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ENTROPY ON EFFECT ALGEBRAS WITH THE RIESZ DECOMPOSITION PROPERTY II: MV–ALGEBRAS

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We study the entropy mainly on special effect algebras with (RDP), namely on tribes of fuzzy sets and σ-complete MV-algebras. We generalize results from [14] and [15] which were known only for special tribes.

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We continue the study of entropy. In Part I, we have introduced basic properties of entropy. Here we concentrate mainly on the state space which allows us to understand the entropy better, Section 8, and we introduce also quotient effect algebras. Section 9 studies shortly the problem of a so-called entropy generator theorem for effect algebras or MV-algebras. The main results of the present part are in Section 10 and Section 11. There we study the entropy on tribes of fuzzy sets and using the Loomis-Sikorski theorem also on σ-complete MV-algebras. That generalizes results from [14] and [15] which were proved only for full tribes. In addition, we give also a solution to Problem 7 from [14].

8. STATE SPACE OF EFFECT ALGEBRAS

Since entropy depends only on the range of the state, to understand entropy better, we will study the range of states in more details.

An effect algebra E is said to be *monotone σ-complete* if, for any sequence $\{a_n\}$ such that $a_n \le a_{n+1}$ for any n, $\bigvee_n a_n \in E$. It can happen that E is monotone σ-complete but not a lattice. According to [11, Prop. 16.9], if $E = \Gamma(G,u)$ is with (RDP), then E is monotone σ-complete iff (G,u) is monotone σ-complete, i.e., if $g_n \le g_{n+1} \le g$ for a sequence $\{g_n\}$ from G, there is $\bigvee_n g_n \in G$.

Given a subset A of E, there is a sub-effect algebra of E, $E_{ea}(A)$, generated by A.

Indeed, if A is a subset of an effect algebra E, we set $A' := \{a' : a \in A\}$ and $A + A := \{a + b \in E : a, b \in A\}$. We define $E_0 = A \cup A' \cup \{0, 1\}$, and for any $n \geq 1$ we put $E_{n+1} := (E_n + E_n)'$.

$$E_{ea}(A) = \bigcup_n E_n. \tag{8.1}$$

Hence, if A is countable, so is $E_{ea}(A)$.

We recall that the definition of a state was given in Section 3. A state s on E is *σ-additive* if $s(a) = \lim_n s(a_n)$ whenever $a_n \nearrow a$ (i.e., $a_n \leq a_{n+1}$ and $a = \bigvee_n a_n$). A *state* on a unital po-group (G, u) is any mapping $\hat{s} : G \to \mathbb{R}$ such that (i) $\hat{s}(g) \geq 0$ for any $g \in G^+$, (ii) $\hat{s}(g + h) = \hat{s}(g) + \hat{s}(h)$ for all $g, h \in G$, and (iii) $\hat{s}(u) = 1$. We denote by $\mathcal{S}(E)$ and $\mathcal{S}(G, u)$ the sets of all states on E and (G, u), respectively. According to [11, Cor. 4.4], if (G, u) is an interpolation group, then $\mathcal{S}(G, u)$ is nonempty, consequently by remarks after Theorem 2.1 [2], if E satisfies (RDP), then $\mathcal{S}(E) \neq \emptyset$.

By [11, Lem. 4.21], $\hat{s}(G) = \{\hat{s}(g) : g \in G\}$ is a subgroup of the group $\mathbb{R}$ of all real numbers which is either cyclic or dense in $\mathbb{R}$. In the first case $\hat{s}$ is said to be *discrete*. In such a case $\hat{s}(G) = \frac{1}{n}\mathbb{Z}$ for some integer $n \geq 1$.

A state s on an effect algebra E is said to be *discrete* if $s(E) = \{s(a) : a \in E\} \subseteq \{0, 1/n, 2/n, \ldots, n/n\}$ for some integer $n \geq 1$. It can happen that $s(E)$ is a proper subset of $\{0, 1/n, 2/n, \ldots, n/n\}$. Indeed, let $E = \{0, a, a', 1\}$, and let $s(a) = 0.3$ and $s(a') = 0.7$.

We now show that there is a one-to-one correspondence among discrete states on E and (G, u), respectively.

Proposition 8.1. Let $E = \Gamma(G, u)$ be an effect algebra with (RDP). Then a state s on E is discrete if and only if its extension $\hat{s}$ to (G, u) is discrete.

Proof. If $\hat{s}$ is discrete, it can be easily seen that s is discrete. Conversely, let s be discrete. That is $s(E) \subseteq \{0, 1/n, 2/n, \ldots, n/n\}$ for some integer $n \geq 1$; let n be the smallest one. We suppose that $s(E) = \{0, k_1/n, \ldots, k_m/n, 1\}$, where $1 \leq k_1 < \cdots < k_m \leq n$. Since n is minimal, this implies that the greatest common divisor of $n, k_1, \ldots, k_m$ is 1. The elementary arithmetic yields that there are integers $a_0, a_1, \ldots, a_m \in \mathbb{Z}$ such that $a_0 n + a_1 k_1 + \cdots + a_m k_m = 1$. Therefore, $1/n \in \hat{s}(G)$, i.e., $\hat{s}(G) = \frac{1}{n}\mathbb{Z}$. $\square$

A state s on E (on (G, u)) is said to be *extremal* if the equality $s = \lambda s_1 + (1 - \lambda) s_2$, where $0 < \lambda < 1$ and s_1, s_2 are states on E (on (G, u)), yields $s = s_1 = s_2$. We denote by $\mathrm{Ext}_{\mathcal{S}}(E)$ and $\mathrm{Ext}_{\mathcal{S}}(G, u)$ the sets of all extremal states on E and (G, u), respectively.

We say that a net of states, $\{s_\alpha\}$, *converges weakly* to a state s if $s_\alpha(a) \to s(a)$ for any $a \in E$ ($a \in G$). If E satisfies (RDP), $\mathcal{S}(E)$ and $\mathcal{S}(G, u)$ are nonempty compact sets and by Krein–Mil'man theorem [11, Thm. 5.17], every state on E (on (G, u)) is a weak limit of a net of convex combinations of extremal states. Hence, $\mathrm{Ext}_{\mathcal{S}}(E)$ and $\mathrm{Ext}_{\mathcal{S}}(G, u)$ are nonempty sets.

We note that if E is a lattice effect algebra with (RDP), then $\mathrm{Ext}_{\mathcal{S}}(E)$ is a compact set. If E is an effect algebra with (RDP) which is not a lattice then it can happen that $\mathrm{Ext}_{\mathcal{S}}(E)$ is not compact (see [11, Ex. 6.10]).

We recall that a state s on $E = \Gamma(G,u)$ is extremal iff $\hat{s}$ is extremal on (G,u). For extremal states on Abelian unital po-groups we have the following criteria, see [11, Cor. 6.21, Thm. 12.14, Thm. 12.18].

Theorem 8.2. Let (G,u) be an interpolation Abelian unital po-group. A discrete state s on (G,u) is extremal if and only if given any $x,y \in G^+$, there exists $z \in G^+$ such that $z \le x, z \le y$ and

$$s(z) = \min\{s(x), s(y)\}. \tag{8.2}$$

A state s is extremal if and only if given any $x,y \in G^+$

$$\min\{s(x), s(y)\} = \sup\{s(z) : z \in G^+, z \le x,\ z \le y\}. \tag{8.3}$$

If, in addition, (G,u) is an ℓ-group, then a state s is extremal if and only if

$$s(x \wedge y) = \min\{s(x), s(y)\} \tag{8.4}$$

holds for all $x,y \in G^+$.

We recall that if E is a lattice with (RDP), then s on E is extremal iff (8.4) holds for all $x,y \in E$. For a general case of E without (RDP), extremal states need not satisfy criteria (8.2) and (8.3) restricted for $x,y \in E$ as it was mentioned by the anonymous referee; as an example can serve $MO2$ (= horizontal sum of two copies of 2^2).

Proposition 8.3. If s is an extremal state on an Abelian po-group (G,u) with interpolation, then for any $x,y,v \in G^+$, we have

$$\min\{s(x), s(y), s(v)\} = \sup\{s(z) : z \in G^+,\ z \le x,\ z \le y, z \le v\}. \tag{8.5}$$

Proof. Indeed, we trivially have $\sup\{s(z) : z \in G^+, z \le x,\ z \le y, z \le v\} \le \min\{s(x), s(y), s(v)\}$. Let $\epsilon > 0$ be given. Due to (8.3), there are two elements $z_1, z_2 \in G^+$ with $z_1 \le x, y$, $z_2 \le x, v$ such that

$$\min\{s(x), s(y)\} - \epsilon/2 < s(z_1) \le \min\{s(x), s(y)\},$$
$$\min\{s(x), s(v)\} - \epsilon/2 < s(z_2) \le \min\{s(x), s(v)\}.$$

Applying again (8.3) to $s(z_1)$ and $s(z_2)$, we find an element $z_3 \in G^+$ with $z_3 \le z_1, z_2$ such that

$$\min\{s(z_1), s(z_2)\} - \epsilon/2 < s(z_3) \le \min\{s(z_1), s(z_2)\}.$$

Then $\min\{s(x), s(y), s(v)\} - \epsilon < s(z_3) \le \min\{s(x), s(y), s(v)\}$, which proves (8.5).

$\square$

We recall that a *rational convex combination* is any convex combination in which all the coefficients are rational numbers.

Proposition 8.4. A state s on an effect algebra E with (RDP) is discrete if and only if s is a rational convex combination of discrete extremal states.

Proof. It follows directly from [11, Prop. 6.22], Theorem 2.1 [2] and Proposition 8.1. □

We have seen that if s is a discrete state on E with (RDP), then it can happen that the range of s, $s(E)$, is a proper subset of $\{0, 1/n, 2/n, \dots, 1\}$ whereas $\hat{s}(G) = \frac{1}{n}\mathbb{Z}$. We now show that for discrete extremal states we have the equality $s(E) = \{0, 1/n, 2/n, \dots, 1\}$, and for other extremal states their range is dense in $[0,1]$.

If an effect algebra E has the property that, for infinitely many integers $n \geq 1$, there is an element $v \in E$ such that $nv = 1$, then E has no discrete state. Indeed, for any state s of E we have $1/n \in s(E)$ for infinitely many integers n.

Proposition 8.5. Let s be a discrete extremal state on an effect algebra E with (RDP). Then

$$s(E) = \{0, 1/n, 2/n, \dots, 1\} \tag{8.6}$$

for some integer $n \geq 1$.

If s is a non-discrete extremal state, then the range $s(E)$ is dense in $[0,1]$.

Proof. Suppose that s is an extremal discrete state on E with (RDP) such that $\hat{s}(G) = \frac{1}{n}\mathbb{Z}$, where $E = \Gamma(G,u)$. We assert that $i/n \in s(E)$ for any $i = 1, \dots, n-1$. Indeed, there exists an element $g \in G$ such that $i/n = \hat{s}(g)$. Then $g = g_1 - g_2$, where $g_1, g_2 \in G^+$. According to (8.2), there exists $g_0 \leq g_1, g_2$ with $g_0 \geq 0$, such that $\hat{s}(g_0) = \min\{\hat{s}(g_1), \hat{s}(g_2)\} = \hat{s}(g_2)$. Then $i/n = \hat{s}(g) = \hat{s}(g_1) - \hat{s}(g_0) = \hat{s}(g_1 - g_0)$. Applying again (8.2) to $g_1 - g_0 > 0$ and u, we have that there exists $a \in E$ such that $i/n = \min\{\hat{s}(g_1 - g_0), \hat{s}(u)\} = \hat{s}(g) = s(a)$.

Now let s be a non-discrete extremal state on E. Take $g \in G$ such that $0 < \hat{s}(g) < 1$. Then $g = g_1 - g_2$, where $g_1, g_2 \geq 0$. By (8.3), $\hat{s}(g_2) = \min\{\hat{s}(g_1), \hat{s}(g_2)\} = \sup\{\hat{s}(z) : z \in G^+,\ z \leq g_1,\ z \leq g_2\}$, which proves $\hat{s}(g) = \lim_n \hat{s}(g_1 - z_n)$ for some sequence of positive elements $\{z_n\}$ under g_1. Applying again (8.3) to positive elements $g_1 - z_n$ and u for any $n \geq 1$, we find a sequence of elements of E, $\{a^n_m\}_m$, such that $\hat{s}(g_1 - z_n) = \lim_m s(a^n_m)$, which proves that every $\hat{s}(g)$ from the real interval $[0,1]$ can be approximated by values from $s(E)$. Since the range $\hat{s}(G)$ is dense in $\mathbb{R}$, then the range $s(E)$ is dense in the real interval $[0,1]$. □

Question: When $s(E) = \hat{s}(G) \cap [0,1]$? This is true for any unital ℓ-group (G,u) and any extremal state on an effect algebra E with (RDP) which is a lattice. Further cases are presented below.

Proposition 8.6. Let s be a σ-additive non-discrete extremal state on a monotone σ-complete effect algebra E with (RDP). Then $s(E) = [0,1]$.

Proof. Let $t \in (0,1)$. Due to density of $s(E)$ in $[0,1]$, by Proposition 8.5, there is a sequence of elements, $\{a_n\}$, from E such that $s(a_n) \geq s(a_{n+1}) \geq t$ and

$\lim_n s(a_n) \doteq t$. By (8.3), there is an element $z_1 \in E$, $z_1 \leq a_1, a_2$ such that $t \leq s(z_1)$. By induction, we can find a sequence of elements, $\{z_n\}$, such that $z_n \leq a_n$, $z_{n+1} \leq z_n$ and $t \leq s(z_n)$. Hence, $t = \lim_n s(z_n)$. Since the sequence $\{z_n\}$ is monotone, there is $z = \bigwedge_n z_n \in E$. It is evident that $s(z) = \lim_n s(z_n) = t \in s(E)$. $\square$

To extend the result of Proposition 8.6, we introduce the following facts about ideals on effect algebras.

We say that a poset X satisfies the *countable interpolation property* provided that, for any two sequences $\{x_i\}$ and $\{y_j\}$ such that $x_i \leq y_j$ for all i, j, there exists an element $z \in X$ such that $x_i \leq z \leq y_j$ for all i, j.

We recall that if $E = \Gamma(G, u)$ for some interpolation unital po-group (G, u), then E has countable interpolation iff G has countable interpolation, [11, Prop. 16.3], and if E is monotone σ-complete, then E has countable interpolation (see [11, Thm. 16.10, Prop 16.9]).

An *ideal* of an effect algebra E is a non-empty subset I of E such that (i) $x \in E$, $y \in I$, $x \leq y$ imply $x \in I$, and (ii) if $x, y \in I$ and $x+y$ is defined in E, then $x+y \in I$. An ideal I is said to be a *Riesz ideal* if, for $x \in I$, $a, b \in E$ and $x \leq a+b$, there exist $a_1, b_1 \in I$ such that $x \leq a_1 + b_1$ and $a_1 \leq a$ and $b_1 \leq b$.

For example, if E is with (RDP), then any ideal of E is Riesz.

A proper ideal I (i. e. $I \neq E$) is *maximal* if it is not contained in any proper ideal. Every effect algebra has at least one maximal ideal.

If s is a state on E, then the *kernel* of the state s, i. e. the set

$$\mathrm{Ker}(s) := \{a \in E : s(a) = 0\},$$

is an ideal of E.

Let P be a (proper) ideal of an effect algebra E with (RDP). We define a relation $\sim_P$ on E via $a \sim_P b$ iff $a - e = b - f$ for some $e, f \in P$. According to [9, Sec 3.1], we have that $\sim_P$ is an equivalence such that (i) $a + b \in E$, $a_1 + b_1 \in E$, $a \sim_P a_1$, $b \sim_P b_1$ imply $(a+b) \sim_P (a_1 + b_1)$, (ii) $a \sim_P b$ implies $a' \sim_P b'$, (iii) $a + b \in E$, $c \sim_P a$ imply there exists an element $d \in E$ such that $d \sim_P b$ and $d + c \in E$, (iv) $a + b, a_1 + b_1 \in E$, $a_1 \sim_P a, (a_1 + b_1) \sim_P (a+b)$ imply $b_1 \sim_P b$. If we define $a/P := [a] := [a]_P := \{b \in E : b \sim_P a\}$, then $E/P := \{[a]_P : a \in E\}$ is an effect algebra, where $[a] + [b] = [c]$ iff there exist $a_1 \in [a], b_1 \in [b], c_1 \in [c]$ such that $a_1 + b_1 = c_1$. For the constant elements in E/P we take $[0]$ and $[1]$. We recall that

$$[a]_P \leq [b]_P \text{ in } E/P \quad \Leftrightarrow \quad \text{there exists } a_1 \in [a]_P \text{ such that } a_1 \leq b. \tag{8.7}$$

We recall (i) if E satisfies (RDP), then E/P is with (RDP); (ii) if s is a state of E with (RDP), then the function $\bar{s}$ defined on $E/\mathrm{Ker}(s)$ via $\bar{s}([a]) := s(a)$, $[a] \in E/\mathrm{Ker}(s)$, is a state on $E/\mathrm{Ker}(s)$ such that $s(E) = \bar{s}(E/\mathrm{Ker}(s))$; (iii) s is extremal on E with (RDP) iff $\bar{s}$ is extremal on $E/\mathrm{Ker}(s)$; (iv) if E with (RDP) satisfies countable interpolation, then E/P has countable interpolation (compare with [11, Prop. 16.4]), and (v) if (E, s, T) is a dynamical system (E has (RDP)), then $(E/\mathrm{Ker}(s), \bar{s}, \overline{T})$ is also a dynamical system, where $\overline{T} : E/\mathrm{Ker}(s) \to E/\mathrm{Ker}(s)$ is an $\bar{s}$-preserving mapping defined by $\overline{T}([a]) := [T(a)]$, $[a] \in E/\mathrm{Ker}(s)$.

Proposition 8.7. Let s be a non-discrete extremal state on an effect algebra E satisfying (RDP) and countable interpolation. Then $s(E) = [0,1]$.

Proof. The proof is divided into two steps.

Claim 1. If s is an extremal state such that $\mathrm{Ker}(s) = \{0\}$, then E is a σ-complete lattice.

Let $\{a_i\}$ be any sequence of elements of E, and let $U(\{a_i\})$ be the set of all upper bounds of the sequence $\{a_i\}$; $U(\{a_i\})$ is nonempty. Let $\alpha := \inf\{s(b) : b \in U(\{a_i\})\}$, and choose a countable subset $\{b_j\}$ of $U(\{a_i\})$ such that $\alpha = \inf\{s(b_j)\}$. By countable interpolation, there exists an element $x \in E$ such that $a_i \leq x \leq b_j$ for all i, j. Given any $v \geq a_i$, countable interpolation provides an element $y \in E$ such that $a_i \leq y \leq x, v$. Then $y \in U(\{a_i\})$ and $s(y) = \alpha$, whence $s(x - y) = 0$, i.e., $x = y$. This implies x is the supremum of $\{a_i\}$.

Passing to complements, we see that E is a σ-complete lattice.

Step 2. Let s be any extremal state on E. Since E has countable interpolation, so has $E/\mathrm{Ker}(s)$, and $\bar{s}$ is an extremal state on $E/\mathrm{Ker}(s)$. Therefore, by Step 1, $E/\mathrm{Ker}(s)$ is a complete lattice while the state $\bar{s}$ has the property $\mathrm{Ker}(\bar{s}) = \{0\}$.

It is well-known [9, Thm. 7.1.1] that a state on an effect algebra with (RDP) which is a lattice (= MV-algebra) is extremal iff its kernel is a maximal ideal. Since $\bar{s}$ is extremal, then $(E/\mathrm{Ker}(s))/\mathrm{Ker}(\bar{s}) = E/\mathrm{Ker}(s)$ is isomorphic to some MV-subalgebra of the real interval $[0,1]$. Since $E/\mathrm{Ker}(s)$ is σ-complete, then $E/\mathrm{Ker}(s) \cong [0,1]$. The isomorphism in question is in view of (8.4) the mapping $[a] \mapsto s(a)$, $[a] \in E/\mathrm{Ker}(s)$. Consequently, $s(E) = \bar{s}(E/\mathrm{Ker}(s)) = [0,1]$. □

We recall that in both Propositions 8.6 and 8.7 we have $s(E) = \hat{s}(G) \cap [0,1]$. In view of the later proposition, the σ-additivity of a state in Proposition 8.6 is superfluous:

Proposition 8.8. Let s be a non-discrete extremal state on a monotone σ-complete effect algebra E with (RDP). Then $s(E) = [0,1]$.

Proof. The statement follows from Proposition 8.7 while every monotone σ-complete effect algebra with (RDP) has according to [11, Prop. 16.9, Thm 16.10] countable interpolation. □

9. ENTROPY AND GENERATORS

From the classical entropy theory we know that if $\mathcal{A}$ is a partition of a dynamical system $(\Omega, \mathcal{S}, P, T)$ such that $\bigcup_{n=0}^{\infty} T^{-n}(\mathcal{A})$ generates the σ-algebra $\mathcal{S}$, then $h(T) = h(\mathcal{A}, T)$; this is the so-called entropy generator theorem. An analogous problem is studied in the present section giving some partial answers.

For example, if $E = [0,1]$ and $\mathcal{A} = \{t, 1-t\}$, where t is an irrational number in the MV-algebra E, or if $E = \{0, 1/n, 2/n, \ldots, 1\}$ and $\mathcal{A} = \{1/n, 1/n, \ldots, 1/n\}$ (see Example 5.1 and Example 5.3 from [2]), then $\bigcup_{n=0}^{\infty} T^{n}(\mathcal{A})$ generates E as a σ-complete MV-algebra, and $h_*^{\mathcal{R}}(T) = h_*^{\mathcal{R}}(\mathcal{A}, T) = h_{\mathcal{R}}^*(\mathcal{A}, T) = h_{\mathcal{R}}^*(T) = 0$.

Theorem 9.1. Let (E, s, T) be a dynamical system, where E is an effect algebra with (RDP) such that $s(E)$ is finite.

Then, for any partition $\mathcal{A}$ in E, we have

$$h_*^{\mathcal{R}}(\mathcal{A}, T) = h_{\mathcal{R}}^*(\mathcal{A}, T) = 0,$$

and

$$h_*^{\mathcal{R}}(T) = h_{\mathcal{R}}^*(T) = 0.$$

Proof. Let $s(E) = \{0, \alpha_1, \ldots, \alpha_k, 1\}$, where $0 < \alpha_1 < \cdots < \alpha_k < 1$. There is an integer $q \geq 1$ such that $q\alpha_1 \leq 1$ and $(q+1)\alpha_1 > 1$. Then $1/(q+1) < \alpha_1 \leq 1/q$.

Let now $\mathcal{A}$ be any partition in E, and let $\mathcal{C}$ be any Riesz refinement of partitions $\mathcal{A}, T(\mathcal{A}), \ldots, T^{n-1}(\mathcal{A})$. Then $H(\mathcal{C}) \leq \log q$ and $H_*^n(\mathcal{A}, T)_{\mathcal{R}} \leq H_n^*(\mathcal{A}, T)_{\mathcal{R}} \leq \log q$, i.e., $0 \leq h_*^{\mathcal{R}}(\mathcal{A}, T) \leq h_{\mathcal{R}}^*(\mathcal{A}, T) = 0$. Consequently, $h_*^{\mathcal{R}}(T) = h_{\mathcal{R}}^*(T) = 0$. □

Theorem 9.1 holds also if E is finite or if s is a discrete state.

Proposition 9.2. Let (E, s, T) be a dynamical system, where E is an effect algebra with (RDP). Then for the dynamical system $(E/\mathrm{Ker}(s), \bar{s}, \overline{T})$ we have

$$h_*^{\mathcal{R}}(T) = h_*^{\mathcal{R}}(\overline{T}), \quad h_{\mathcal{R}}^*(T) = h_{\mathcal{R}}^*(\overline{T}). \tag{9.1}$$

Proof. *Step 1.* If $\mathcal{A} = \{a_i\}$ is a partition in E, then $[\mathcal{A}] := \{[a_i]\}$ is a partition in $E/\mathrm{Ker}(s)$, and $H(\mathcal{A}, T) = H([\mathcal{A}], \overline{T})$.

Step 2. If $[a_1], \ldots, [a_n]$ are summable elements in $E/\mathrm{Ker}(s)$, i.e., $[a_1] + \cdots + [a_n] = [b]$, then there are summable elements $\tilde{a}_1, \ldots, \tilde{a}_n \in E$ such that $\tilde{a}_i \in [a_i]$ for any i and $[\tilde{a}_1 + \cdots + \tilde{a}_n] = [b]$.

Indeed, by (8.7), we have that $[a_{n-1}] \leq [a_n']$. Therefore, there is $\tilde{a}_{n-1} \in [a_{n-1}]$ such that $\tilde{a}_{n-1} \leq a_n'$, i.e., $\tilde{a}_{n-1} + a_n$ is defined in E. Since $[a_{n-2}] \leq ([a_{n-1}] + [a_n])' = [(\tilde{a}_{n-1} + a_n)']$, there is $\tilde{a}_{n-2} \in [a_{n-2}]$ such that $\tilde{a}_{n-2} \leq (\tilde{a}_{n-1} + a_n)'$, i.e., $\tilde{a}_{n-2} + \tilde{a}_{n-1} + a_n \in E$. Using mathematical induction, we see that there are elements $\tilde{a}_i \in [a_i]$ such that $[\tilde{a}_1 + \cdots + \tilde{a}_n] = [a_1] + \cdots + [a_n]$.

In particular, if $\{[a_i]\}$ is a partition in $E/\mathrm{Ker}(s)$, then there is a partition $\{\tilde{a}_i\}$ in E such that $[\tilde{a}_i] = [a_i]$. In fact, after finding summable elements $\{\hat{a}_i\}$ in E such that $[\hat{a}_i] = [a_i]$ for $i = 1, \ldots, n$, we set $a_0 = (\hat{a}_1 + \cdots + \hat{a}_n)'$ and put $\tilde{a}_i = \hat{a}_i$ for $i = 1, \ldots, n-1$, and $\tilde{a}_n = \hat{a}_n + a_0$.

Step 3. Let $\{[c_{ij}]\}$ be a Riesz refinement of partitions $\{[a_i]\}$ and $\{[b_j]\}$. By Step 2 we can assume that $\{a_i\}$ and $\{b_j\}$ are partitions in E, and $\{c_{ij}\}$ is their Riesz refinement in E.

Summarizing all steps, we see that

$$\begin{aligned} H_*^n(\mathcal{A}, T)_{\mathcal{R}} &= H_*^n([\mathcal{A}], \overline{T})_{\mathcal{R}}, \\ h_*^{\mathcal{R}}(\mathcal{A}, T) &= h_*^{\mathcal{R}}([\mathcal{A}], \overline{T}), \\ h_*^{\mathcal{R}}(T) &= h_*^{\mathcal{R}}(\overline{T}). \end{aligned}$$

In a similar manner we proceed also with the second equality in (9.1). □

As a direct corollary of Proposition 9.2 we have that if (E,s,T) is a dynamical system, where E is a lattice effect algebra with (RDP), i.e. an MV-algebra, then $E/\mathrm{Ker}(s)$ is an Archimedean MV-algebra, i.e., isomorphic to a Bold algebra of fuzzy sets. We recall that a *Bold algebra* is a system $\mathcal{F} \subseteq [0,1]^\Omega$ of fuzzy sets defined on a nonempty set Ω such that (i) $1 \in \mathcal{F}$, (ii) if $f \in \mathcal{F}$, then $1-f \in \mathcal{F}$, and (iii) if $f,g \in \mathcal{F}$, then $f \oplus g \in \mathcal{F}$, where $(f \oplus g)(\omega) = \min\{f(\omega)+g(\omega),1\}$, $\omega \in \Omega$. Every Bold algebra is an MV-algebra.

Therefore, to calculate entropy of a dynamical system (E,s,T), where E is an MV-algebra, it is enough to assume that E is a Bold algebra of fuzzy sets.

A partial answer to the problem from the beginning of the present section is as follows.

Theorem 9.3. Let (E,s,T) be a dynamical system where s is an extremal state on an effect algebra E satisfying (RDP) and countable interpolation. Then $h_*^{\mathcal{R}}(T)=0$, and $h_{\mathcal{R}}^*(T)=0$ if s is discrete and $h_{\mathcal{R}}^*(T)=\infty$ if s is non-discrete. In addition, there is a partition $\tilde{\mathcal{A}}$ of $E/\mathrm{Ker}(s)$ such that $\bigcup_{n=0}\overline{T}^n(\tilde{\mathcal{A}})$ generates $E/\mathrm{Ker}(s)$, and $h_*^{\mathcal{R}}(T)=h_*^{\mathcal{R}}(\overline{T})=h_*^{\mathcal{R}}(\tilde{\mathcal{A}},\overline{T})$ and $h_{\mathcal{R}}^*(T)=h_{\mathcal{R}}^*(\overline{T})=h_{\mathcal{R}}^*(\tilde{\mathcal{A}},\overline{T})$.

The same is true if E is monotone σ-complete with (RDP).

Proof. If s is a discrete state, the statement follows from Theorem 9.1.

Suppose now s is non-discrete. According to Claim 1 of Proposition 8.7, $E/\mathrm{Ker}(s)$ is a σ-complete lattice effect algebra with (RDP) (= σ-complete MV-algebra) and according to Step 2 of the same proposition, $E/\mathrm{Ker}(s)$ is isomorphic to the MV-algebra of the real interval $[0,1]$ from Example 5.3 [2]. Due to Proposition 9.2 and Example 5.3 [2], $h_*^{\mathcal{R}}(T)=h_*^{\mathcal{R}}(\overline{T})=0$ and $h_{\mathcal{R}}^*(T)=h_{\mathcal{R}}^*(\overline{T})=\infty$.

According to [9, Thm. 6.1.43], every MV-subalgebra of $[0,1]$ generated by an irrational number is dense in $[0,1]$, and if it is generated by a rational number, it is isomorphic to $\{0,1/n,2/n,\dots,1\}$ for some $n \geq 1$.

Therefore, if we take an arbitrary partition $\tilde{\mathcal{A}}$ in $E/\mathrm{Ker}(s)$ containing an irrational number, then the σ-complete MV-algebra generated by $\tilde{\mathcal{A}}$ is isomorphic with the whole interval $[0,1]$. By Example 5.3 [2], $h_*^{\mathcal{R}}(\tilde{\mathcal{A}},\overline{T})=h_{\mathcal{R}}^*(\tilde{\mathcal{A}},\overline{T})=0$.

If E is monotone σ-complete, then E has the countable interpolation property.

□

Theorem 9.4. Let (E,s,T) be a dynamical system, where s is an extremal state on a lattice effect algebra E with (RDP). Then $h_*^{\mathcal{R}}(T)=H_*^{\mathcal{R}}(\mathcal{A},T)=0$ for any partition $\mathcal{A}$ in E.

Proof. Since s is an extremal state, due to (8.4), $E/\mathrm{Ker}(s)$ is a lattice effect algebra which is isomorphic with some MV-subalgebra of $[0,1]$. According to (9.1) and Example 5.7 [2], we have the statement in question. □

Proposition 9.5. Let E be any linearly ordered effect algebra. For any dynamical system (E, s, T) we have $h_*^{\mathcal{R}}(T) = H_*^{\mathcal{R}}(\mathcal{A}, T) = 0$ for any partition $\mathcal{A}$ in M.

Proof. Since E a linearly ordered effect algebra, it has a unique state, s, which is therefore extremal. The conclusion follows immediately from Theorem 9.4. □

We present a class of effect algebras satisfying the conditions of Proposition 9.5, for more details see [3].

Example 9.6. Let L and G be two commutative ℓ-groups and let $L \times_{lex} G$ denote the lexicographical product of L and G. Then $L \times_{lex} G$ is an ℓ-group iff L is linear, [10, p. 26 (d)]. In addition, if (L, u) is a linear unital ℓ-group, then $(u, 0)$ is a strong unit in $L \times_{lex} G$, consequently,

$$M(L, G, u) := \Gamma(L \times_{lex} G, (u, 0))$$

is an MV-algebra. The effect algebra $M(L, G, u)$ has a unique state, s, and there is a unique state s_0 on $\Gamma(L, u)$, namely, $s(h, g) := s_0(h), (h, g) \in M(L, G, u)$. In addition if also G is linearly ordered, then $M(L, G, s)$ is linearly ordered.

In any case, if G is an ℓ-group, $M(L, G, u)$ has a unique state, therefore it is extremal, and in view of Theorem 9.4, $h_*^{\mathcal{R}}(T) = H_*^{\mathcal{R}}(\mathcal{A}, T) = 0$ for any s-preserving transformation T (there are infinitely many such ones) and any partition $\mathcal{A}$ in $M(L, G, u)$.

As a particular case of Example 9.6, we have $M_n := \Gamma(\mathbb{Z} \times_{lex} \mathbb{Z}, (n, 0))$ or $\Gamma(\mathbb{R} \times_{lex} \mathbb{R}, (1, 0))$. Then in the first case $s(M_n) = \{0, 1/n, 2/n, \ldots, 1\}$ and in the second one, the range of the state is the whole interval $[0, 1]$.

We recall that some results of this kind are also in Theorem 10.2 and Theorem 11.3. In general, the entropy generator theorem seems to be open even for σ-complete MV-algebras.

10. ENTROPY ON TRIBES

Entropy on tribes (systems of fuzzy sets) was studied in [15, Sec 10] and also in [14]. All these tribes were closed also under the natural product of fuzzy sets. In the present section, we study entropy on tribes without any assumption on the product. The main result is Theorem 10.2 which generalizes [14, Thm. 4.13] proved *only* for full tribes (we recall that if $(\Omega, \mathcal{S})$ is a measurable space, then the system of all $\mathcal{S}$-measurable functions from $[0, 1]^\Omega$ is said to be a *full tribe*). We note that the basic properties of entropy on MV-algebras without any product are in [13].

A *tribe* of fuzzy sets on a set $\Omega \neq \emptyset$ is a system $\mathcal{T} \subseteq [0, 1]^\Omega$ such that

(i) $1_\Omega \in \mathcal{T}$,

(ii) if $a \in \mathcal{T}$, then $1_\Omega - a \in \mathcal{T}$,

(iii) if $\{a_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathcal{T}$, then

$$\min\left\{\sum_{n=1}^{\infty} a_n, 1\right\} \in \mathcal{T}.$$

Every tribe $\mathcal{T}$ is a σ-complete MV-algebra, where $(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}, \omega \in \Omega$, moreover, $\mathcal{T}$ is a σ-complete MV-algebra such that if $\{f_n\}$ is a sequence from $\mathcal{T}$ and $f_n \nearrow f$ (pointwisely), then $f \in \mathcal{T}$.

Let $\mathcal{T}$ be a tribe and denote by

$$\mathcal{S}_0(\mathcal{T}) := \{A \subseteq \Omega : \chi_A \in \mathcal{T}\}. \tag{10.1}$$

Then (see [1], [9, Thm. 7.1.7] or [15])

(1) $\mathcal{S}_0(\mathcal{T})$ is a σ-algebra of crisp subsets of Ω. The set of all central elements of $\mathcal{T}$ is the set $C(\mathcal{T}) = \{\chi_A : A \in \mathcal{S}_0(\mathcal{T})\}$.

(2) If $f \in \mathcal{T}$, then f is $\mathcal{S}_0(\mathcal{T})$-measurable.

(3) $\mathcal{T}$ contains all $\mathcal{S}_0(\mathcal{T})$-measurable fuzzy functions on Ω if and only if $\mathcal{T}$ contains all constant functions with values in $[0,1]$.

(4) If s is a σ-additive state on $\mathcal{T}$, then there exists a unique probability measure P on $\mathcal{S}_0(\mathcal{T})$ such that

$$s(f) = \int_{\Omega} f(\omega)\, \mathrm{d}P(\omega), \quad f \in \mathcal{T}, \tag{10.2}$$

(theorem of Butnariu–Klement, [15, Thm. 8.1.12]).

In what follows, we extend the result of Riečan [15, Sec. 10.3] from a full tribe (i. e. a tribe of all measurable fuzzy sets on a measurable probability space $(\Omega, \mathcal{S}, P)$) which is closed with respect to the natural product of fuzzy sets to an arbitrary tribe which is not necessarily closed under the product. For example, if $\mathcal{T}_n = \{0, 1/n, 2/n, \dots, 1\}$, then $\mathcal{T}_n$, $n \geq 2$, is a tribe which is not closed under the natural product.

We recall that if $(\Omega, \mathcal{S}, P)$ is a probability space, f is any fuzzy set measurable with respect to $\mathcal{S}$ and $\mathcal{S}_0$ any σ-subalgebra of $\mathcal{S}$, then $E(f|\mathcal{S}_0)$ denotes the conditional expectation of f with respect to $\mathcal{S}_0$. If $\mathcal{B} = \{B_1, \dots, B_n\}$ is a measurable partition with respect to $(\Omega, \mathcal{S}, P)$ such that $P(B_j) > 0$ for all j, then

$$E(f|\sigma(\mathcal{B})) = \sum_{j=1}^{n} \left(\frac{1}{P(B_j)} \int_{B_j} f(\omega)\, \mathrm{d}P(\omega)\right) \chi_{B_j},$$

where $\sigma(\mathcal{B})$ is a σ-algebra generated by $\mathcal{B}$.

In what follows, we suppose that the tribe $\mathcal{T}$, the σ-additive state s on $\mathcal{T}$ and a probability measure P on $\mathcal{S}_0(\mathcal{T})$ connected with s via (10.2) are fixed.

Proposition 10.1. Let $\mathcal{A} = \{a_1, \dots, a_m\}$ and $\mathcal{B} = \{\chi_{B_1}, \dots, \chi_{B_n}\}$ be a partition and a Boolean partition of a tribe $\mathcal{T}$ with a σ-additive state s. Let $\sigma(\mathcal{B})$ be a σ-algebra generated by $\{B_1, \dots, B_n\}$ in $\mathcal{S}_0(\mathcal{T})$. Then

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{m} \int_\Omega \phi\left(E(a_i|\sigma(\mathcal{B}))\right) \,\mathrm{d}P.$$

Proof. If χ_B is a central element of $\mathcal{T}$ and a an arbitrary element of $\mathcal{T}$, then $a \wedge \chi_B = a \cdot \chi_B$. Therefore, if $\mathcal{C} = \{c_{ij}\}$ is a (unique) Riesz refinement of $\mathcal{A}$ and $\mathcal{B}$, we have $c_{ij} = a_i \wedge \chi_{B_j} = a_i \cdot \chi_{B_j}$ and we can check

$$\begin{aligned}
H(\mathcal{A}|\mathcal{B}) &= H_\mathcal{C}(\mathcal{A}|\mathcal{B}) = \sum_{ij} \left\{ s(\chi_{B_j})\phi\left(\frac{s(c_{ij})}{s(\chi_{B_j})}\right) : \ s(\chi_{B_j}) > 0 \right\} \\
&= \sum_{ij} s(\chi_{B_j})\phi\left(\frac{s(a_i \cdot \chi_{B_j})}{s(\chi_{B_j})}\right) \\
&= \sum_{ij} \int_{B_j} \phi\left(\frac{s(a_i \cdot \chi_{B_j})}{s(\chi_{B_j})}\right) \mathrm{d}P \\
&= \sum_{ij} \int_{B_j} \phi\left(\frac{1}{P(B_j)} \int_{B_j} a_i \,\mathrm{d}P\right) \mathrm{d}P \\
&= \sum_{i} \int_{\Omega} \left(\sum_j \phi\left(\frac{1}{P(B_j)} \int_{B_j} a_i \,\mathrm{d}P\right) \chi_{B_j}\right) \mathrm{d}P \\
&= \sum_{i} \int_{\Omega} \phi\left(\sum_j \left(\frac{1}{P(B_j)} \int_{B_j} a_i \,\mathrm{d}P\right) \chi_{B_j}\right) \mathrm{d}P \\
&= \sum_i \phi(E(a_i|\sigma(\mathcal{B})))\,\mathrm{d}P.
\end{aligned}$$

□

In the next theorem we will identify χ_B with B if necessary.

Theorem 10.2. Let $(\mathcal{T}, s, T)$ be a dynamical system, where $\mathcal{T}$ is a tribe of fuzzy sets of a set $\Omega \neq \emptyset$ and s is a σ-additive state on $\mathcal{T}$. Let $\mathcal{B}$ be a Boolean partition of Ω such that $\sigma(\bigcup_n T^n(\mathcal{B})) = \mathcal{S}_0(\mathcal{T})$. Then, for every partition $\mathcal{A} = \{a_1, \dots, a_m\}$, we have

$$\begin{aligned}
h_*^{\mathcal{R}}(\mathcal{A}, T) &\leq h_*^{\mathcal{R}}(\mathcal{B}, T) + \int_\Omega \left(\sum_{i=1}^{m} \phi(a_i)\right) \mathrm{d}P, \\
h_{\mathcal{R}}^*(\mathcal{A}, T) &\leq h_{\mathcal{R}}^*(\mathcal{B}, T) + \int_\Omega \left(\sum_{i=1}^{m} \phi(a_i)\right) \mathrm{d}P.
\end{aligned}$$

Proof. The condition $\sigma(\bigcup_n T^n(\mathcal{B})) = \mathcal{S}_0(\mathcal{T})$ implies that every $T^n(\mathcal{B})$ is a Boolean partition, too. Put $\mathcal{B}_n = \bigvee_{i=0}^{n-1} T^i(\mathcal{B})$ for any $n \geq 1$. Due to the condition, we have $\sigma(\mathcal{B}_n) \nearrow \mathcal{S}_0(\mathcal{T})$.

It is easy to show that $h_*^{\mathcal{R}}(\mathcal{B},T) = h_*^{\mathcal{R}}(\bigvee_{j=0}^{k} T^j(\mathcal{B}),T)$ for any k. Therefore by (7.19) of [2], we have

$$h_*^{\mathcal{R}}(\mathcal{A},T) \leq h_*^{\mathcal{R}}(\mathcal{B},T) + H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}_n).$$

Using the martingale convergence theorem and Proposition 10.1, it is possible to show that:

$$\lim_n H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}_n) = \lim_n \sum_{i=1}^{m} \int_\Omega \phi\left(E(a_i|\sigma(\mathcal{B}_n))\right) \mathrm{d}P = \sum_{i=1}^{m} \int_\Omega \phi\left(E(a_i|\sigma(\mathcal{S}_0(\mathcal{T})))\right) \mathrm{d}P$$
$$= \sum_{i=1}^{m} \int_\Omega \phi(a_i)\, \mathrm{d}P = \int_\Omega \left(\sum_{i=1}^{m} \phi(a_i)\right) \mathrm{d}P.$$

In a similar way we obtain also the second inequality. □

As a corollary of Theorem 10.2 we have the known result of Kolmogorov–Sinai which determines $h_B(T)$, namely

$$h_B(T) = h_*^{\mathcal{R}}(\mathcal{B},T) = h_{\mathcal{R}}^*(\mathcal{B},T)$$

if $\mathcal{B}$ satisfies the conditions of Theorem 10.2 because $\phi(a_i) = 0$ for any central element a_i and then $h_*^{\mathcal{R}}(\mathcal{A},T) \leq h_*^{\mathcal{R}}(\mathcal{B},T)$ for any Boolean partition $\mathcal{A} = \{a_i\}$.

11. ENTROPY ON σ–COMPLETE MV–ALGEBRAS

In this Section, we generalize Theorem 10.2 from a tribe to any σ-complete MV-algebra, Theorem 11.3. The basic tool is the Loomis–Sikorski representation of a σ-complete MV-algebra as a σ-homomorphic image of a tribe [4, 12]. We recall that an MV-algebra M is *semisimple* if it is isomorphic with some Bold algebra.

Denote by $\mathrm{Ext}_{\mathcal{S}}(M)$ the set of all extremal states on M. Then by [9, Thm. 6.1.30],

$$\mathrm{Ext}_{\mathcal{S}}(M) \neq \emptyset,$$

and it is a compact Hausdorff space with respect to the weak topology of states (i. e., $m_\alpha \to m$ iff $m_\alpha(a) \to m(a)$ for any $a \in M$), and any state m on M is in the closure of the convex hull of $\mathrm{Ext}_{\mathcal{S}}(M)$. If $a \in M$, then by $\hat{a}$ we denote the function defined on $\Omega = \mathrm{Ext}_{\mathcal{S}}(M)$ such that $\hat{a}(\omega) = \omega(a)$, $\omega \in \Omega$. Then

$$\widehat{M} = \{\hat{a} : a \in M\}$$

is the Bold algebra isomorphic with M.

It is evident, that if $\mathcal{A} = \{a_1, \dots, a_m\}$ is a partition in M so is $\widehat{\mathcal{A}} = \{\hat{a}_1, \dots, \hat{a}_m\}$ in $\widehat{M}$, and $\mathcal{A}$ is Boolean if and only if $\widehat{\mathcal{A}} = \{\chi_{A_1}, \dots, \chi_{A_m}\}$ where $A_1, \dots, A_m$ is a partition in Ω.

If M is a σ-complete MV-algebra, then M is semisimple and $\mathrm{Ext}_{\mathcal{S}}(M)$ is basically disconnected (i. e. the closure of every open F_σ subset of Ω is open).

Denote by $\mathcal{T} = \mathcal{T}(M)$ the tribe of fuzzy sets on Ω which is generated by $\widehat{M}$. If $f, g \in \mathcal{T}$, we write $f \sim g$ if $\{\omega : f(\omega) \neq g(\omega)\}$ is a meager set. Then $\mathcal{I} = \{f \in \mathcal{T} : f \sim \hat{0}\}$ is a σ-ideal of $\mathcal{T}$, and the mapping $h : \mathcal{T} \to M$ defined by $h(f) = a$ if and only if $f \sim \hat{a}$ $(f \in \mathcal{T}, a \in M)$, is a surjective σ-homomorphism [4] such that h maps $\mathcal{S}_0(\mathcal{T})$ onto $B(M)$, the Boolean elements of M [5], and $\mathcal{T}/\mathcal{I} \cong M$. That is $h(f) = a$ if and only if $\{\omega : f(\omega) \neq \hat{a}(\omega)\}$ is meager. This is a base of the original proof of Loomis–Sikorski's theorem for σ-complete MV-algebras.

The triplet $(\Omega, \mathcal{T}, h)$ is said to be the *canonical representation* of the σ-complete MV-algebra M.

For example, if M is a σ-complete MV-algebra which is weakly divisible, then M has no discrete state. Therefore, $\widehat{M}$ consists of all continuous functions defined on $\Omega = \mathrm{Ext}_{\mathcal{S}}(M)$, $\mathcal{T}$ is the set of all Baire measurable fuzzy sets on Ω, and $C(\mathcal{T})$ is the Baire σ-algebra, i. e. the σ-algebra generated by compact G_δ sets on Ω, or equivalently, by $\{f^{-1}([a,\infty)) : f \in C(\Omega),\ a \in \mathbb{R}\}$ [9, Prop. 7.1.11].

If now s is a σ-additive state on the σ-complete MV-algebra M, then the mapping $\hat{s}$ defined on $\mathcal{T}$ by

$$\hat{s}(f) = s(h(f)), \quad f \in \mathcal{T}, \tag{11.1}$$

is a σ-additive state on $\mathcal{T}$ and by the previous section, there is a unique probability measure P on $\mathcal{S}_0(\mathcal{T})$ such that $\hat{s}(f) = \int_\Omega f(\omega)\,\mathrm{d}P(\omega)$, $f \in \mathcal{T}$.

Therefore, we have the following statements concerning entropies taken in M and $\mathcal{T}$, respectively.

(i) $H(\mathcal{A}) = H(\widehat{\mathcal{A}})$.

(ii) If $\mathcal{C} \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$, then $\widehat{\mathcal{C}} \in \mathrm{Ref}_{\mathcal{R}}(\widehat{\mathcal{A}}_1, \dots, \widehat{\mathcal{A}}_n)$.
If $\mathcal{D} \in \mathrm{Ref}_{\mathcal{R}}(\widehat{\mathcal{A}}_1, \dots, \widehat{\mathcal{A}}_n)$, then $h(\mathcal{D}) \in \mathrm{Ref}_{\mathcal{R}}(\mathcal{A}_1, \dots, \mathcal{A}_n)$. $H(\mathcal{D}) = H(h(\mathcal{D}))$.

(iii)

$$\begin{aligned} H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) &:= H_*^{\mathcal{R}}(\widehat{\mathcal{A}}_1 \vee \cdots \vee \widehat{\mathcal{A}}_n), \\ H^*_{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n) &:= H^*_{\mathcal{R}}(\widehat{\mathcal{A}}_1 \vee \cdots \vee \widehat{\mathcal{A}}_n). \end{aligned}$$

(iv) $H_{\mathcal{C}}(\mathcal{A}|\mathcal{B}) = H_{\widehat{\mathcal{C}}}(\widehat{\mathcal{A}}|\widehat{\mathcal{B}})$.

(v)

$$\begin{aligned} H_*^{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) &= H_*^{\mathcal{R}}(\widehat{\mathcal{A}}_1 \vee \cdots \vee \widehat{\mathcal{A}}_n|\widehat{\mathcal{B}}_1 \vee \cdots \vee \widehat{\mathcal{B}}_m), \\ H^*_{\mathcal{R}}(\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_n|\mathcal{B}_1 \vee \cdots \vee \mathcal{B}_m) &= H^*_{\mathcal{R}}(\widehat{\mathcal{A}}_1 \vee \cdots \vee \widehat{\mathcal{A}}_n|\widehat{\mathcal{B}}_1 \vee \cdots \vee \widehat{\mathcal{B}}_m). \end{aligned}$$

Let T be an s-preserving transformation on M, and let $\hat{T}$ be the mapping from $\widehat{M}$ into itself induced by T, i. e., $\hat{T}(\hat{a}) = \widehat{T(a)}$. Then $\hat{s}(\hat{T}(\hat{a})) = \hat{s}(\hat{a}) = s(a) = s(T(a))$.

(vi)

$$\begin{aligned} H^n_*(\widehat{\mathcal{A}},\hat{T})_{\mathcal{R}} &:= H^{\mathcal{R}}_*(\widehat{\mathcal{A}}\vee\hat{T}(\widehat{\mathcal{A}})\vee\cdots\vee\hat{T}^{n-1}(\widehat{\mathcal{A}})),\\ H^*_n(\widehat{\mathcal{A}},\hat{T})_{\mathcal{R}} &:= H^*_{\mathcal{R}}(\widehat{\mathcal{A}}\vee\hat{T}(\widehat{\mathcal{A}})\vee\cdots\vee\hat{T}^{n-1}(\widehat{\mathcal{A}})). \end{aligned}$$

(vii)

$$\begin{aligned} h^{\mathcal{R}}_*(\mathcal{A},T) &= h^{\mathcal{R}}_*(\widehat{\mathcal{A}},\hat{T}) := \lim_n \frac{1}{n}H^n_*(\widehat{\mathcal{A}},\hat{T})_{\mathcal{R}},\\ h^*_{\mathcal{R}}(\widehat{\mathcal{A}},\hat{T}) &= h^*_{\mathcal{R}}(\widehat{\mathcal{A}},\hat{T}) := \lim_n \frac{1}{n}H^*_n(\widehat{\mathcal{A}},\hat{T})_{\mathcal{R}}. \end{aligned}$$

(viii)

$$\begin{aligned} h^{\mathcal{R}}_*(T) &= h^{\mathcal{R}}_*(\hat{T}) := \sup\{h^{\mathcal{R}}_*(\widehat{\mathcal{A}},\hat{T})\},\\ h^*_{\mathcal{R}}(T) &= h^*_{\mathcal{R}}(\hat{T}) := \sup\{h^*_{\mathcal{R}}(\widehat{\mathcal{A}},\hat{T})\}. \end{aligned}$$

Proposition 11.1. Let $\mathcal{A} = \{a_1,\dots,a_m\}$ and $\mathcal{B}$ be a partition and a Boolean partition in a σ-complete MV-algebra M with a σ-additive state s. Let $\sigma(\widehat{\mathcal{B}})$ be a σ-algebra generated by $\widehat{\mathcal{B}}$ in $\mathcal{S}_0(\mathcal{T})$. Then

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{m}\int_\Omega \phi\left(E(\hat{a}_i|\sigma(\widehat{\mathcal{B}}))\right)\,\mathrm{d}P. \tag{11.2}$$

Proof. Let $\mathcal{B} = \{b_1,\dots,b_n\}$ and let $\mathcal{C} = \{c_{ij} = a_i\wedge b_j\}$ be a unique Riesz refinement of $\mathcal{A}$ and $\mathcal{B}$. Then $\widehat{\mathcal{C}} = \{\hat{c}_{ij} = \hat{a}_i\wedge\hat{b}_j\}$ is a unique Riesz refinement of $\widehat{\mathcal{A}}$ and Boolean $\widehat{\mathcal{B}}$ in $\widehat{M}$ as well as in $\mathcal{T}$. Take the σ-additive state $\hat{s}$ on $\mathcal{T}$, then $H(\mathcal{A}|\mathcal{B}) = H(\widehat{\mathcal{A}}|\widehat{\mathcal{B}})$, and applying Proposition 10.1, we obtain (11.2). □

We now extend the result of Theorem 10.2 from tribes to σ-complete MV-algebras. This result extends also the result of Riečan [15, Thm. 10.3.4] which was proved only for special tribes.

We recall that if M is a σ-complete MV-algebra, then $C(M)$, the center of M, is a Boolean σ-algebra. If A is a subset of $C(M)$, by $\sigma(A)$ we mean the σ-algebra generated by A in $C(M)$.

Lemma 11.2. Let M be a σ-complete MV-algebra and let $(\Omega,\mathcal{T},h)$ be the canonical representation of M. Then the σ-algebra generated by $C(\widehat{M})$ is equal to $\mathcal{S}_0(\mathcal{T})$.

Proof. It is clear that $C(\widehat{M}) = \{\hat{a} : a\in C(M)\}$. Since $\Omega := \mathrm{Ext}_{\mathcal{S}}(M)$, then $\hat{a} = \chi_A$ for some $A\in\mathcal{S}_0(\mathcal{T})$, and in addition, A is a clopen subset of Ω.

We define by $\mathcal{S}_0$ the set of all such that $A\in\mathcal{S}_0(\mathcal{T})$. We claim that $\mathcal{S}_0 = \mathcal{S}_0(\mathcal{T})$.

According to [11, Thm. 8.14], the space $\mathrm{Ext}_{\mathcal{S}}(M)$ is homeomorphic with the set of all extremal states, $\mathrm{Ext}_{\mathcal{S}}(C(M))$, on the Boolean σ-algebra $C(M)$. In addition, any restriction of $m \in \mathrm{Ext}_{\mathcal{S}}(M)$ gives an element of $\mathrm{Ext}_{\mathcal{S}}(C(M))$, and conversely, any element of $\mathrm{Ext}_{\mathcal{S}}(C(M))$ can be uniquely extended to an extremal state on M, and this correspondence defines the mentioned homeomorphism.

Consequently, by the proof of the classical Loomis–Sikorski theorem, $\mathcal{S}_0$ is a σ-algebra of crisp subsets of Ω, and due to the definition of h, its restriction onto $\mathcal{S}_0$ defines a σ-homomorphism from $\mathcal{S}_0$ onto $C(M)$.

It is evident that $\mathcal{S}_0 \subseteq \mathcal{S}_0(\mathcal{T})$. On the other hand, $A \in \mathcal{S}_0(\mathcal{T})$ iff $\chi_A \in \mathcal{T}$, i.e., $h(\chi_A) = a$ for some $a \in M$. Since χ_A is a Boolean element of $\mathcal{T}$, so is a in M. Consequently, $A \in \mathcal{S}_0$. □

Theorem 11.3. Let (M, s, T) be a dynamical system, where M is a σ-complete MV-algebra and s is a σ-complete state on M. Let $\mathcal{B}$ be a Boolean partition of M such that $\sigma(\bigcup_n T^n(\mathcal{B})) = C(M)$. Then for every partition $\mathcal{A} = \{a_1, \dots, a_m\}$, we have

$$\begin{aligned} h_*^{\mathcal{R}}(\mathcal{A},T) &\le h_*^{\mathcal{R}}(\mathcal{B},T) + \int_\Omega \left(\sum_{i=1}^m \phi(\hat{a}_i)\right) \mathrm{d}P, \\ h_{\mathcal{R}}^*(\mathcal{A},T) &\le h_{\mathcal{R}}^*(\mathcal{B},T) + \int_\Omega \left(\sum_{i=1}^m \phi(\hat{a}_i)\right) \mathrm{d}P. \end{aligned}$$

Proof. The condition $\sigma(\bigcup_n T^n(\mathcal{B})) = C(M)$ implies that every $T^n(\mathcal{B})$ is a Boolean partition, too. So is $\hat{T}^n(\widehat{\mathcal{B}})$ in $\mathcal{T}$. Put $\widehat{\mathcal{B}}_n = \bigvee_{i=0}^{n-1} \hat{T}^i(\widehat{\mathcal{B}})$ for any $n \ge 1$. Due to Lemma 11.2, we have $\sigma(\widehat{\mathcal{B}}_n) \nearrow \mathcal{S}_0(\mathcal{T})$. Using properties of entropies in M and $\mathcal{T}$, (i) – (vii) from the previous section, we can prove following the proof of Theorem 10.2 the following inequalities

$$h_*^{\mathcal{R}}(\mathcal{A},T) = \hat{h}_*^{\mathcal{R}}(\widehat{\mathcal{A}},\hat{T}) \le \hat{h}_*^{\mathcal{R}}(\widehat{\mathcal{B}},\hat{T}) + H_*^{\mathcal{R}}(\widehat{\mathcal{A}}|\widehat{\mathcal{B}}_n) = h_*^{\mathcal{R}}(\mathcal{B},T) + H_*^{\mathcal{R}}(\mathcal{A}|\mathcal{B}_n).$$

Using the martingale convergence theorem and applying Proposition 11.1, we can prove in the same way as in Theorem 10.2 that

$$\lim_n H_*^{\mathcal{R}}(\widehat{\mathcal{A}}|\widehat{\mathcal{B}}_n) = \int_\Omega \left(\sum_{i=1}^m \phi(\hat{a}_i)\right) \mathrm{d}P.$$

This implies the inequality in question because $\hat{h}_*^{\mathcal{R}}(\widehat{\mathcal{B}},\hat{T}) = h_*^{\mathcal{R}}(\mathcal{B},T)$. □

We note that all above results from Sections 10–11 do not need any concept of a product MV-algebra while the refinement of a partition with a Boolean partition is unique and it coincides with the product of these two partitions (if the product is defined on the MV-algebra). The same is true also for effect algebras (product effect algebras are introduced in [8]). In particular, we have a solution to the Problem 7 from [14] where the authors are asking how we can proceed with entropy not assuming the product on the MV-algebra.

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