Two-dimensional self-avoiding walks

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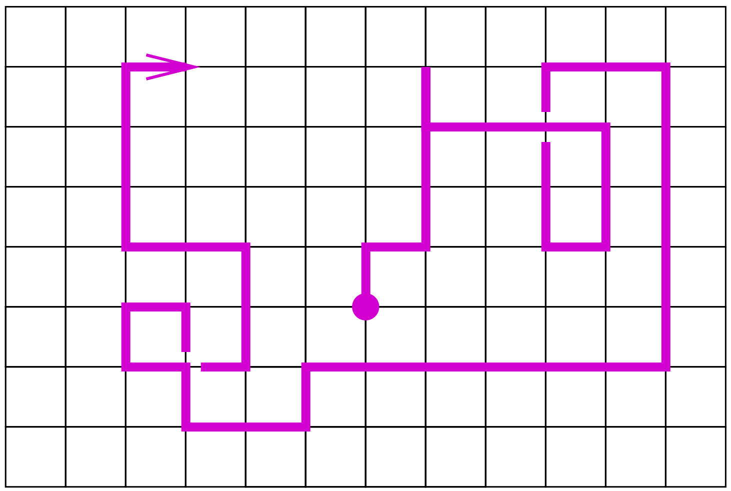
Self-avoiding walks (SAWs)

A walk
with $n = 47$ steps



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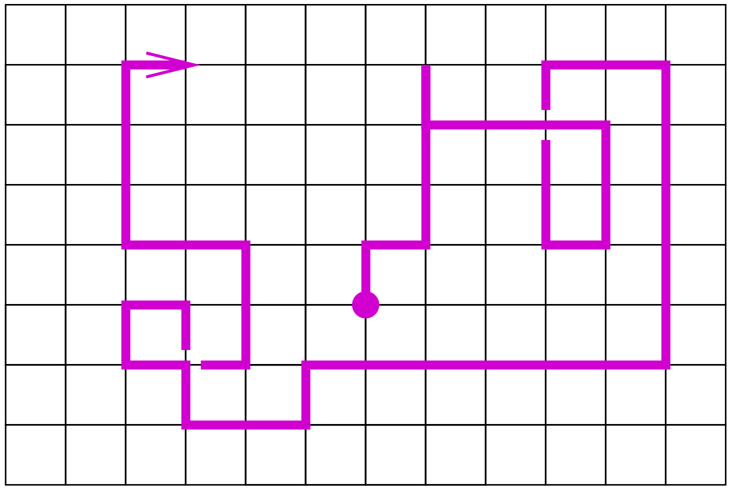
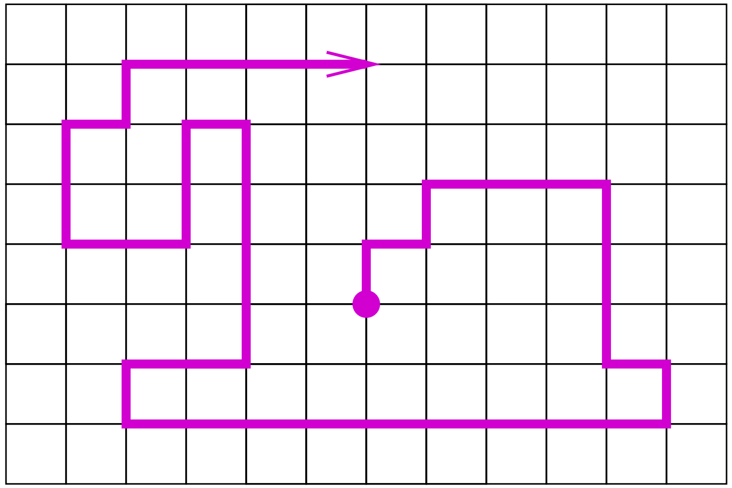


A self-avoiding walk
with $n = 40$ steps



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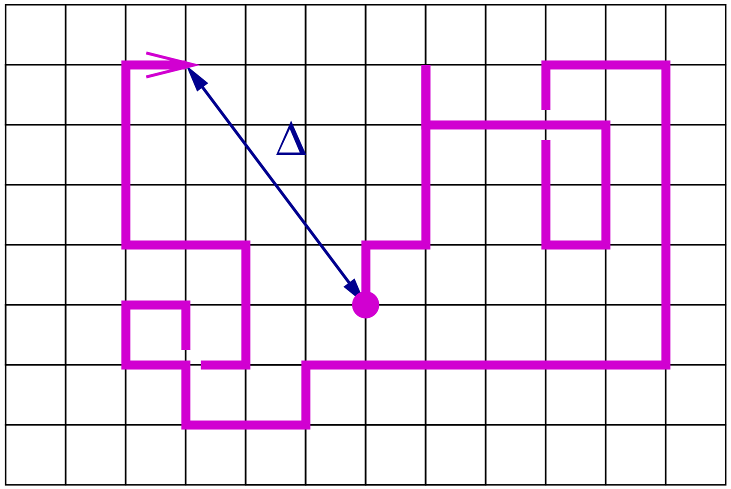

End-to-end distance:

$$\Delta = \sqrt{3^2 + 4^2} = 5$$

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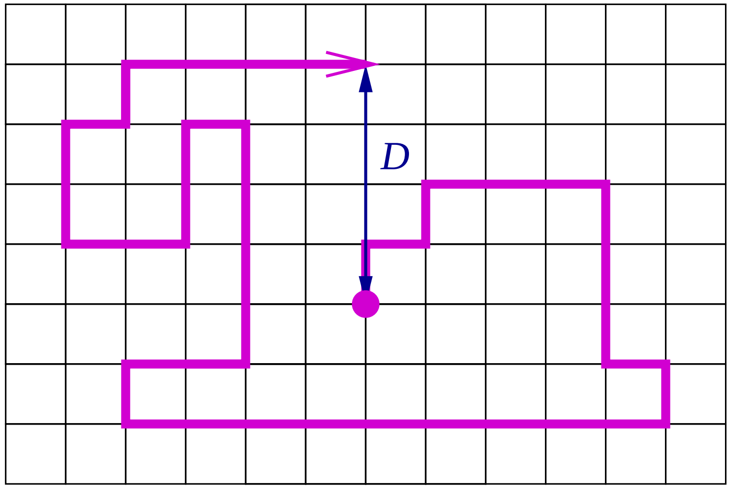


End-to-end distance:

$$D = 4$$

Some natural questions

General walks

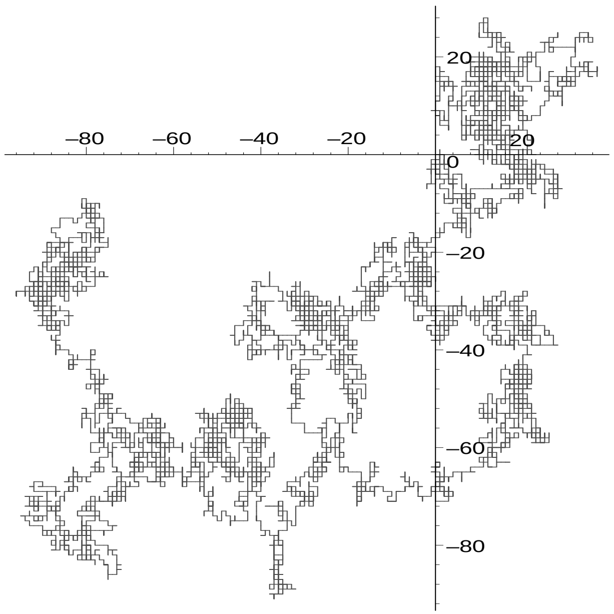
• Number:

$$a_n = 4^n$$

• End-to-end distance:

$$\mathbb{E}(\Delta_n) \sim (\kappa)\, n^{1/2}$$

• Limiting object: The (uniform) random walk converges to the Brownian motion



Some natural (but hard) questions

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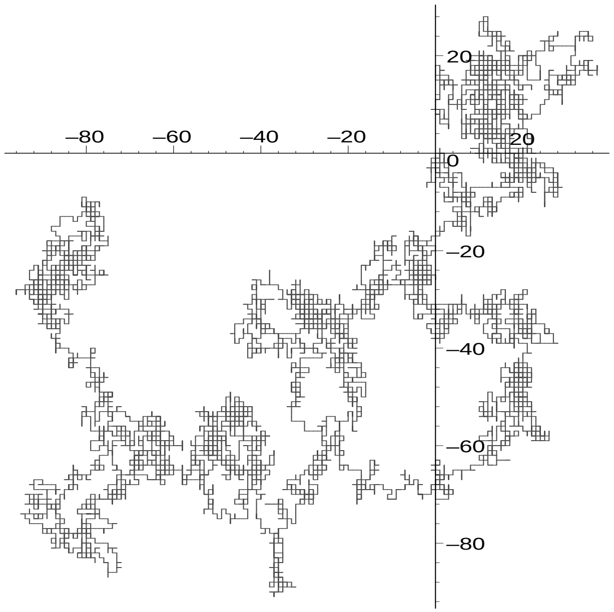


Self-avoiding walks

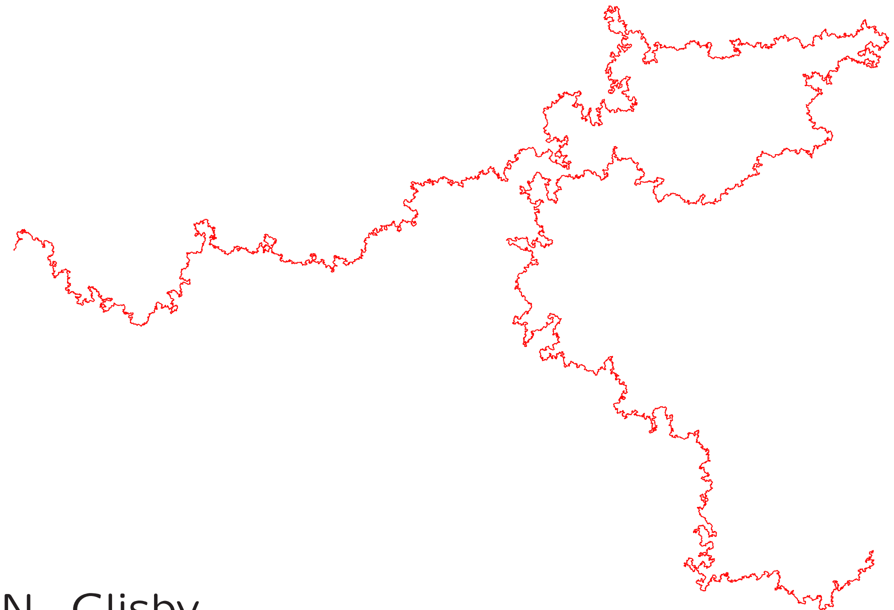
- Number:

$$c_n = ?$$

- End-to-end distance:

$$\mathbb{E}(D_n) \sim ?$$

- Limit of the random uniform SAW?



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The number of n-step SAWs: predictions vs. theorems

- Predicted: The number of n-step SAWs behaves asymptotically as:

$$c_n \sim \mu^n \, n^\gamma$$

where $\gamma = 11/32$ for all 2D lattices (square, triangular, honeycomb) [Nienhuis 82]

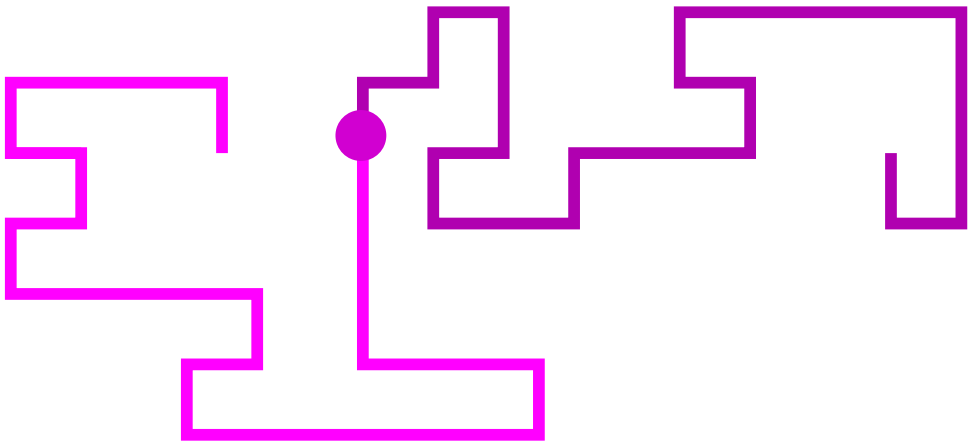
The probabilistic meaning of the exponent γ

- Predicted: The number of n-step SAWs behaves asymptotically as:

$$c_n \sim \mu^n n^\gamma$$

$\Rightarrow$ The probability that two n-step SAWs starting from the same point do not intersect is

$$\frac{c_{2n}}{c_n^2} \sim n^{-\gamma}$$



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- Known: there exists a constant μ, called growth constant, such that

$$c_n^{1/n} \to \mu$$

and a constant α such that

$$\mu^n \leq c_n \leq \mu^n \alpha^{\sqrt{n}}$$

[Hammersley 57], [Hammersley-Welsh 62]

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[Hammersley 57], [Hammersley-Welsh 62]

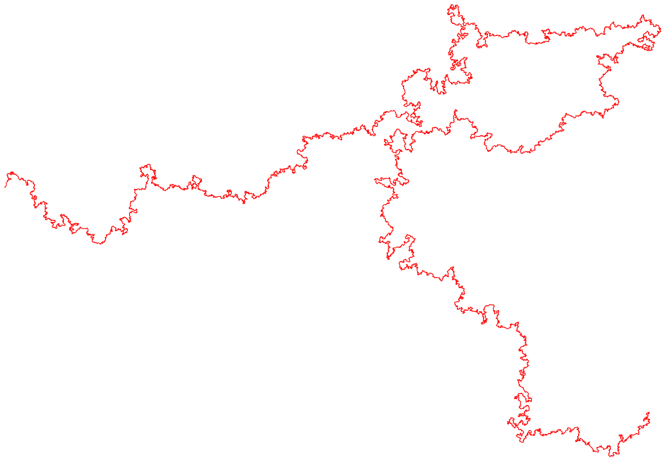
- c_n is only known up to $n = 71$ [Jensen 04]

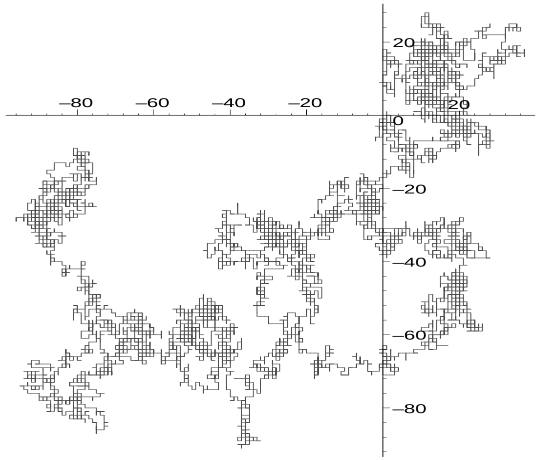
The end-to-end distance: predictions vs. theorems

- Predicted: The end-to-end distance is on average

$\mathbb{E}(D_n) \sim n^{3/4}$ (vs. $n^{1/2}$ for a simple random walk)

[Flory 49, Nienhuis 82]





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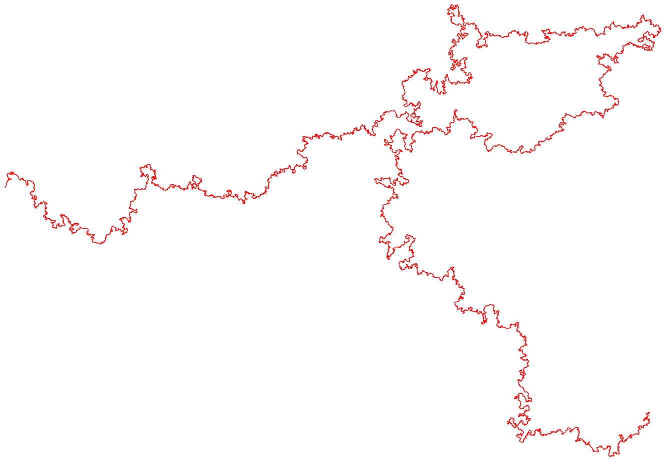
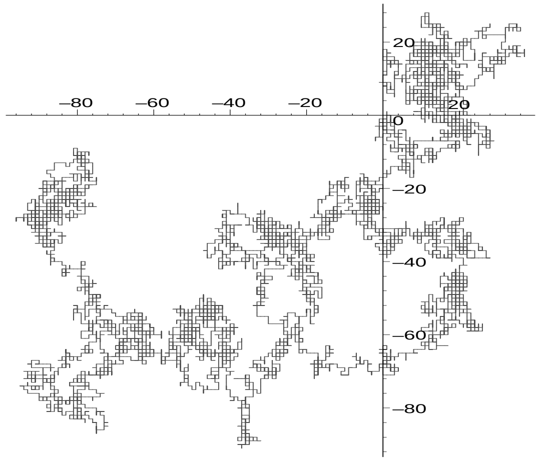




- Known [Madras 2012], [Duminil-Copin & Hammond 2012]:

$$n^{1/4} \leq \mathbb{E}(D_n) \ll n^1$$

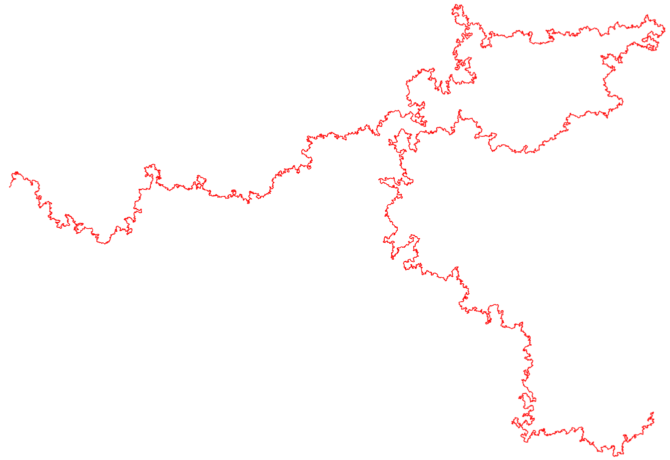
The scaling limit: predictions vs. theorems

• Predicted: The limit of SAW is $\mathrm{SLE}_{8/3}$, the Schramm-Loewner evolution process with parameter 8/3.

• Known: true if the limit of SAW exists and is conformally invariant [Lawler, Schramm, Werner 02]

Confirms the predictions

$$c_n \sim \mu^n n^{11/32} \quad \text{and} \quad \mathbb{E}(D_n) \sim n^{3/4}$$



Outline

I. Self-avoiding walks (SAWs): Generalities, predictions and results

II. The growth constant on honeycomb lattice is $\mu = \sqrt{2+\sqrt{2}}$
[Duminil-Copin & Smirnov 10]

What else?

Outline

I. Self-avoiding walks (SAWs): Generalities, predictions and results

II. The growth constant on honeycomb lattice is $\mu = \sqrt{2+\sqrt{2}}$
[Duminil-Copin & Smirnov 10]

What else?

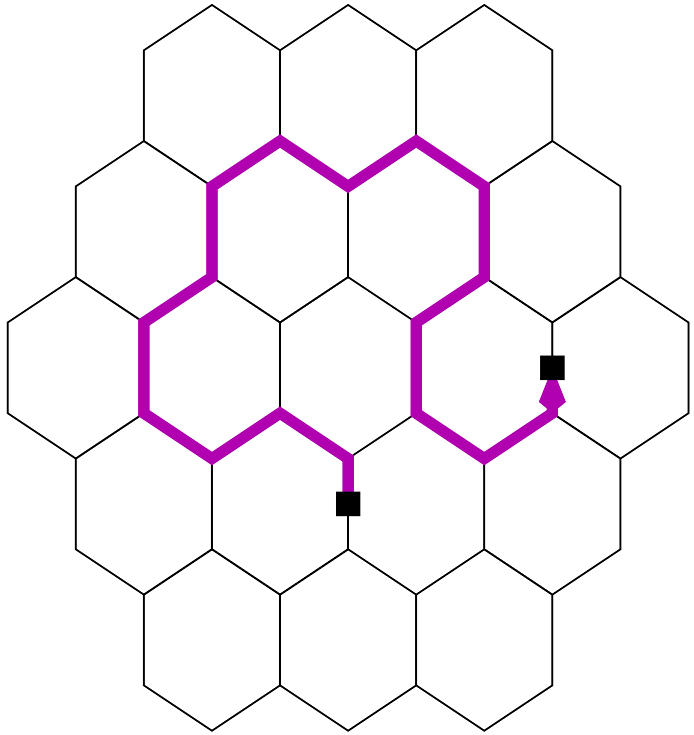
III. The $1+\sqrt{2}$-conjecture: SAWs in a half-plane interacting with the boundary (honeycomb lattice) [Beaton, MBM, Duminil-Copin, de Gier & Guttmann 12]

IV. The ???-conjecture: The mysterious square lattice
(d'après [Cardy & Ikhlef 09])

II. The growth constant

on the honeycomb lattice:

The $\mu = \sqrt{2+\sqrt{2}}$ ex-conjecture



[Duminil-Copin & Smirnov 10]

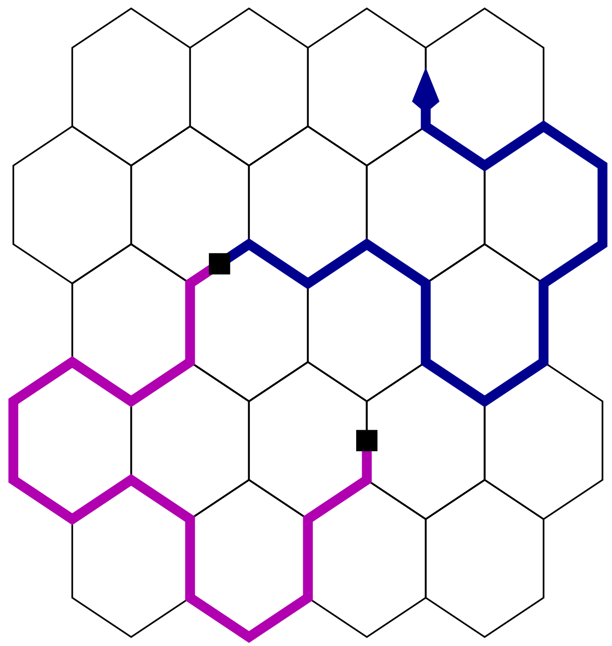
The growth constant

Clearly,

$$c_{m+n} \leq c_m \, c_n$$

$\Rightarrow$ $\lim_n c_n^{1/n}$ exists and

$$\mu := \lim_n c_n^{1/n} = \inf_n c_n^{1/n}$$



Theorem [Duminil-Copin & Smirnov 10]: the growth constant is

$$\mu = \sqrt{2+\sqrt{2}}$$

(conjectured by Nienhuis in 1982)

Growth constants and generating functions

- Let $C(x)$ be the length generating function of SAWs:

$$C(x) = \sum_{n \geq 0} c_n x^n.$$

- The radius of convergence of $C(x)$ is

$$\rho = 1/\mu,$$

where

$$\mu = \lim_n c_n^{1/n}$$

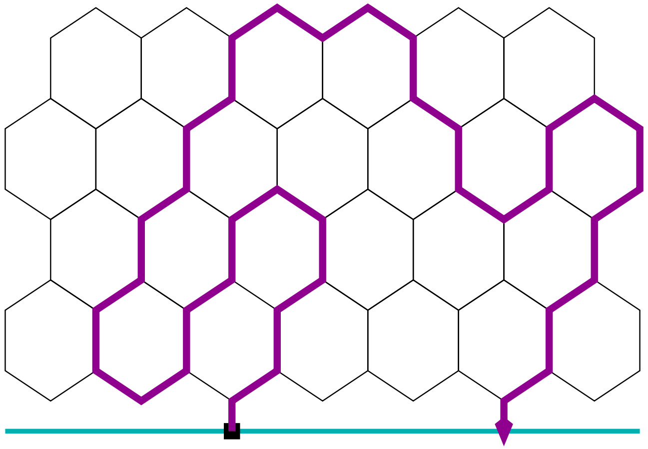
is the growth constant.

- Notation: $x^* := 1/\sqrt{2+\sqrt{2}}$. We want to prove that $\rho = x^*$.

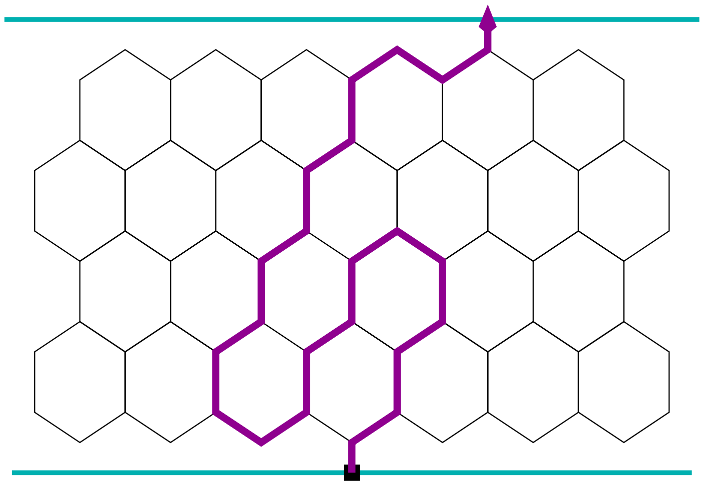
Many families of SAWs have the same radius ρ

For instance...

Arches



Bridges

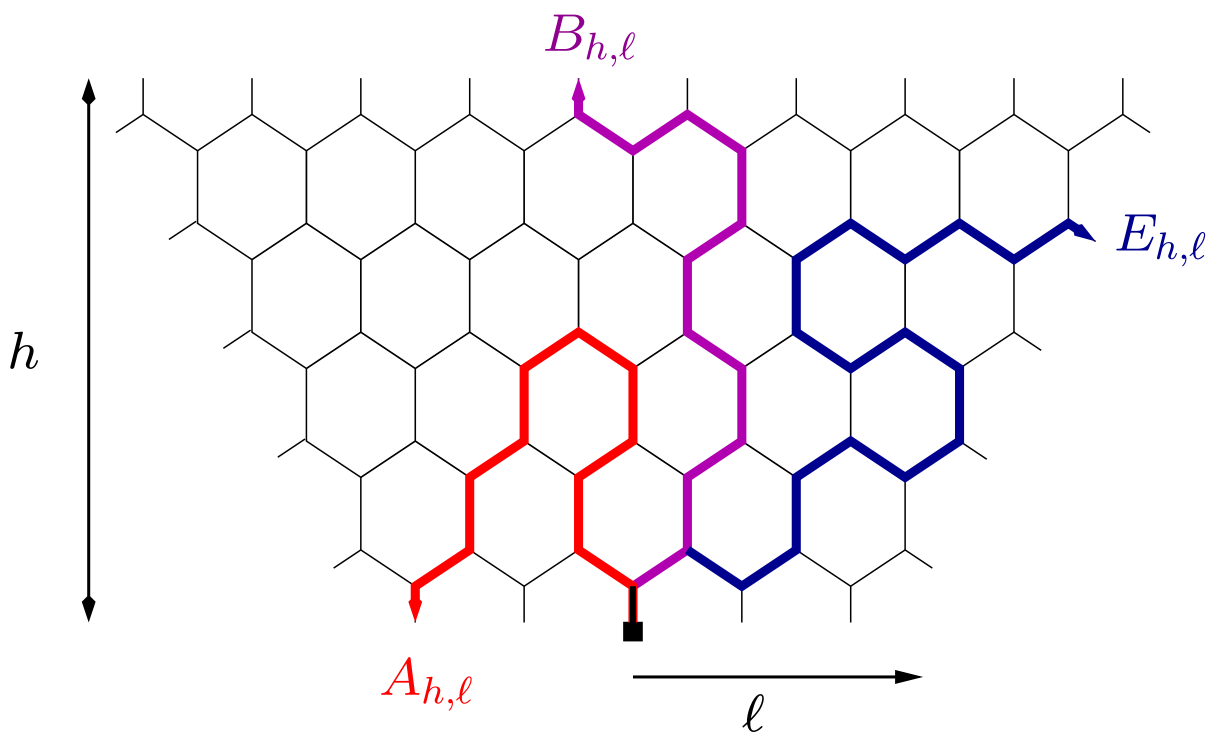


[Hammersley 61]

To prove: $A(x)$ (or $B(x)$) has radius $x^* := 1/\sqrt{2+\sqrt{2}}$.

1. Duminil-Copin and Smirnov's "global" identity

Consider the following finite domain $D_{h,\ell}$.



$A_{h,\ell}$ arches

$B_{h,\ell}$ bridges

$E_{h,\ell}$ ...

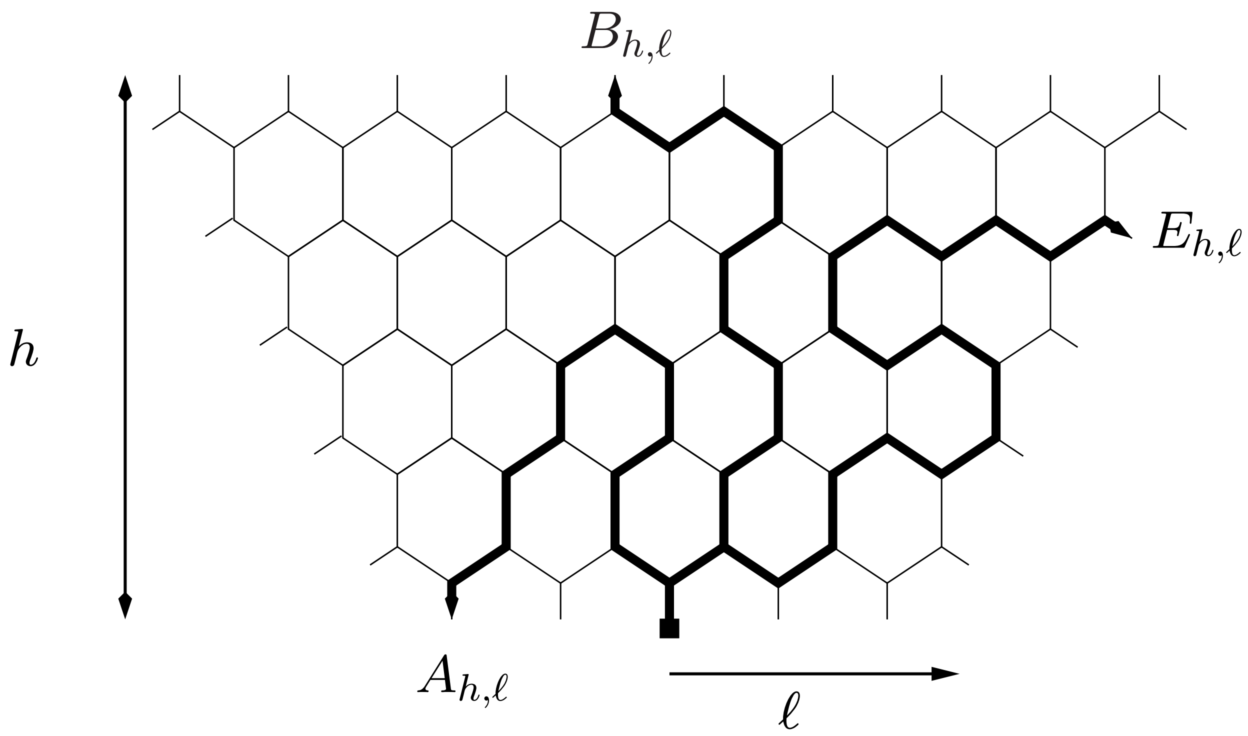
Let $A_{h,\ell}(x)$ (resp. $B_{h,\ell}(x)$, $E_{h,\ell}(x)$) be the generating function of SAWs that start from the origin and end on the bottom (resp. top, right/left) border of the domain $D_{h,\ell}$. These series are polynomials in x.

1. Duminil-Copin and Smirnov's "global" identity

At $x^* = 1/\sqrt{2+\sqrt{2}}$, and for all h and ℓ,

$$\alpha A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \varepsilon E_{h,\ell}(x^*) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$ and $\varepsilon = \frac{1}{\sqrt{2}}$.



$A_{h,\ell}$ arches

$B_{h,\ell}$ bridges

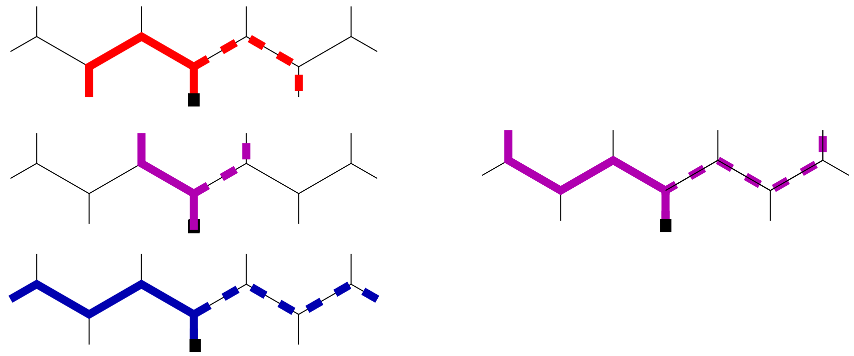
$E_{h,\ell}$ ...

Example: the domain $D_{1,1}$

$$A(x) = 2x^3$$

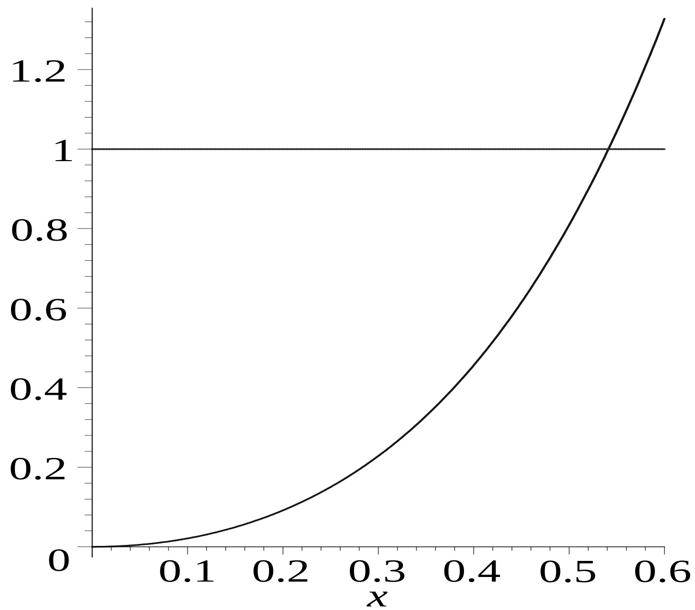
$$B(x) = 2x^2 + 2x^4$$

$$E(x) = 2x^4$$



$$\Longrightarrow \alpha A(x) + B(x) + \varepsilon E(x) = 2x^2 + 2\alpha x^3 + 2x^4(1+\varepsilon)$$

and this polynomial equals 1 at $x^* = 1/\sqrt{2+\sqrt{2}} \simeq 0.54$



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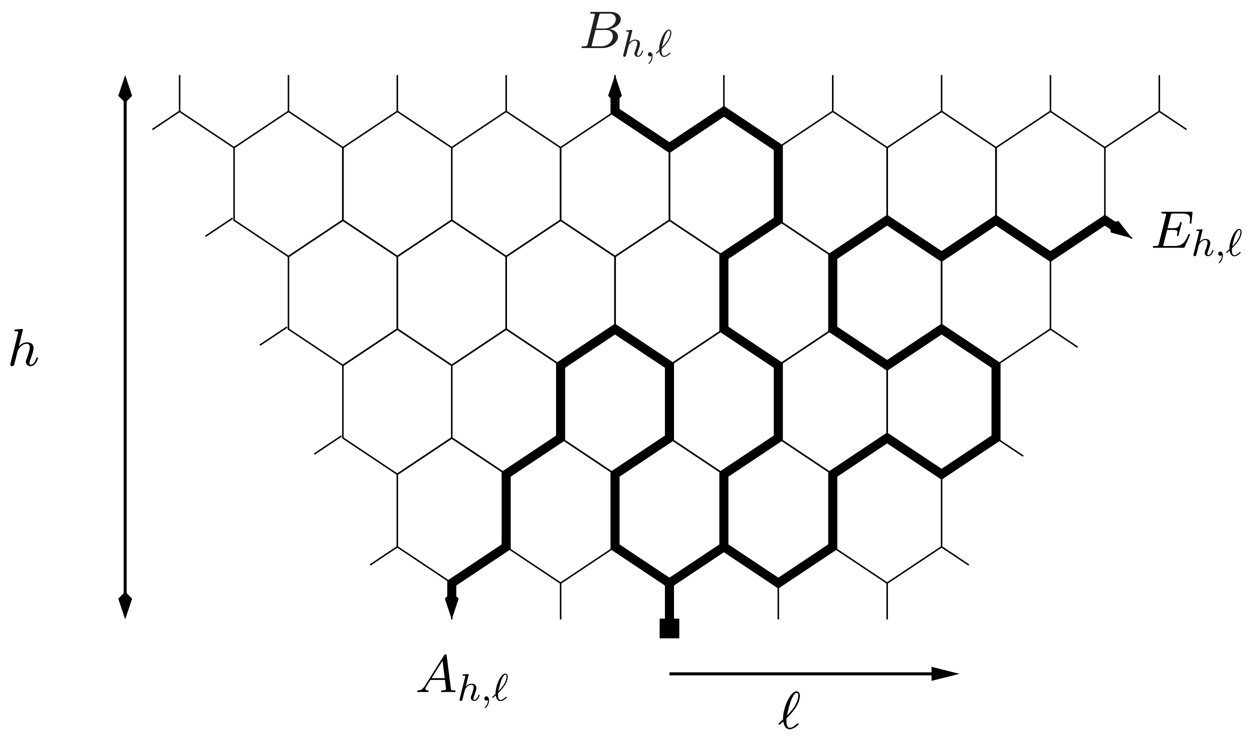


$A_{h,\ell}$ arches

$B_{h,\ell}$ bridges

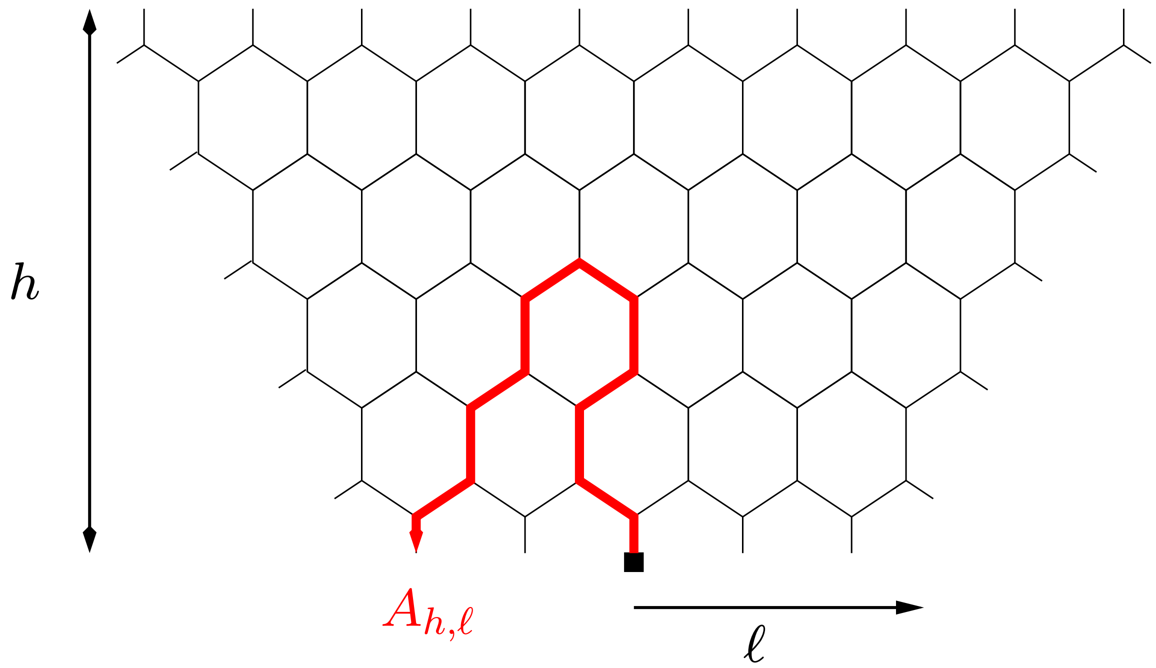
$E_{h,\ell}$ ...

2. A lower bound on ρ

$$\alpha A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \varepsilon E_{h,\ell}(x^*) = 1$$

As h and ℓ tend to infinity, $A_{h,\ell}(x^*)$ counts more and more arches, but remains bounded (by $1/\alpha$): thus it converges, and its limit is the GF $A(x)$ of all arches, taken at $x = x^*$.

This series is known to have radius ρ. Since it converges at x^*, we have $x^* \leq \rho$.



3. An upper bound on ρ

$$\alpha A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \varepsilon E_{h,\ell}(x^*) = 1$$

...

$\rho \leq x^*$: Not much harder. Thus:

$$\rho = x^* = 1/\sqrt{2+\sqrt{2}}$$

4. Where does the global identity come from?

$$\frac{\sqrt{2-\sqrt{2}}}{2}\, A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}}\, E_{h,\ell}(x^*) = 1$$

From a local identity that is re-summed over all vertices of the domain.

A local identity

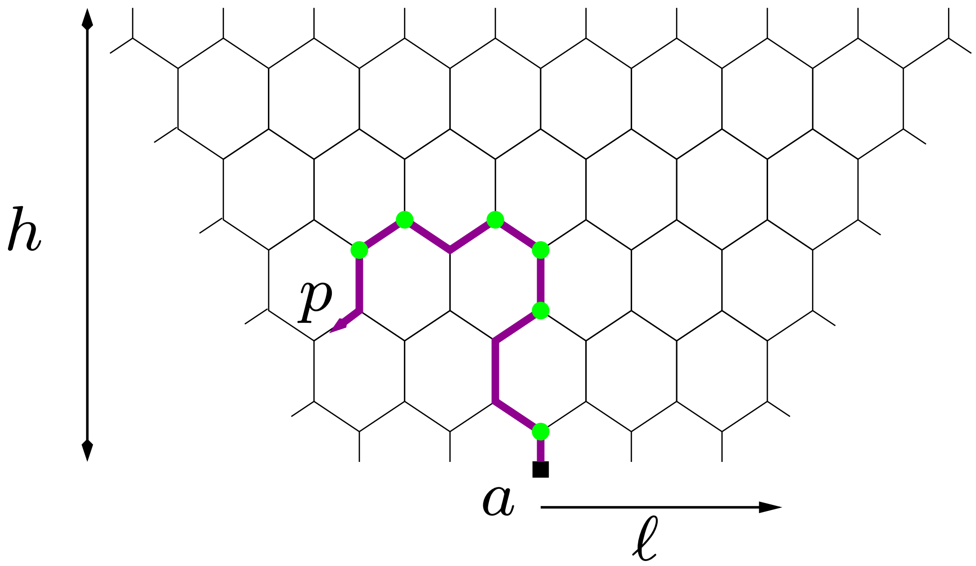
Let $D \equiv D_{h,\ell}$ be our domain, a the origin of the walks, and p a mid-edge in the domain. Let

$$F(p) \equiv F(x,\theta;p) = \sum_{\omega:a \rightsquigarrow p} x^{|\omega|} e^{i\theta W(\omega)},$$

where $|\omega|$ is the length of ω, and $W(\omega)$ its winding number:

$$W(\omega) = \text{left turns} - \text{right turns}.$$

Example:



$$W(\omega) = 6 - 4 = 2$$

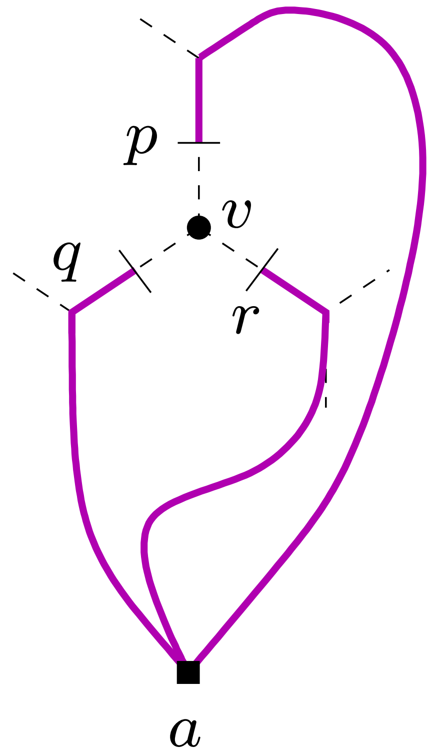
A local identity

Let

$$F(p) \equiv F(x, \theta; p) = \sum_{\omega: a \rightsquigarrow p \text{ in } D} x^{|\omega|} e^{i\theta W(\omega)},$$

If p, q and r are the 3 mid-edges around a vertex v of the honeycomb lattice, then, for $x = x^*$ and $\theta = -5\pi/24$,

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$



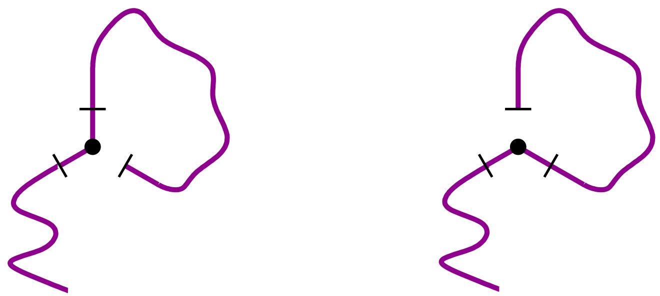
Rem: $(p - v)$ is here a complex number!

First Kirchhoff law

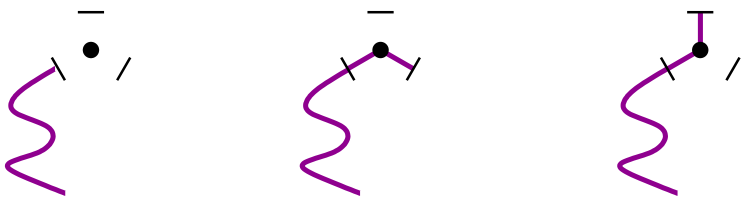
A local identity

Proof: Group walks that only differ in the neighborhood of v:

- Walks that visit all mid-edges:



- Walks that only visit one or two mid-edges:



The contribution of all walks in a group is zero.

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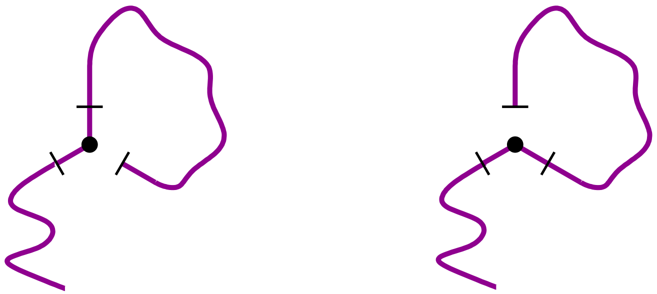

$$e^{-i\pi/3}e^{-4i\theta} + ie^{4i\theta} = 0$$

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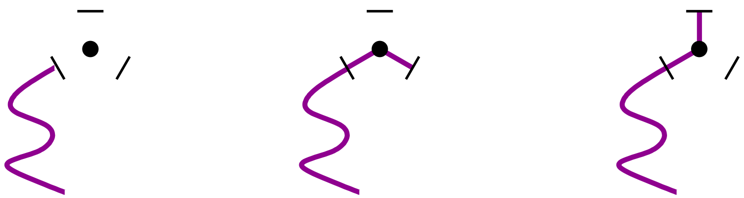


$$e^{-2i\pi/3} + e^{-i\pi/3}e^{-i\theta}x + ie^{i\theta}x = 0$$

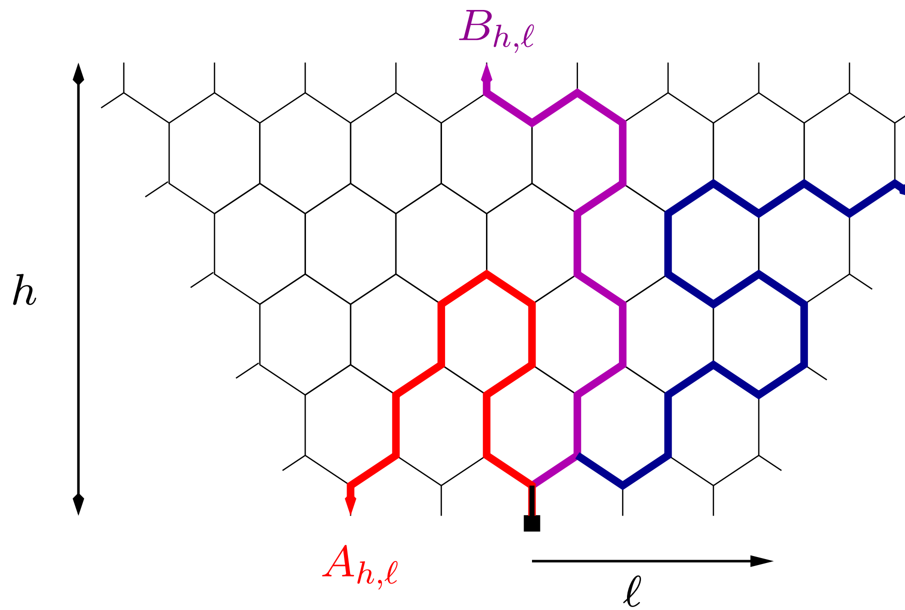
The contribution of all walks in a group is zero.

Proof of the global identity

Sum the local identity

$$(p-v)F(p)+(q-v)F(q)+(r-v)F(r)=0$$

over all vertices v of the domain $D_{h,\ell}$.



- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.
- The domain has a right-left symmetry.

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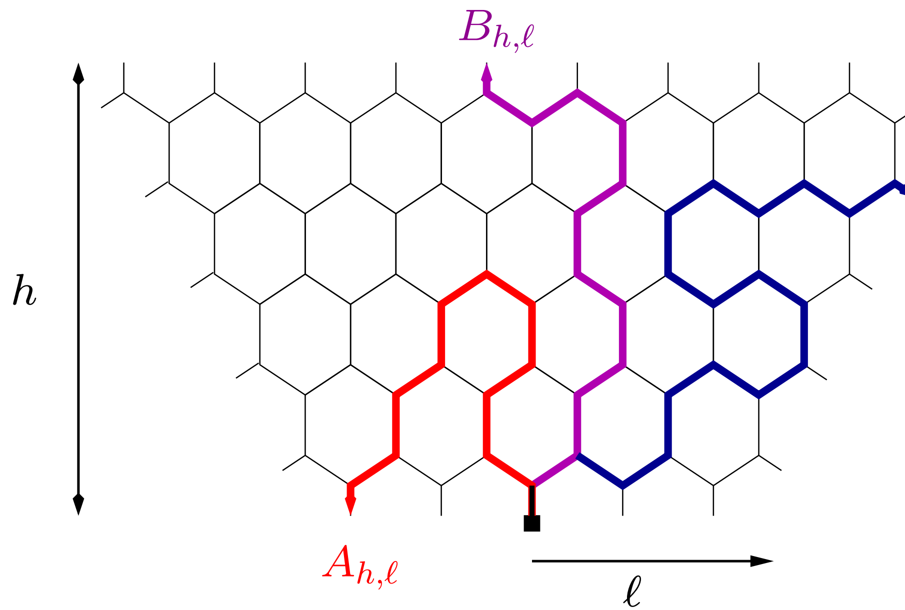

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This gives:

$$\frac{\sqrt{2-\sqrt{2}}}{2} A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}} E_{h,\ell}(x^*) = 1.$$

The $\sqrt{2+\sqrt{2}}$-conjecture is proved...

What else?

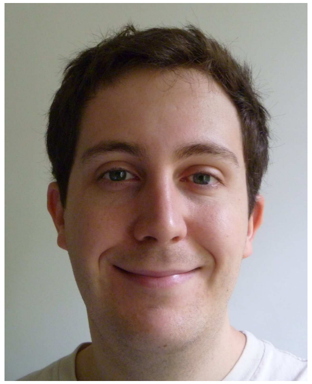
III. The $1+\sqrt{2}$-conjecture: SAWs on the honeycomb lattice interacting with a boundary

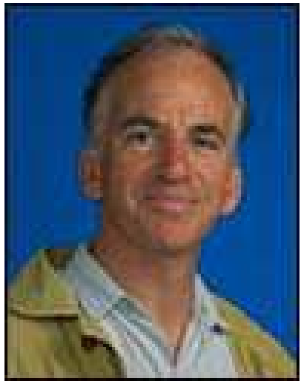
Conjecture of [Batchelor & Yung, 95]

joint work with

Nick Beaton, Hugo Duminil-Copin, Jan de Gier and

Tony Guttmann



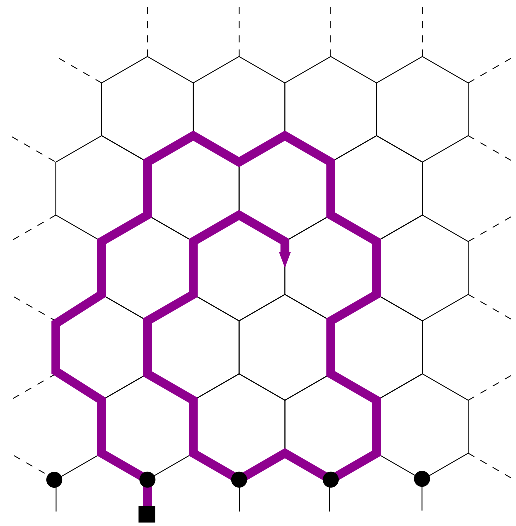



Walks in a half-plane interacting with a "surface"

- Enumeration by contacts of n-step walks:

$$\bar{c}_n(y) = \sum_{|\omega|=n} y^{\text{contacts}(\omega)}$$

y^3



In statistical physics, the parameter y is called "fugacity"

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- Generating function

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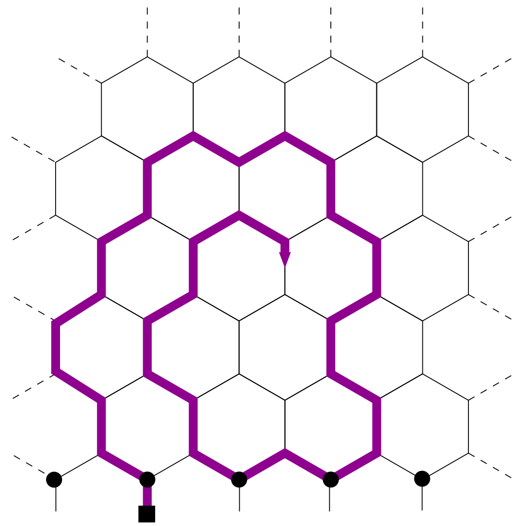

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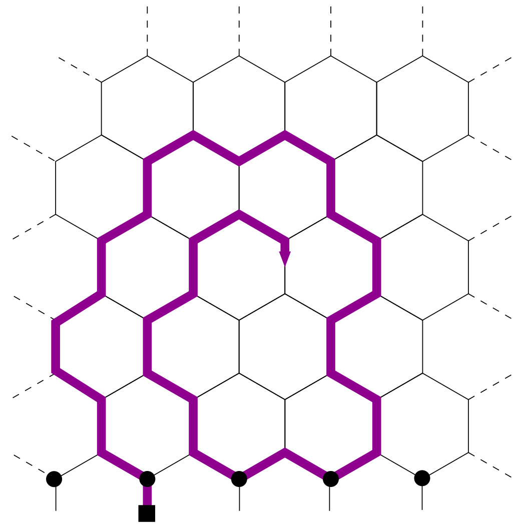
- Radius and growth constant ($y > 0$ fixed):

$$\rho(y) = \frac{1}{\mu(y)} = \lim_n \bar{c}_n(y)^{-1/n}$$

y^3



[Hammersley, Torrie and Whittington 82]

In statistical physics, the parameter y is called "fugacity"

The critical fugacity y_c

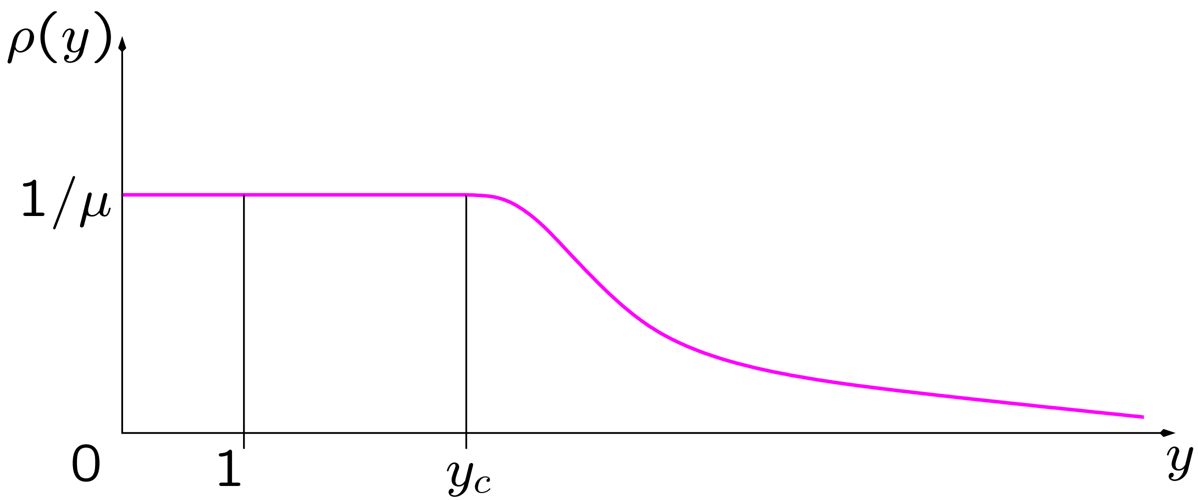
- Radius and growth constant: for $y > 0$,

$$\rho(y) = \frac{1}{\mu(y)} = \lim_n \bar{c}_n(y)^{-1/n}$$

Proposition: $\rho(y)$ is a continuous, weakly decreasing function of $y \in (0, \infty)$. There exists $y_c > 1$ such that

$$\rho(y) \begin{cases} = 1/\mu & \text{if } y \leq y_c, \\ < 1/\mu & \text{if } y > y_c, \end{cases}$$

where μ is the growth constant of (unrestricted) SAWs.
[Whittington 75, Hammersley, Torrie and Whittington 82]

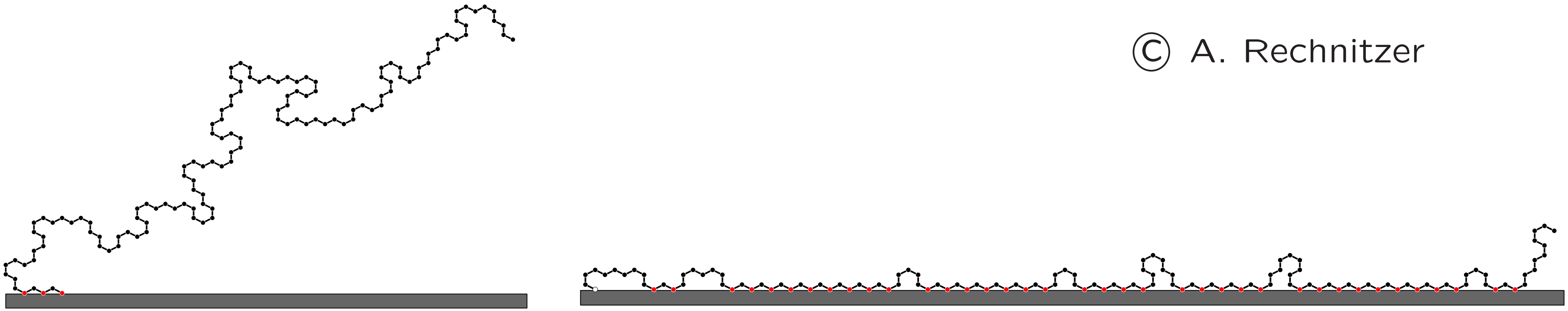


The critical fugacity: probabilistic meaning

Take half-space SAWs of length n under the Boltzmann distribution

$$\mathbb{P}_n(\omega) = \frac{y^{\text{contacts}(\omega)}}{\bar{c}_n(y)}.$$

Then for $y < y_c$, the walk escapes from the surface. For $y > y_c$, a positive fraction of its vertices lie on the surface.



© A. Rechnitzer

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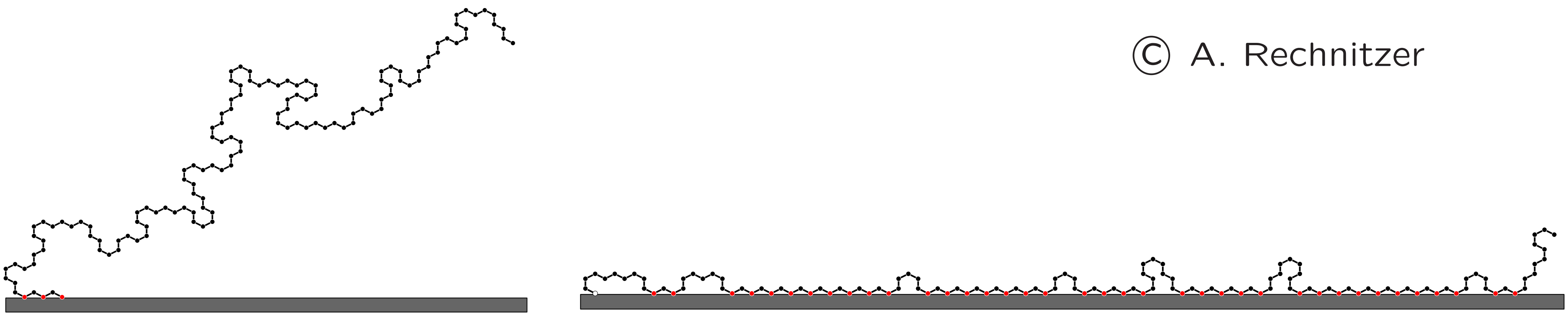


Theorem [B-BM-dG-DC-G 12]: this phase transition occurs at

$$y_c = 1 + \sqrt{2}$$

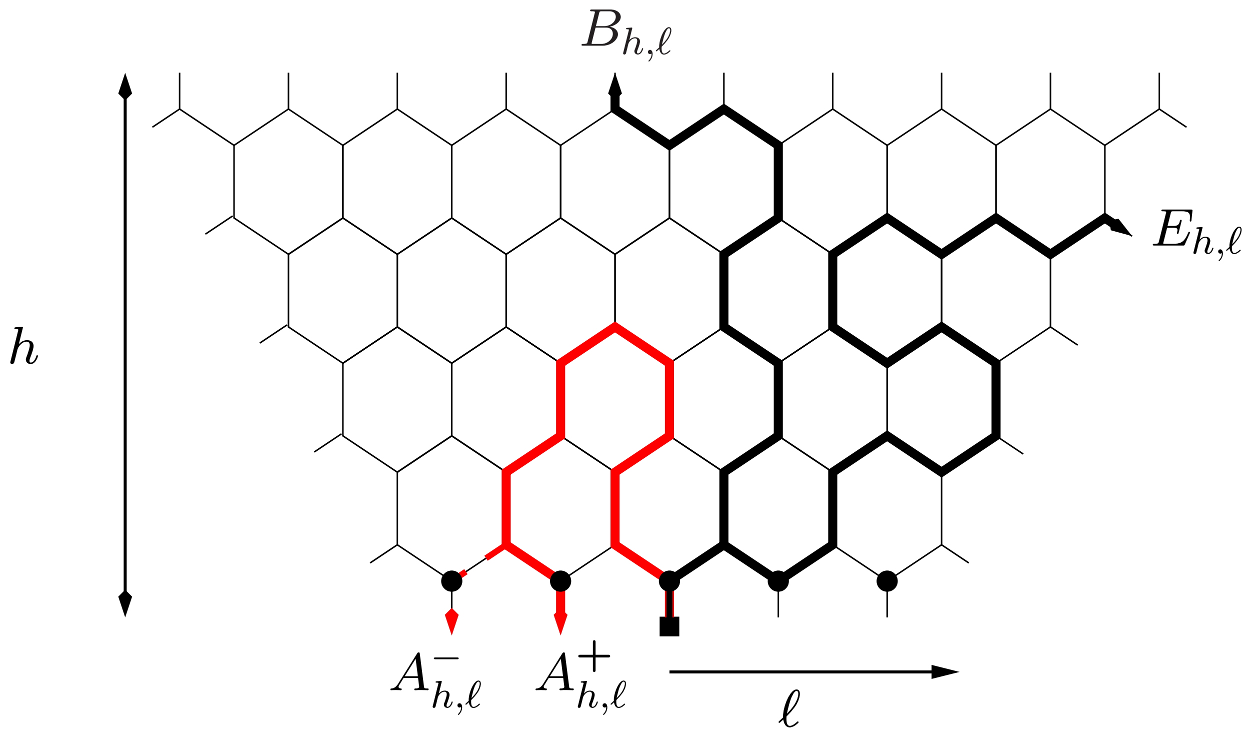
(conjectured by Batchelor and Yung in 1995)

0. Duminil-Copin and Smirnov's "global" identity: refinement with lower contacts

For $x^* = 1/\sqrt{2+\sqrt{2}}$, and for any y,

$$\alpha \frac{\sqrt{2}-y}{y(\sqrt{2}-1)} A^-_{h,\ell}(x^*,y) + \alpha A^+_{h,\ell}(x^*,y) + B_{h,\ell}(x^*,y) + \varepsilon E_{h,\ell}(x^*,y) = y$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$.



$A_{h,\ell}$ arches

$B_{h,\ell}$ bridges

$E_{h,\ell}$ ...

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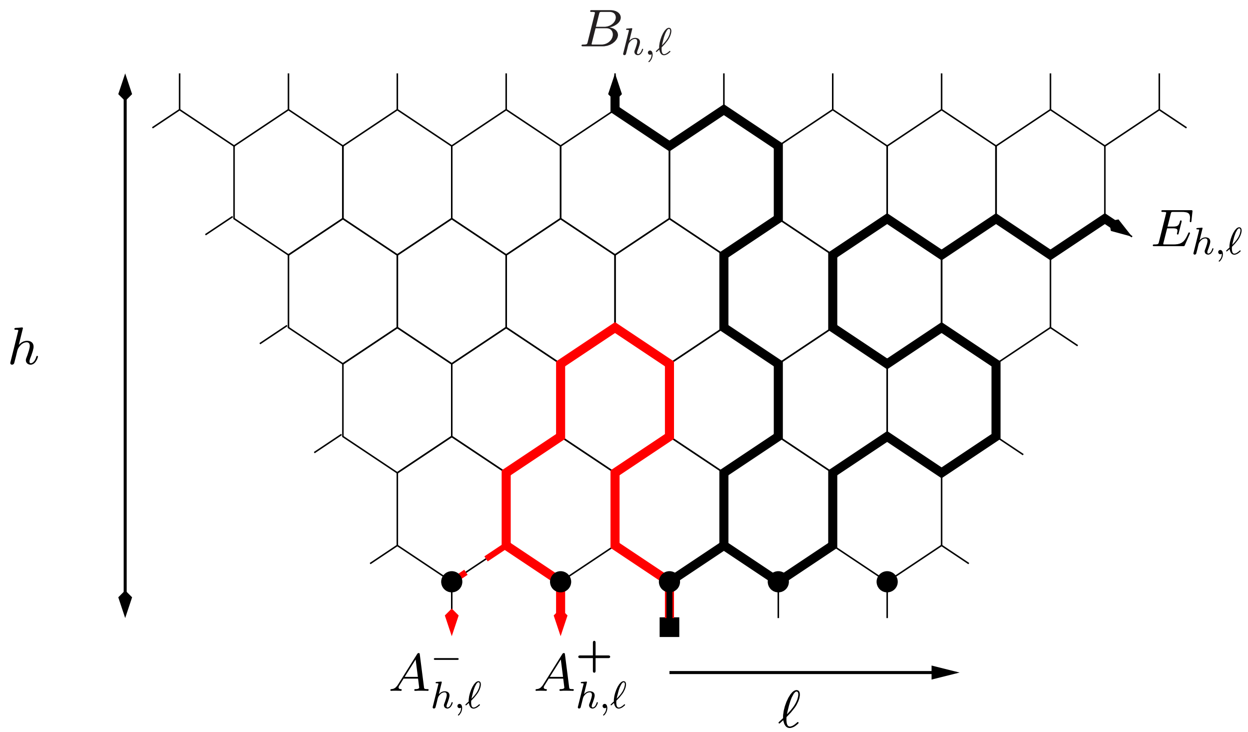
So what?



$A_{h,\ell}$ arches

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$E_{h,\ell}$ ...

1. Duminil-Copin and Smirnov's "global" identity: refinement with *upper* contacts

For $x^* = 1/\sqrt{2+\sqrt{2}}$, and for any y,

$$\alpha A_{h,\ell}(x^*,y) + \frac{y^*-y}{y(y^*-1)} B_{h,\ell}(x^*,y) + \varepsilon E_{h,\ell}(x^*,y) = 1$$

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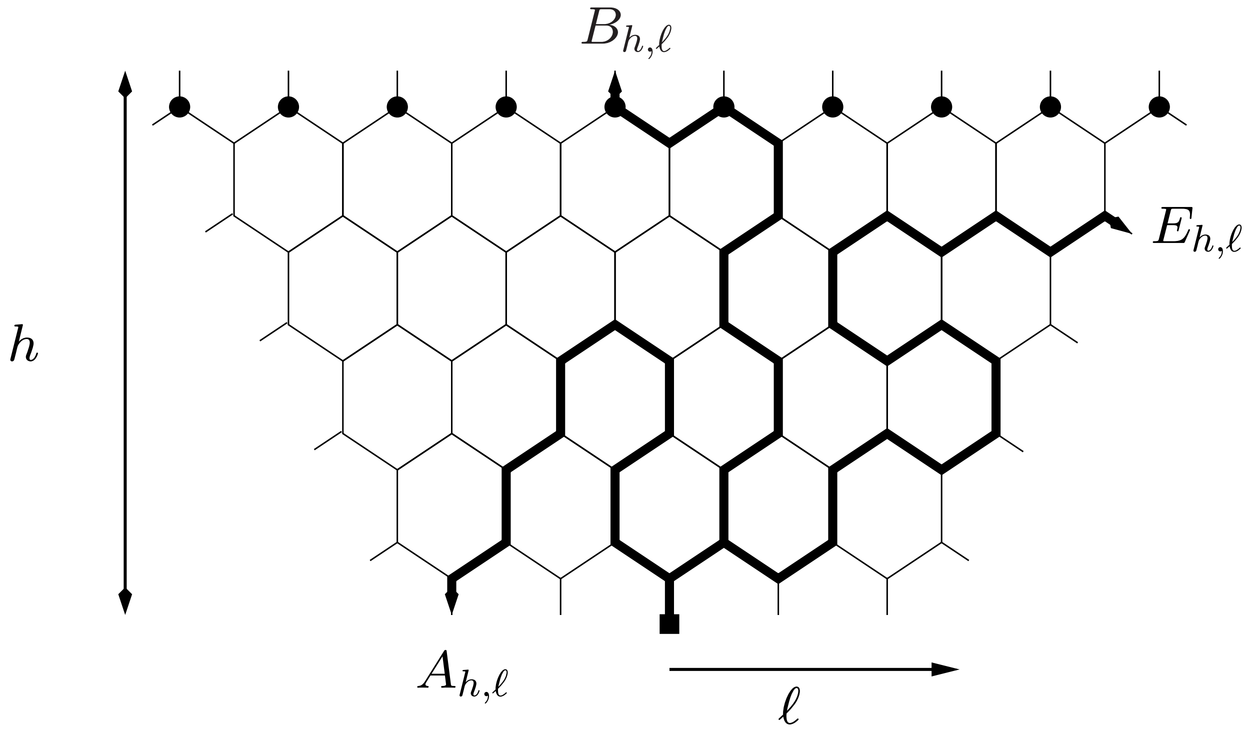


$A_{h,\ell}$ arches

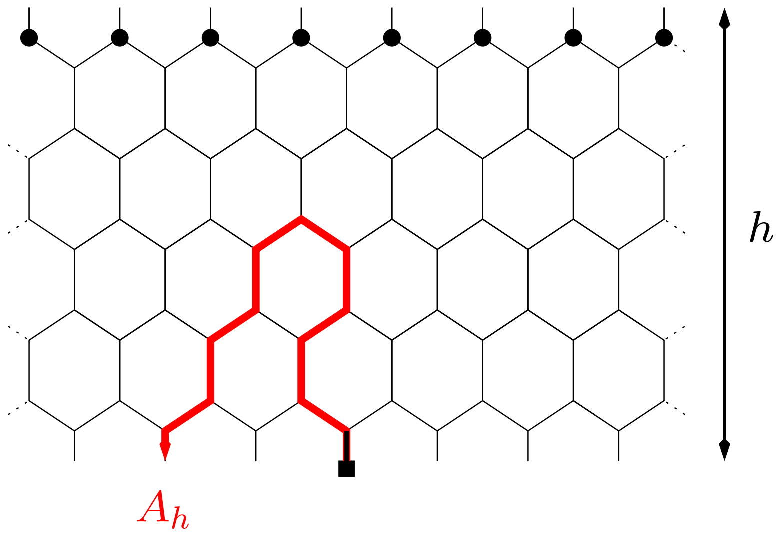
$B_{h,\ell}$ bridges

$E_{h,\ell}$ ...

2. An alternative description of the critical fugacity y_c

Proposition: Let $A_h(x, y)$ be the (rational[1]) generating function of arches in a strip of height h, counted by length and upper contacts.
Let y_h be the radius of convergence[2] of $A_h(x^*, y)$.
Then, as $h \to \infty$,

$$y_h \searrow y_c.$$

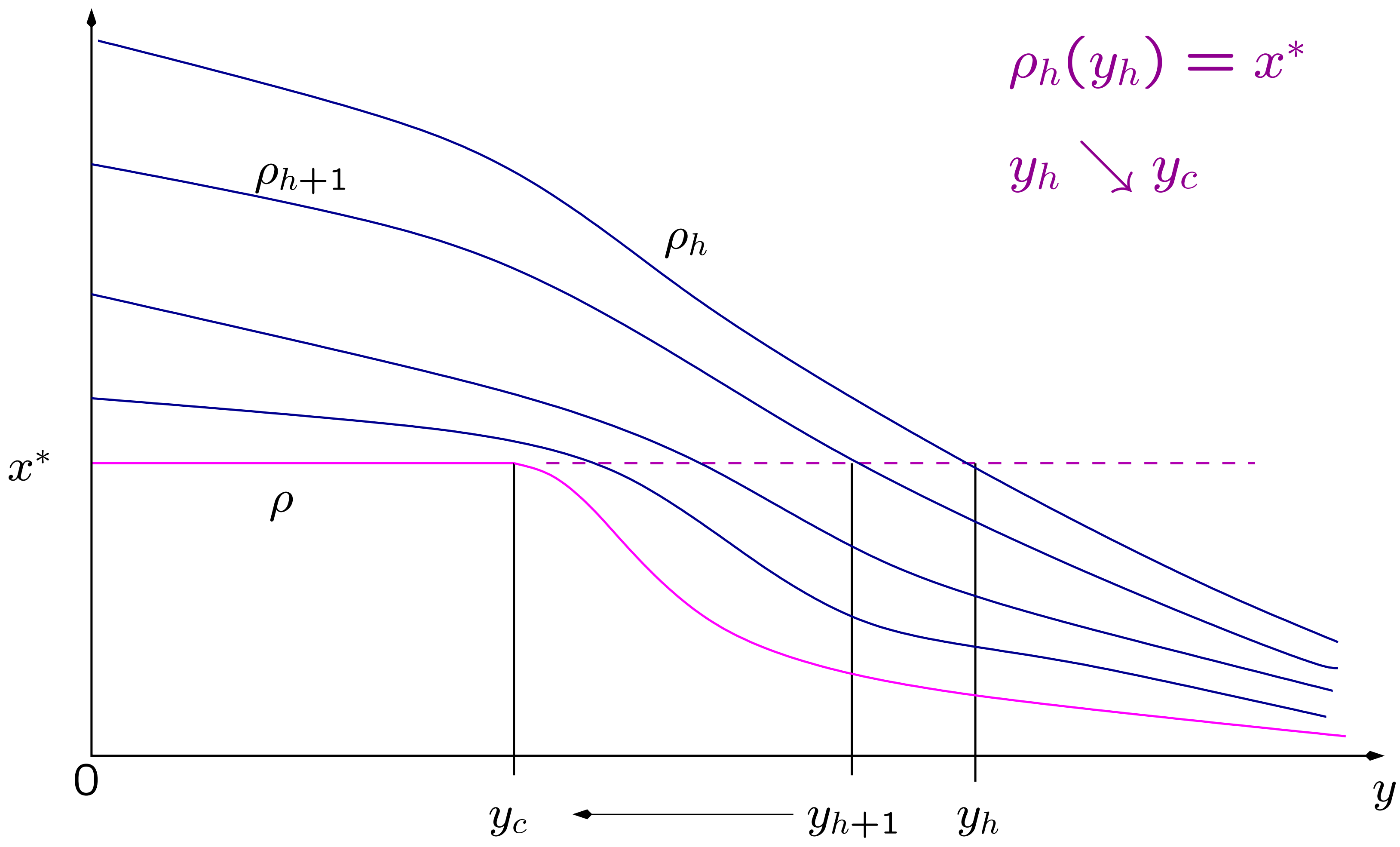


(uses [van Rensburg, Orlandini and Whittington 06])

1. [Rechnitzer 03]
2. For all k, the coefficient of y^k in $A_h(x, y)$ is finite at $x^* = 1/\mu$

The complete picture

For $y > 0$ fixed, let $\rho_h(y)$ be the radius of $A_h(x, y)$.



3. A lower bound on y_c

- For $x^* = 1/\sqrt{2+\sqrt{2}}$, and for any y,

$$\alpha A_{h,\ell}(x^*, y) + \frac{y^* - y}{y(y^* - 1)} B_{h,\ell}(x^*, y) + \varepsilon E_{h,\ell}(x^*, y) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.

- Set $y = y^*$.

3. A lower bound on y_c

- For $x^* = 1/\sqrt{2+\sqrt{2}}$,

$$\alpha A_{h,\ell}(x^*, y^*) + \qquad 0 \qquad + \varepsilon E_{h,\ell}(x^*, y^*) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1 + \sqrt{2}$.

- Set $y = y^*$.

3. A lower bound on y_c

• For $x^* = 1/\sqrt{2+\sqrt{2}}$,

$$\alpha A_{h,\ell}(x^*,y^*) + \qquad 0 \qquad + \varepsilon E_{h,\ell}(x^*,y^*) = 1$$

with $\alpha = \frac{\sqrt{2-\sqrt{2}}}{2}$, $\varepsilon = \frac{1}{\sqrt{2}}$ and $y^* = 1+\sqrt{2}$.

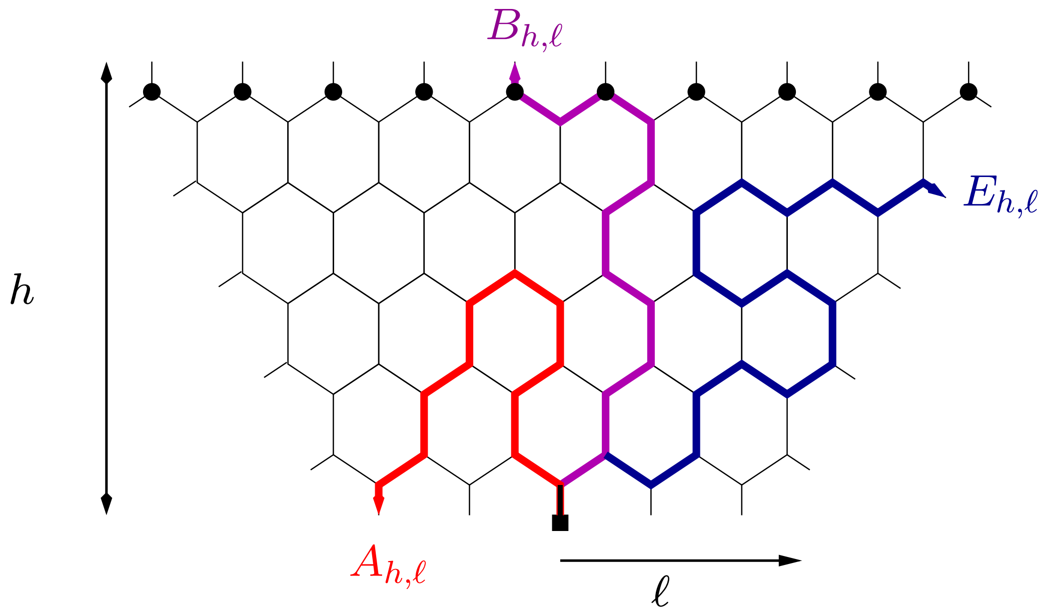
• Set $y = y^*$. For h fixed, $A_{h,\ell}(x^*,y^*)$ increases with ℓ but remains bounded: its limit is $A_h(x^*,y^*)$ (arches in an h-strip), and is finite.

Since the radius of $A_h(x^*,y)$ is y_h,

$$y^* \le y_h,$$

and since y_h decreases to y_c,

$$y^* \le y_c.$$



4. An upper bound on y_c

$$\alpha A_{h,\ell}(x^*,y) + \frac{y^*-y}{y(y^*-1)} B_{h,\ell}(x^*,y) + \varepsilon E_{h,\ell}(x^*,y) = 1$$

Harder! Uses a third ingredient:

Proposition: The length generating function $B_h(x,1)$ of bridges of height h, taken at $x^* = 1/\mu$, satisfies

$$B_h(x^*,1) \to 0 \quad \text{as } h \to \infty.$$

Inspired by [Duminil-Copin & Hammond 12], "The self-avoiding walk is sub-ballistic"

Conjecture (from SLE):

$$B_h(x^*,1) \simeq h^{-1/4}$$

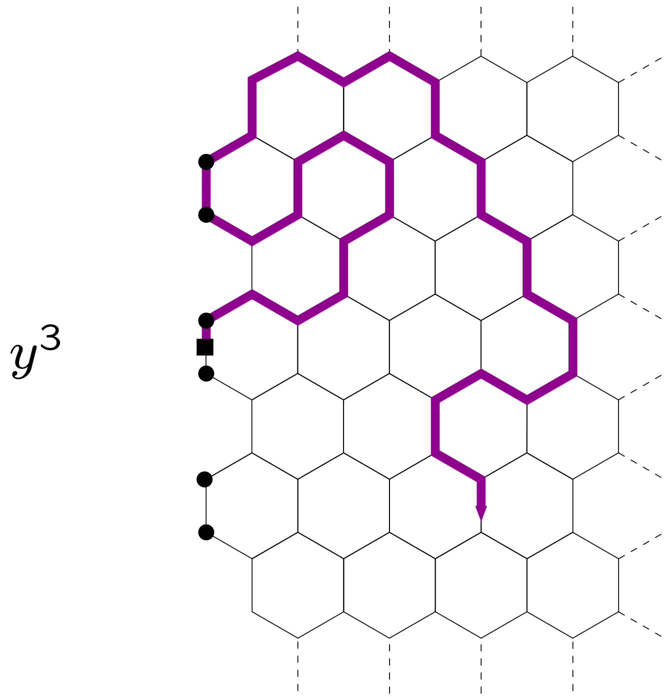
More about this?

The $\sqrt{\frac{2+\sqrt{2}}{1+\sqrt{2}-\sqrt{2+\sqrt{2}}}}$ conjecture

(due to [Batchelor, Bennett-Wood and Owczarek 98], proved by Nick Beaton)

- A similar result for SAWs confined to the half-plane $\{x \geq 0\}$ (rather than $\{y \geq 0\}$).

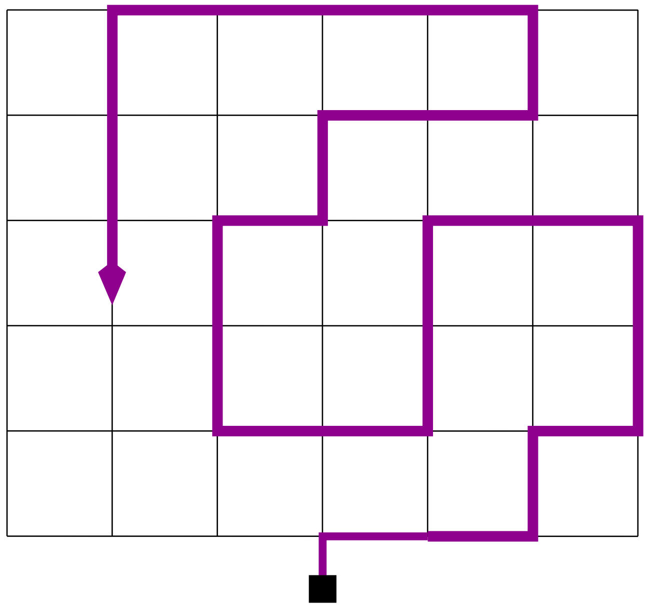
See Nick's poster on Tuesday!



IV. The mysterious square lattice

A $\mu = \frac{\sqrt{182+26\sqrt{30261}}}{26}$ conjecture?

[Jensen & Guttmann 99], [Clisby & Jensen 12]



Looking for a local identity

Let

$$F(p) \equiv F(x,t,\theta;p) = \sum_{\omega:a \rightsquigarrow p \text{ in } D} x^{|\omega|} t^{s(\omega)} e^{i\theta W(\omega)},$$

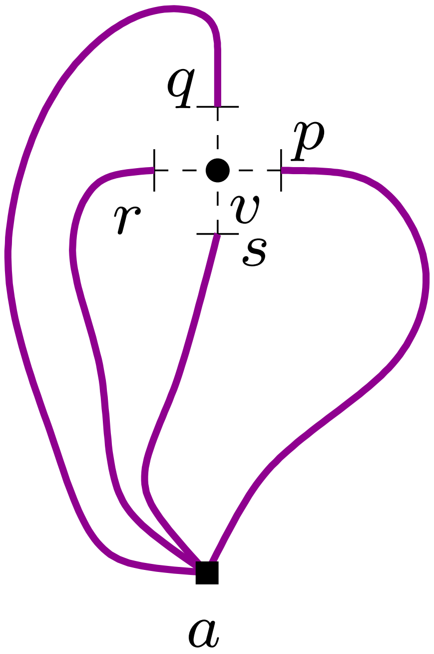
where $|\omega|$ is the length of ω, $s(\omega)$ the number of vertices where ω goes straight and $W(\omega)$ the winding number:

$$W(\omega) = \text{left turns} - \text{right turns}.$$

Could it be that

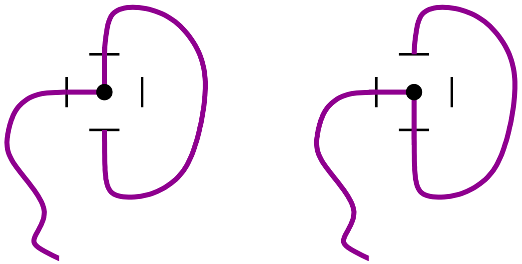
$$(p-v)F(p) + (q-v)F(q) + (r-v)F(r) + (s-v)F(s) = 0$$

for an appropriate choice of x, t and θ?

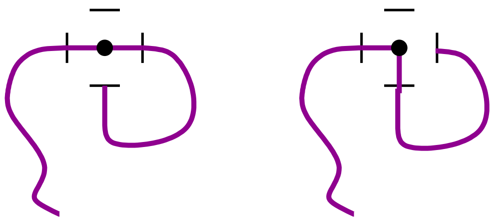


Group walks that only differ in the neighborhood of v

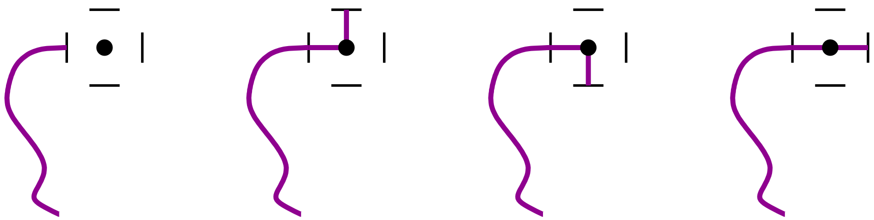
- Walks that visit three mid-edges (type 1):



- Walks that visit three mid-edges (type 2):



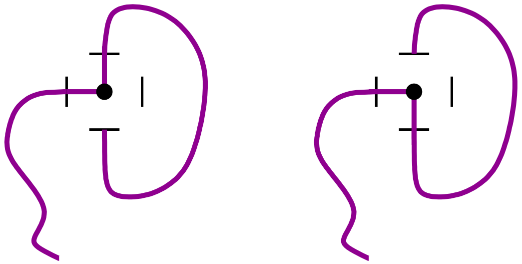
- Walks that only visit one or two mid-edges:



The contribution of all walks in a group should be zero.

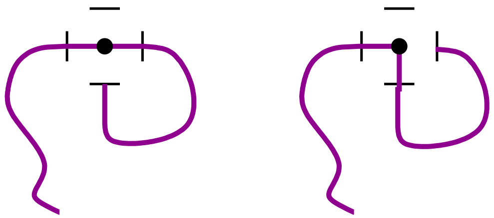
Group walks that only differ in the neighborhood of v

- Walks that visit three mid-edges (type 1):



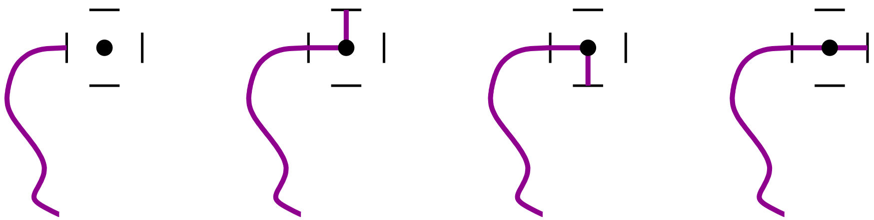
$$-ie^{-3i\theta} + ie^{3i\theta} = 0$$

- Walks that visit three mid-edges (type 2):



$$-ite^{-3i\theta} + e^{2i\theta} = 0$$

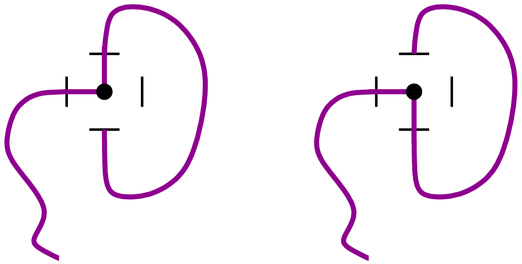
- Walks that only visit one or two mid-edges:



$$-1 + ixe^{i\theta} - ixe^{-i\theta} + tx = 0$$

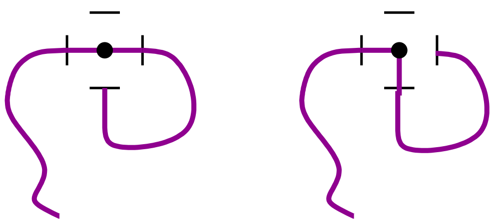
Group walks that only differ in the neighborhood of v

- Walks that visit three mid-edges (type 1):



$$-ie^{-3i\theta} + ie^{3i\theta} = 0$$

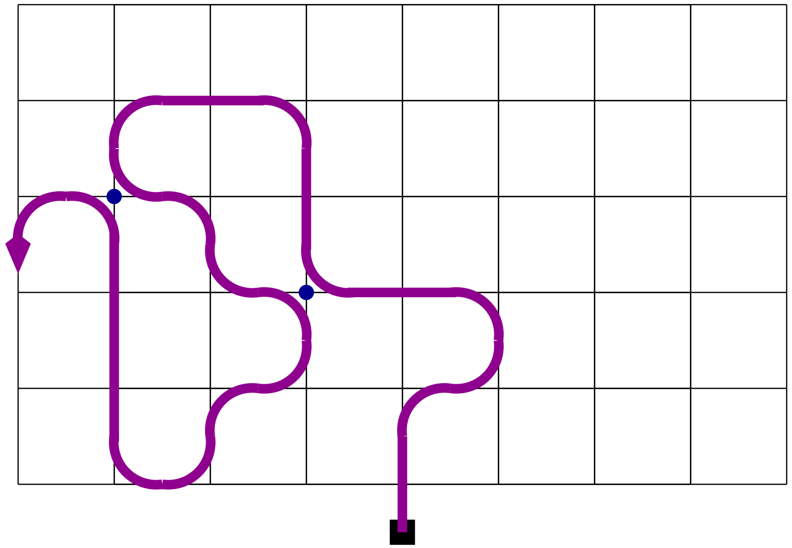
- Walks that visit three mid-edges (type 2):



$$-ite^{-3i\theta} + e^{2i\theta} = 0$$

No solution with t real

A generalization of self-avoiding walks: osculating walks



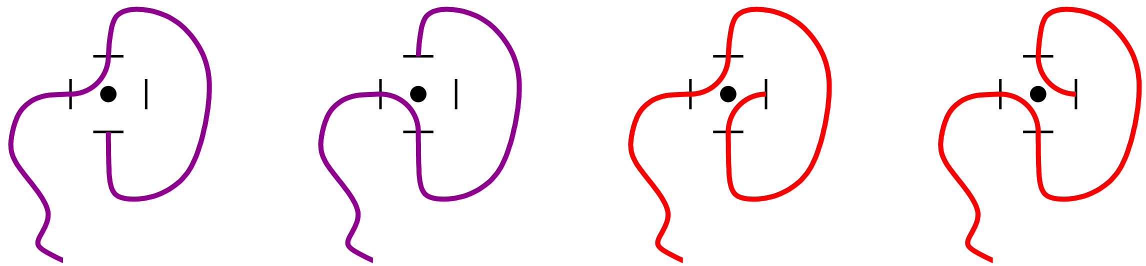
$$F(p) \equiv F(x,t,y,\theta;p) = \sum_{\omega: a \rightsquigarrow p \text{ in } D} x^{|\omega|} t^{s(\omega)} y^{c(\omega)} e^{i\theta W(\omega)},$$

where $|\omega|$ is the length of ω, $s(\omega)$ the number of vertices where ω goes straight, $c(\omega)$ the number of contacts, and $W(\omega)$ the winding number.

[Cardy-Ikhlef 09]

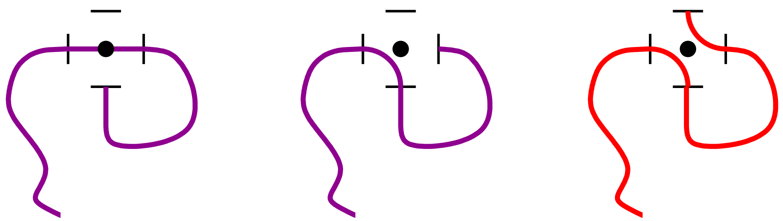
Group walks that only differ in the neighborhood of v

- Walks that visit three or four mid-edges (type 1):



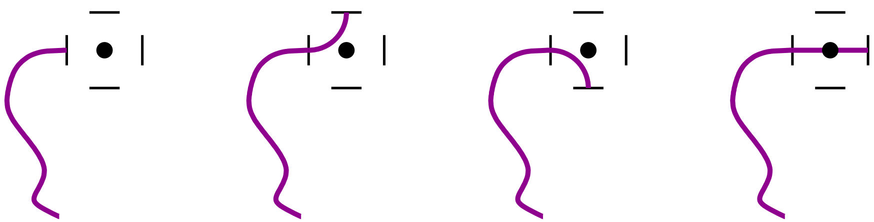
$$-ie^{-3i\theta}+ie^{3i\theta}+xye^{-4i\theta}+xye^{4i\theta}=0$$

- Walks that visit three or four mid-edges (type 2):



$$-ite^{-3i\theta}+e^{2i\theta}+ixye^{i\theta}=0$$

- Walks that only visit one or two mid-edges:



$$-1+ixe^{i\theta}-ixe^{-i\theta}+tx=0$$

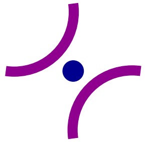
Four (real and non-negative) solutions

θ	t 	xy 	x^{-1}
$-\frac{\pi}{2}$	0	1	2
$\frac{\pi}{16}$	$\sqrt{2}\cos\frac{\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16}$	$\sqrt{2}\cos\frac{\pi}{16} - 2\sin\frac{\pi}{16}$
$-\frac{5\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16} + 2\cos\frac{3\pi}{16}$
$-\frac{7\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16}$	$\sqrt{2}\cos\frac{3\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16} + 2\cos\frac{\pi}{16}$

Note:

$$\cos\frac{\pi}{16} = \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \quad \text{and} \quad \sin\frac{\pi}{16} = \frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

Four (real and non-negative) solutions

θ	t 	xy 	x^{-1}	
$-\frac{\pi}{2}$	0	1	2	
$\frac{\pi}{16}$	$\sqrt{2}\cos\frac{\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16}$	$\sqrt{2}\cos\frac{\pi}{16} - 2\sin\frac{\pi}{16}$	
$-\frac{5\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16} + 2\cos\frac{3\pi}{16}$	(3)
$-\frac{7\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16}$	$\sqrt{2}\cos\frac{3\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16} + 2\cos\frac{\pi}{16}$	

- Four local identities $\Rightarrow$ proof for (weighted) growth constants?

Four (real and non-negative) solutions

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$-\frac{\pi}{2}$	0	1	2
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$-\frac{5\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16}$	$\sqrt{2}\sin\frac{3\pi}{16} + 2\cos\frac{3\pi}{16}$ (3)
$-\frac{7\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16}$	$\sqrt{2}\cos\frac{3\pi}{16}$	$\sqrt{2}\sin\frac{\pi}{16} + 2\cos\frac{\pi}{16}$

- Four local identities $\Rightarrow$ proof for (weighted) growth constants?

$\Rightarrow$ cf. [Glazman 13] for a proof in Case (3), and an asymmetric model wich interpolates between (3) and the honeycomb lattice.

Some questions

- Another global identity: for $x^* = 1/\sqrt{2+\sqrt{2}}$,

$$\frac{\sqrt{2-\sqrt{2}}}{2}\, A_{h,\ell}(x^*) + B_{h,\ell}(x^*) + \frac{1}{\sqrt{2}}\, E_{h,\ell}(x^*) = 1$$

Some questions

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This value of x is supposed to correspond to a dense phase of SAWs. Meaning, and proof?

Some questions

- Another global identity: for $x^* = 1/\sqrt{2-\sqrt{2}}$,

$$-\frac{\sqrt{2+\sqrt{2}}}{2}\, A_{h,\ell}(x^*) + B_{h,\ell}(x^*) - \frac{1}{\sqrt{2}}\, E_{h,\ell}(x^*) = 1$$

This value of x is supposed to correspond to a dense phase of SAWs. Meaning, and proof?

- A global identity for the $O(n)$ loop model [Smirnov 10] $\Rightarrow$ critical point?

References

• Smirnov's lecture/paper at the 2010 ICM for a general view of discrete preholomorphic functions and their use in physics/combinatorics/probability theory

Duminil-Copin and Smirnov, The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$, arXiv:1007.0575

• SAWs in a half-plane interacting with the boundary:

Beaton, MBM, Duminil-Copin, de Gier and Guttmann, The critical fugacity for surface adsorption of SAW on the honeycomb lattice is $1+\sqrt{2}$, arXiv:1109.0358

Beaton, The critical surface fugacity of self-avoiding walks on a rotated honeycomb lattice, arXiv:1210.0274

• Global quasi-identities and numerical estimates:

Beaton, Guttmann and Jensen, A numerical adaptation of SAW identities from the honeycomb to other 2D lattices, arXiv:1110.1141.

Beaton, Guttmann and Jensen, Two-dimensional self-avoiding walks and polymer adsorption: Critical fugacity estimates arXiv:1110.6695.

In 5 dimensions and above: Brownian behaviour

- The critical exponents are those of the simple random walk:

$$c_n \sim \mu^n n^{0}, \qquad \mathbb{E}(D_n) \sim n^{1/2}.$$

- The limit exists and is the d-dimensional Brownian motion

[Hara-Slade 92]