on $\langle t_0, \infty)$. Then for any stabilizing feedback control

$$u(t) = \int_{-h}^{0} L(t, \tau) \,.\, x_t(\tau) \, \mathrm{d}m_0(\tau)$$

the inequality

(23) $$C_s^\infty(u, \varphi) \geqq V_{(s)}(\varphi)$$

holds for any $s \in \langle t_0, \infty)$ and $\varphi \in \boldsymbol{L}_1^n(m_0)$.

d) Suppose that there exists a constant $\delta > 0$ such that for any $t \in \langle t_0, \infty)$ and $x \in R^n$ the inequality

(24) $$x' \,.\, Q \,.\, x \geqq \delta \,.\, \|x\|^2$$

holds. Then $W(t)$ is the only nonnegative bounded continuous solution of the system (5) on $\langle t_0, \infty)$.

Proof. Let $t_0 \leqq s \leqq T < \infty$. Suppose that $V(t)$ is the continuous nonnegative solution of (5) and that $x(t)$ is a solution of (1) for a control function $u(t)$ and initial condition $x_s = \varphi \in \boldsymbol{L}_1^n(m_0)$. Calculating as in [1] or [2] we get

(25) $$c(t, x(t), u(t)) + \frac{\mathrm{d}[V_{(t)}(x_t)]}{\mathrm{d}t} = \left[u(t) - \int_{-h}^{0} U(t, \tau)\, x_t(\tau)\, \mathrm{d}m_0(\tau)\right]' \,.$$

$$.\, Q_2(t) \left[u(t) - \int_{-h}^{0} U(t, \tau)\, x_t(\tau)\, \mathrm{d}m_0(\tau)\right] \geqq 0$$

where

$$U(t, \tau) = -B'(t) \,.\, Q_2^{-1}(t) \,.\, V(t, 0, \tau)$$

hence

(26) $$C_s^T(u, x) \geqq V_{(s)}(\varphi) - V_{(T)}(\varphi)\,.$$

If we suppose that $V(t)$ is bounded and u is a stabilizing feedback control we get that

$$\lim_{t \to \infty} x(t) = 0\,, \quad \lim_{T \to \infty} V_{(T)}(x_T) = 0\,.$$

 Hence the statement c) is proved.

Now we prove a). Let $x(t)$ be a solution of (1) for some control function u and initial condition $x_s = \varphi \in \mathbf{L}_1^n(m_0)$ For any $T \geqq s$ we have

$$C_s^T(u, x) \geqq W_{(s)}^T(\varphi)$$

hence

$$C_s^\infty(u, x) \geqq W_{(s)}(\varphi) .$$

From (25) we get

$$C_s^T(u^*, \varphi) = W_{(s)}(\varphi) - W_{(T)}(\varphi) \leqq W_{(s)}(\varphi)$$

and (22) is fulfiled.

b) For

$$u(t) = \int_{-h}^{0} U(t, \tau)\, x_t(\tau)\, \mathrm{d}m_0(\tau) ,$$

where U is as above, we have from (25)

$$C_s^T(u, \varphi) = V_{(s)}(\varphi) - V_{(T)}(x_T) \leqq V_{(s)}(\varphi) .$$

But

$$W_{(s)}^T(\varphi) \leqq C_s^T(u, \varphi) \leqq V_{(s)}(\varphi) .$$

Therefore

$$W_{(s)} \leqq V_{(s)} . \tag{27}$$

d) We show that (24) implies that the control $u^*(t)$ is stabilizable. For the solution $x^*(t)$ with $x_s^* = \varphi$ we have (making use of (18))

$$\int_s^\infty \|x^*(t)\|^2\, \mathrm{d}t \leqq 1/\delta \int_s^\infty x^{*\prime}(t) \,.\, Q_1(t) \,.\, x^*(t)\, \mathrm{d}t \leqq$$

$$\leqq 1/\delta \,.\, \int_s^\infty c(t, x^*(t), u^*(t))\, \mathrm{d}t = 1/\delta \,.\, W_{(s)}(x) \leqq 1/\delta \,.\, K_3 \,.\, \|\varphi\|_1^2 \,.$$

According (23)

$$W_{(s)}(x) = C_s^\infty(u^*, x) \geqq V_{(s)}(\varphi)$$

Combining with (27) we get $V_{(s)} = W_{(s)}$.

Remark. a) If all the functions $A(t, \tau)$, $B(t)$, $Q_1(t)$, $Q_2(t)$ are periodic in t with the same period d the functions $W^T(t)$ fulfil the equations

$$W^{T+d}(t + d) = W^T(t) .$$

Therefore the functions $W(t, \tau, \varrho)$ and $L^*(t, \tau)$ are periodic in t with the period d

b) If all the functions A, B, Q_1, Q_2 are constant in t then $W(t, \tau, \varrho)$ and $L^*(t,\tau)$ are constant in t. The function $W(\tau, \varrho) : \tau, \varrho \in \langle -h, 0\rangle$ is the solution of the simplified system

$$A_0' \,.\, W_0 + W_0 \,.\, A_0 + W_1(0) + W_1'(0) - W_0 \,.\, B_1 \,.\, W_0 + Q_1 = 0 \tag{28a}$$

$$\frac{\mathrm{d}W_1(\tau)}{\mathrm{d}\tau} = W_0 \,.\, A_1(\tau) + A_0 \,.\, W_1(\tau) + W_2(0, \tau) - W_0 \,.\, B_1 \,.\, W_1(\tau) \tag{28b}$$

$$\frac{\mathrm{d}W_2(\tau, d + \tau)}{\mathrm{d}\tau} = A_1'(\tau) \,.\, W_1(d + \tau) + W_1'(\tau) \,.\, A_1(d + \tau) - \tag{28c}$$

$$- W_1'(\tau) \,.\, B_1 \,.\, W_1(d + \tau) \quad \text{for} \quad -h \leqq \tau \leqq \tau + d \leqq 0$$

$$W_1(-h) = W_0 \,.\, A_2 \tag{28d}$$

$$W_2(-h, \tau) = A_2' \,.\, W_1(\tau) \tag{28e}$$

$$W_2(\tau, \varrho) = W_2'(\varrho, \tau)\,. \tag{28f}$$

The function W can be obtained in the form

$$W(\tau, \varrho) = \lim_{t \to -\infty} V(t, \tau, \varrho) \tag{29}$$

where V is the solution of the system (5) on $\langle t_0, \infty)$ with the initial condition $V(0) = 0$.

(Received May 5, 1979.)

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