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OPTIMAL CONTROL OF A BATCH SERVICE QUEUEING SYSTEM WITH BOUNDED WAITING TIME

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The following batch service queueing system is considered. Customers arrive according to a Poisson process with given rate and are served in batches by a single server with no capacity constraints. Costs are charged for serving the customers and for holding them in the queue. Each time of service is subject to control and its choice is restricted by the requirement that customers waiting times cannot exceed a given constant T. Viewing the system as a Markov decision process with unbounded costs, we show that the policy which minimizes the expected average cost per unit time over an infinite time horizon is of the following form:
when the system is in state (i, s), $s < T$ (i is the number of customers waiting in the queue and s is the length of time since the first arrival in a given cycle), serve if and only if the server is free and i is at least as large as some control limit $i^*(T)$ and whenever the system reaches state (i, T) a service commences immediately for any positive integer i. The formula for the expected average cost in the class of bounded control limit policies is obtained in the explicit analytical form, its properties are examined and an inequality for finding the optimal control limit is derived in the general case.

1. INTRODUCTION

Suppose customers arrive according to a Poisson process with a given rate $\lambda > 0$ and are served in batches by a single server with no capacity constraints. The service times are independent random variables with an identical distribution $B(t) = \mathsf{P}(B < t)$ which is independent of batch size. The waiting time of any customer is bounded and cannot exceed a given constant $T > 0$; we assume that $B(T) = 1$. The following cost structure is considered. The waiting cost of i customers in the queue per unit time is $w(i)$ where $w(\cdot)$ is a nonnegative real-valued function. The cost of serving i customers is $K + ci$ where $K > 0$ and c is any real constant. This cost is charged at the beginning of a service. The foregoing model with no restrictions on waiting times was examined e.g. in [2], [5] and [7]. Obviously, it is the limiting case of our model for $T \to +\infty$. Deb and Serfozo [2] proved that in this case the optimal operating policy is a control limit policy, (i.e., a service commences if and only if the server is free and the number of customers waiting in the queue is at least as large as some control limit).

Although the systems governed by control limit policies without any guarantee on waiting times may work well in case the input intensities are high, this operating doctrine is not a very attractive policy for systems with low input intensities. Weiss [8] analysed a two terminal shuttle system and concluded that if the expected queue length under a control limit policy is large then scheduled service may be preferable even though it is not optimal. The same holds for one terminal systems. On the other hand, there are usually some restrictions on customers waiting times in most real systems, such as transportation of perishable items, transhipment of mail and military supplies, charter airline flights and shuttle buses, (other application areas can be found in [2]). Batch service queueing systems with zero service times and bounded waiting times, which are of interest by themselves, were treated in [4].

Let 1 and 2 denote the decision "do not serve" and "serve a batch consisting of all customers waiting in the queue," respectively. The system is reviewed continuously in time and the state of the system is a pair (i, s) where $i \in I = \{1, 2, \ldots\}$, is the number of customers waiting in the queue and s is the length of time since the first arrival after the last clearing. Obviously, $s \leqq T$ and whenever the system is in state (i, T) for any $i \in I$, a service commences immediately. The next service can be initiated only after the previous one was completed and if there is at least one customer waiting in the queue. Let π be an arbitrary control policy such that $\pi(i, T) = 2$ for all $i \in I$ and let $C(\pi; i, s; t)$ denote the total expected cost incurred up to time t following policy π when the system starts with initial state (i, s). The expected average cost over infinite time horizon associated with policy π is

$$\Phi(\pi; i, s) = \limsup_{t \to +\infty} t^{-1} C(\pi; i, s; t) . \tag{1.1}$$

Our objective is to find the average cost optimal policy, i.e., such a policy π^* for which

$$\Phi(\pi^*; i, s) = \inf_{\pi} \Phi(\pi; i, s) \quad \text{for all} \quad (i, s) . \tag{1.2}$$

The optimal operating policy is presented in Theorem 3.2 and is of the following form: when the system is in state (i, s), $s < T$, a service begins if and only if the server is free and i is at least as large as some value $i^*(T)$ which does not depend on s. The optimal value $i^*(T)$ can be easily computed using the result presented in Theorem 4.2. Such a simple rule is ideally suited for practical applications and can be used instead of the more traditional policy of scheduled periodic service. The resulting saving will be substantial especially for the systems with small input intensities.

2. PRELIMINARIES

First assume that the system is reviewed at times $\eta_1 + (j + j_0) d$ where $d > 0$ is such that $T = nd$ for some positive integer n, $j + j_0 \leqq n$, $j_0 = 0$ on $\{\eta_1 \geqq B\}$, $j_0 = \min \{i: \eta_1 + (i - 1) d < B \leqq \eta_1 + id\}$ on $\{\eta_1 < B\}$ where η_1 is the first arrival time in a given cycle and B is the appropriate service time. We say that a cycle is completed each time a service begins. The state of the system is a pair (i, k) where i is the number of customers waiting in the queue and k is the number of steps since the first arrival in a given cycle. Let $\{X_y, y \geqq 0\}$ denote the input Poisson process. In order to insure the finitness of the cost incurred in a cycle, suppose that

$$\mathsf{E}\left(\int_0^T w(X_y + i)\,\mathrm{d}y\right) < +\infty \quad \text{for all} \quad i \in I\,, \tag{2.1}$$

which is satisfied for any polynomial or exponential function $w(\cdot)$. Now consider an equivalent average cost criterion (cf. [9], [6], p. 159) of minimizing

$$\tilde{\Phi}_d(\pi; i, k) = \limsup_{m \to +\infty} \mathsf{E}_\pi\left(\sum_{j=0}^{m} Z_j \mid (i, k)\right) / \mathsf{E}_\pi\left(\sum_{j=0}^{m} \tau_j \mid (i, k)\right) \tag{2.2}$$

where Z_j is the cost incurred during the jth transition interval and τ_j is the length of this interval. It follows from renewal theory that for any stationary policy f, the associated expected average cost is independent of the initial state. The optimal average cost policy belongs to the class of stationary policies and can be obtained from the average cost optimality equation which is in general of the following form (cf. [3])

$$v(x) = \min_{a \in A(x)} \{z(x, a) - g\tau(x, a) + \sum_{r \in \mathscr{X}} p_{x,r}(a)\, v(r)\} \quad \text{for all} \quad x \in \mathscr{X} \tag{$*$}$$

where $\mathscr{X}$ is a countable state space and $A(x)$ for $x \in \mathscr{X}$ denotes the set of actions available in state x. If the system is in state x and action $a \in A(x)$ is taken then the next state of the system is chosen according to the transition probabilities $\{p_{x,r}(a), r \in \mathscr{X}\}$, $\tau(x, a)$ is the expected length of a transition interval when action a is taken in state x and $z(x, a)$ is the expected cost incurred during such a transition interval. Let $\{g, v(x), x \in \mathscr{X}\}$ be any finite solution to the average cost optimality equation $(*)$ and let $f^*(\cdot)$ be such a stationary policy that the action $f^*(x)$ minimizes the right side of $(*)$ for all $x \in \mathscr{X}$. Then, under certain assumptions (cf. [3]), $f^*(\cdot)$ is the optimal average cost policy and g is the optimal average expected cost.

Next, we find the form of the optimality equation for our model. Denote

$$q_m(y) = (\lambda y)^m\, \mathrm{e}^{-\lambda y}/m!\,, \quad p_m(T) = \int_0^T q_m(y)\,\mathrm{d}B(y) \quad \text{for} \quad m = 0, 1, \ldots \tag{2.3}$$

$$g_d = \inf_\pi \tilde{\Phi}_d(\pi; i, k) \tag{2.4}$$

and let $\tau(i, k; a)$ be the expected length of the next transition interval when action a is taken in state (i, k). We have

$$\tau(i, k; 1) = d \quad \text{for} \quad k < n \tag{2.5}$$

$$\tau(i, k; 2) = 1/\lambda + d\sum_{j=1}^{n} j\mathsf{P}(\eta_1 + (j-1)\,d < B \leqq \eta_1 + jd) \equiv \tag{2.6}$$

$$\equiv 1/\lambda + d\sum_{j=1}^{n} j\psi_j \quad \text{for} \quad k \leqq n\,.$$

Obviously, for our model $A(i, k) = \{1, 2\}$ for $k < n$, $A(i, n) = \{2\}$ and the transition probabilities $\{p_{x,r}(a)\}$ are given by

$$p_{(i,k),(i+j,k+1)}(1) = q_j(d) \quad \text{for} \quad k < n$$
$$p_{(i,k),(1,0)}(2) \quad = p_0(T)$$
$$p_{(i,k),(j+1,m)}(2) \quad = \psi_m \, q_j(md) \quad \text{for} \quad j \geqq 0,\ 1 \leqq m \leqq n\,.$$

The expected cost $z(x, a)$ incurred during a transition interval is

$$z(i, k; 1) = \mathsf{E}\left(\int_0^d w(X_u + i)\, \mathrm{d}u\right)$$
$$z(i, k; 2) = K + ci + w(0)\, \mathsf{E}\eta_1 + \sum_{j=1}^{n} \mathsf{E}\left(\int_0^{jd} w(X_u + 1)\, \mathrm{d}u\right) \psi_j\,.$$

Hence, the average cost optimality equation is of the following form

$$(2.7) \qquad v_d(i, n) = K + ci + w(0)/\lambda + \sum_{j=1}^{n} W(jd, 1)\, \psi_j - g_d \tau(i, n; 2) + $$
$$+ \, p_0(T)\, v_d(1, 0) + \sum_{j=0}^{+\infty} \sum_{m=1}^{n} q_j(md)\, \psi_m\, v_d(j + 1, m)\,,$$

$$(2.8) \qquad v_d(i, k) = \min\left\{W(d, i) - g_d d + \sum_{j=0}^{+\infty} q_j(d)\, v_d(i + j, k + 1)\,,\right.$$
$$\left. v_d(i, n)\right\} \quad \text{for} \quad k < n$$

where

$$(2.9) \qquad W(h, i) = \mathsf{E}\left(\int_0^h w(X_u + i)\, \mathrm{d}u\right) = \sum_{j=1}^{+\infty} q_j(h) \sum_{m=0}^{j-1} w(m + i)/\lambda$$

and $\tau(i, k; a)$ is given by (2.5) and (2.6). Whenever both terms on the right of (2.8) are equal, we write $v_d(i, k) = v_d(i, n)$.

In the next section, we examine the average cost optimality equation and find the structure of optimal policies.

3. OPTIMALITY OF BOUNDED CONTROL LIMIT RULES

We examine the structure of optimal policies analysing the optimality equation (2.7)–(2.8) for two important cases:

(a) $w(i) = w$

(b) $w(i + 1) - w(i) \geqq \gamma > 0$

for all $i = 0, 1, 2, \ldots$.

Theorem 3.1. Suppose condition (a) is satisfied. Then the optimal average cost policy is of the following form:

$$(3.1) \qquad f^*(i, s) = 1 \quad \text{for} \quad s < T$$
$$f^*(i, T) = 2$$

for all $i \in I$, i.e., serve if and only if the system reaches state (i, T). The optimal expected average cost is given by

$$g^*(T) = w + \lambda c + \lambda K/(1 + \lambda T) . \tag{3.2}$$

Proof. For $w(i) = w$ and for any $h > 0$, we have from (2.9) $W(h, i) = hw$. Next, (2.6) and (2.7) yield for both cases (a) and (b) $v_d(i + j, n) = v_d(i, n) + cj$ and from (2.8) we get

$$\begin{aligned} v_d(i, n-1) &= \min \{W(d, i) - g_d d + c \sum_{j=0}^{+\infty} j\, q_j(d) + v_d(i, n), v_d(i, n)\} \\ &= \min \{wd - g_d d + \lambda cd + v_d(i, n), v_d(i, n)\} . \end{aligned} \tag{3.3}$$

Obviously, for any $d > 0$, the optimal expected average cost per unit time is of the form (see Ross [6], Theorem 3.16)

$$\begin{aligned} g_d &= \left(w(0)/\lambda + \mathsf{E}\left(\int_0^{T^*_d} w(X_u + 1)\, du\right) + c\mathsf{E}(X_{T^*_d} + 1) + K\right)\Big/(1/\lambda + \mathsf{E}(T_d^*)) \\ &= w + \lambda c + \lambda K/(1 + \lambda \mathsf{E}(T_d^*)) \end{aligned} \tag{3.4}$$

where random variable T_d^* with values in $\{0, d, 2d, \ldots, nd\}$ satisfies $T_d^* + \eta_1 \geqq B$, η_1 is the first arrival time in a given cycle and B is the length of the previous service time. Hence, $g_d > w + \lambda c$ for any $d > 0$ and $v_d(i, n-1) < v_d(i, n)$ for all $i \in I$. Next, (2.8) yields for any $k < n$ and any $d > 0$, $v_d(i, k) \leqq v_d(i, n-1) < v_d(i, n)$ so that (3.1) holds and (3.2) follows from (3.1) and (3.4). □

The optimal operating policy for increasing waiting cost $w(\cdot)$ is presented in the following theorem.

Theorem 3.2. Let (b) hold. Then there is positive integer $i^*(T)$ such that the following is the optimal average cost policy: when the system is in state (i, s), $s < T$ serve if and only if the server is free and $i \geqq i^*(T)$ and whenever the system reaches state (i, T) for any $i \in I$, a service begins immediately. The optimal control limit $i^*(T)$ is determined by

$$i^*(T) = \inf \{i \in I: w(i) + \lambda c \geqq g^*(T)\} \tag{3.5}$$

where $g^*(T)$ is the optimal expected average cost.

Proof. Define for any $d > 0$, such that $T = nd$ for some positive integer n

$$i_d^* = \inf \{i \in I: W(d, i)/d + \lambda c \geqq g_d\} \tag{3.6}$$

where $W(d, i)$ and g_d are given by (2.9) and (2.4), respectively. We first prove that i_d^* is finite. Suppose that this assertion is false. Then for all $i \in I$

$$W(d, i)/d + \lambda c < g_d . \tag{3.7}$$

From the first equality in (3.4) and from (2.1), we get

$$g_d \leqq w(0) + \lambda \left\{K + c + \mathsf{E}\left(\int_0^T w(X_u + 1)\, du\right)\right\} < +\infty . \tag{3.8}$$

On the other hand, (2.9) and condition (b) yield

$$W(d, i)/d \geqq \sum_{j=0}^{+\infty} j\, q_j(d)\, w(i)/(\lambda d) \geqq i\gamma . \tag{3.9}$$

But (3.8) and (3.9) contradict with (3.7) so that i_d^* is finite. Next, it follows from condition (b) that $W(d, i)$ is nondecreasing in i and this together with the first equality in (3.3) yield for $k = n - 1$

$$\begin{aligned} v_d(i, k) &< v_d(i, n) \quad \text{for} \quad i < i_d^* \\ &= v_d(i, n) \quad \text{for} \quad i \geqq i_d^* . \end{aligned} \tag{3.10}$$

We prove that (3.10) holds for any $k < n$. Let this assertion hold for $k = m, m + 1, \ldots, n - 1$ for some positive integer $m < n$. Then we have from (2.8) and (3.6) for any $i \geqq i_d^*$

$$v_d(i, m - 1) = \min \{W(d, i) - g_d d + \lambda c d + v_d(i, n), v_d(i, n)\} = v_d(i, n)$$

and for any $i < i_d^*$

$$v_d(i, m - 1) \leqq W(d, i) - g_d d + \lambda c d + v_d(i, n) < v_d(i, n)$$

so that (3.10) holds also for $k = m - 1$. Finally from (3.6), (3.10) and from $\lim_{d \to 0+} W(d, i)/d = w(i)$, we get the desired result. □

In the last section, we examine the properties of the expected average cost per unit time in the class of bounded control limit policies and derive an inequality for finding the optimal control limit.

4. COMPUTATION OF THE OPTIMAL CONTROL LIMIT

We begin with finding the expected length of a cycle when the system is operated under a bounded control limit policy f_i, $i \in I$ defined as follows:
when the system is in state (j, s), $s < T$, serve if and only if the server is free and $j \geqq i$ and whenever the system is in state (j, T) a service commences immediately for any $j \in I$. We prove the following lemma.

Lemma 4.1. Suppose the system is operated under policy f_i for some $i \in I$. Then the expected length of a cycle is given by

$$T_i = \lambda^{-1}\{1 + \sum_{j=0}^{i-1} j\, q_j(T) + \sum_{j=i}^{+\infty} [(i - 1)\, q_j(T) + (j - i)\, p_j(T)]\} \tag{4.1}$$

where $q_j(T)$ and $p_j(T)$ are defined by (2.3).

Proof. Let η_k denote the kth arrival time in a given cycle, B be the appropriate service time and $\eta'_k \equiv \eta_{k+1} - \eta_1$ for $k = 0, 1, 2, \ldots$. Obviously, the length of a cycle under policy f_i is given by

$$(4.2) \qquad \tau = \max\{\min\{\eta_i, \eta_1 + T\}, B\}.$$

Conditioning on the number of arrivals X_T in the time interval $(\eta_1, \eta_1 + T\rangle$ yields

$$(4.3) \qquad \mathsf{E}_{f_i}(\tau) = 1/\lambda + T\sum_{j=0}^{i-2} q_j(T) + \sum_{j=i-1}^{+\infty} q_j(T)\,\mathsf{E}(\eta'_{i-1} \mid X_T = j) + \\ + \mathsf{E}(B - \eta_i \mid \eta_i < B)\,\mathsf{P}(\eta_i < B).$$

Given that $X_T = k$, the arrival times $\{\eta'_j, j \leqq k\}$ have the same distribution as the order statistics corresponding to k independent random variables uniformly distributed on the interval $\langle 0, T\rangle$ (Theorem 2.3 in [6]). From this, using formula (2.1.6) in [1], we have for $k \geqq i - 1$

$$(4.4) \qquad \mathsf{E}(\eta'_{i-1} \mid X_T = k) = \frac{T^{-k}}{\tilde{B}(i-1, k-i+2)}\int_0^T s^{i-1}(T-s)^{k-i+1}\,\mathrm{d}s = \\ = \frac{T\tilde{B}(i, k-i+2)}{\tilde{B}(i-1, k-i+2)} = \frac{(i-1)\,T}{k+1}$$

where $\tilde{B}(\cdot, \cdot)$ is beta function. Finally

$$(4.5) \qquad \mathsf{E}(B - \eta_i \mid \eta_i < B)\,\mathsf{P}(\eta_i < B) = \int_0^T\int_0^v (v-u)\,\lambda\,q_{i-1}(u)\,\mathrm{d}u\,\mathrm{d}B(v) = \\ = \frac{1}{\lambda}\{\sum_{j=i+1}^{+\infty} j\,p_j(T) - i\sum_{j=i+1}^{+\infty} p_j(T)\} = \frac{1}{\lambda}\sum_{j=i}^{+\infty}(j-i)\,p_j(T)$$

and (4.1) follows from (4.3)–(4.5). □

In the next theorem, we derive the expression for the long run expected average cost per unit time in the explicit analytical form.

Theorem 4.1. Under any f_i policy, $i \in I$ the long run expected average cost per unit time is of the form

$$(4.6) \qquad g(i, T) = \{\lambda K + \sum_{j=0}^{i-1} w(j)\sum_{k=j}^{+\infty} q_k(T) + \sum_{j=i}^{+\infty} w(j)\sum_{k=j+1}^{+\infty} p_k(T)\}/(\lambda T_i) + \lambda c$$

where T_i is given by (4.1).

Proof. It follows from renewal theory that under any stationary policy f the expected average cost is equal to the expected cost incurred in one cycle divided by the expected length of a cycle. In order to find the form of $g(i, T)$, it suffices to evaluate the expected waiting cost $\mathsf{E}_{f_i}(W)$ incurred in one cycle and use (4.1). The same notation will be used as in Lemma 4.1. Conditioning on the number of arrivals X_T

in time interval $(\eta_1, \eta_1 + T\rangle$ yields for any $i \in I$

$$(4.7) \qquad \mathsf{E}_{f_i}(W) = w(0)/\lambda + \sum_{k=i-1}^{+\infty} \mathsf{E}\left(\int_0^{\eta'_{i-1}} w(X_y + 1)\,\mathrm{d}y \mid X_T = k\right) q_k(T) + \\ + \sum_{k=0}^{i-2} q_k(T)\,\mathsf{E}\left(\int_0^T w(X_y + 1)\,\mathrm{d}y \mid X_T = k\right) + \\ + \mathsf{E}\left(\int_0^{B-\eta_i} w(X_y + i)\,\mathrm{d}y \mid \eta_i < B\right) \mathsf{P}(\eta_i < B).$$

Next, we have for any $k \geqq i - 1$

$$(4.8) \quad \mathsf{E}\left(\int_0^{\eta'_{i-1}} w(X_y + 1)\,\mathrm{d}y \mid X_T = k\right) = \sum_{j=0}^{i-2} w(j + 1)\,\mathsf{E}(\eta'_{j+1} - \eta'_j \mid X_T = k) = \\ = \frac{T}{k+1} \sum_{j=1}^{i-1} w(j),$$

where the second equality follows from (4.4). Similarly,

$$(4.9) \qquad \mathsf{E}\left(\int_0^T w(X_y + 1)\,\mathrm{d}y \mid X_T = k\right) = \frac{T}{k+1} \sum_{j=1}^{k+1} w(j).$$

For the last term on the right of (4.7), we get

$$(4.10) \qquad \mathsf{E}\left(\int_0^{B-\eta_i} w(X_y + i)\,\mathrm{d}y \mid \eta_i < B\right) \mathsf{P}(\eta_i < B) = \\ = \int_0^T \int_0^v \mathsf{E}\left(\int_0^{v-u} w(X_y + i)\,\mathrm{d}y\right) \lambda q_{i-1}(u)\,\mathrm{d}u\,\mathrm{d}B(v) = \\ = \sum_{j=0}^{+\infty} w(i + j) \sum_{k=j+1}^{+\infty} \int_0^T \int_0^v q_k(v - u)\, q_{i-1}(u)\,\mathrm{d}u\,\mathrm{d}B(v) = \\ = \sum_{j=0}^{+\infty} \frac{w(i + j)}{\lambda} \sum_{k=j+1}^{+\infty} p_{k+i}(T) = \sum_{j=i}^{+\infty} \frac{w(j)}{\lambda} \sum_{k=j+1}^{+\infty} p_k(T)$$

and (4.6) follows from Lemma 4.1 and from (4.7)–(4.10) after some algebraic manipulations. □

Denote

$$(4.11) \qquad b = \mathsf{E}(B), \quad p_k = \lim_{T \to +\infty} p_k(T).$$

Corollary 4.1. For any $i \in I$, $\lim_{T \to +\infty} g(i, T)$ exists and is equal to

$$(4.12) \quad g(i) = \{\lambda K + \sum_{k=1}^{+\infty} p_k \sum_{j=0}^{k-1} w(j) + \sum_{k=0}^{i-1} p_k \sum_{j=0}^{i-k-1} w(j + k)\} / \{\lambda b + \sum_{j=0}^{i-1} (i - j)\, p_j\} + \lambda c$$

which is the formula obtained in [5].

Proof. From (2.3) and (4.11), we get

$$\sum_{j=0}^{+\infty} p_j = 1\,, \quad \sum_{j=0}^{+\infty} j p_j = \lambda b\,, \quad \lim_{T \to +\infty} q_k(T) = 0 \tag{4.13}$$

and (4.12) follows from (4.1), (4.6) and (4.13) after some algebra. □

In the following lemma, we examine the properties of the expected average cost in the class of bounded control limit policies $\{f_i, i \in I\}$.

Lemma 4.2. Let condition (b) hold. Then for any $T > 0$ $-g(i, T)$ is unimodal in i with mode $i^*(T)$ defined by (3.5), i.e., $g(i + 1, T) < g(i, T)$ for all $i < i^*(T)$ and $g(i + 1, T) \geqq g(i, T)$ for all $i \geqq i^*(T)$.

Proof. From (4.1), we have after some algebraic manipulations

$$T_{i+1} = T_i + \lambda^{-1} H_i \tag{4.14}$$

and (4.6) and (4.14) yield

$$g(i + 1, T) - g(i, T) = H_i\{w(i) + \lambda c - g(i, T)\}/(\lambda T_{i+1}) \tag{4.15}$$

where

$$H_i = \sum_{j=i}^{+\infty} q_j(T) - \sum_{j=i+1}^{+\infty} p_j(T)\,.$$

From (3.5), we have for all $i < i^*(T)$

$$w(i) + \lambda c < g^*(T) \leqq g(i, T) \tag{4.16}$$

where $g^*(T)$ is the optimal expected average cost.Next, for any $i \in I$, $H_i > \mathsf{P}(X_T \geqq i + 1) - \mathsf{P}(X_B \geqq i + 1) \geqq 0$ and from (4.15) and (4.16) we get $g(i + 1, T) - g(i, T) < 0$ for any $i < i^*(T)$. It follows from Theorem 3.2 that $g^*(T) = g(i^*(T), T)$ and $w(i^*(T)) + \lambda c \geqq g(i^*(T), T)$ so that $g(i^*(T) + 1, T) - g(i^*(T), T) \geqq 0$. Assume that $w(i) + \lambda c \geqq g(i, T)$ for some $i \geqq i^*(T)$. We prove that $w(i + 1) + \lambda c \geqq g(i + 1, T)$. Condition (b), (4.14) and (4.15) yield

$$\begin{aligned} w(i + 1) + \lambda c - g(i + 1, T) &> w(i) + \lambda c - g(i, T) - (g(i + 1, T) - g(i, T)) = \\ &= \{w(i) + \lambda c - g(i, T)\}\,(1 - H_i/(\lambda T_{i+1})) = \\ &= \{w(i) + \lambda c - g(i, T)\}\, T_i/T_{i+1} \geqq 0\,, \end{aligned}$$

so that $g(j + 1, T) \geqq g(j, T)$ for all $j \geqq i^*(T)$. □

From Theorems 3.2, 4.1 and from Lemma 4.2, we get the following result.

Theorem 4.2. Let (b) hold. The optimal control limit $i^*(T)$ is determined by

$$i^*(T) = \inf\{i \in I: w(i) + \lambda c \geqq g(i, T)\} \tag{4.17}$$

where $g(i, T)$ is given by (4.6). The optimal expected average cost per unit time is $g^*(T) = g(i^*(T), T)$.

Corollary 4.2. Let (b) hold. Then the optimal control limit $i^*(T)$ is nonincreasing in T and the limiting value i^* is given by (cf. [5])

$$i^* = \inf\{i \in I: w(i) + \lambda c \geqq g(i)\} \tag{4.18}$$

where $g(i)$ is given by (4.12).

Proof. Obviously, for any $T_1 < T_2$ $g^*(T_1) \geqq g^*(T_2)$ and we have from (3.5) $i^*(T_1) \geqq i^*(T_2)$ and

$$i^* = \inf\{i \in I: w(i) + \lambda c \geqq g^*\} \tag{4.19}$$

where $g^* = \lim_{T\to+\infty} g^*(T)$. Next, it was proved in [5] that $-g(i)$ is unimodal in i and we have from (4.12)

$$g(i+1) - g(i) = \left(w(i) + \lambda c - g(i)\right) \sum_{k=0}^{i} p_k / (\lambda b + \sum_{j=0}^{i} (i + 1 - j)\, p_j)$$

so that (4.18) holds. □

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