Winning fast in sparse graph construction games

Ohad N. Feldheim* Michael Krivelevich†

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Abstract

A Graph Construction Game is a Maker-Breaker game. Maker and Breaker take turns in choosing previously unoccupied edges of the complete graph K_N. Maker's aim is to claim a copy of a given target graph G while Breaker's aim is to prevent Maker from doing so. In this paper we show that if G is a d-degenerate graph on n vertices and $N > d^{11}2^{2d+9}n$, then Maker can claim a copy of G in at most $d^{11}2^{2d+7}n$ rounds. We also discuss a lower bound on the number of rounds Maker needs to win, and the gap between these bounds.

1 Introduction to graph construction games

Let $H = (V, E)$ be a hypergraph, that is, $V = V(H)$ is a finite set (the vertices of H) and $E = E(H)$ is a family of subsets of V (the hyperedges of H). Two players *Maker* and *Breaker* play the following game:

Both players take turns in claiming the vertices of the board H (each vertex can be claimed only once). If at any time during the game Maker claims a complete hyperedge of H, called a winning set, we say Maker won the game. If the entire board is claimed and yet Maker fails to claim a complete hyperedge of H – we say Breaker wins.

Such a game is called a *Weak Positional Game* on the hypergraph H; it is called *Weak* to distinguish it from the *Strong* game where both players compete to be the first to claim a complete hyperedge of H. The hypergraph along with the sets of Maker's and Breaker's claimed vertices is called a *Game Position.* A pair of consecutive turns – the first played by Maker and the second played by Breaker is called a *round.* For a survey of the general theory of positional games the interested reader is referred to a paper by Beck [4], or to a recent extensive monograph [5].

An important class of weak positional games is the class of *Graph Construction Games.* In a graph construction game two players, Maker and Breaker, take turns in claiming the edges of the

*School of Mathematical Sciences, Tel Aviv University, Tel Aviv, 69978, Israel.

†School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel. Email: krivelev@post.tau.ac.il.
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complete graph K_n. Maker's goal is to claim a copy of some graph G while Breaker's goal is to prevent him from doing so. Since this paper deals solely with graph construction games, let us define these formally:

Definition 1.1. (Graph construction game) *Let G be a graph. The graph construction game of G on K_N, denoted (K_N, G) is defined to be the weak game on the hypergraph H, whose vertices are edges of the complete graph K_N, and whose hyperedges are the edge sets of copies of G in K_N.*

When one encounters a weak game, two of the most natural questions that arise are "Who wins?", and – if Maker wins – "How quickly can victory be achieved?". It is not difficult to see that if for some N_0, the game (K_{N_0}, G) can be won by Maker, then for every $N > N_0$ Maker can also win (K_N, G) at least as quickly using the same winning strategy and ignoring the extra vertices and edges. This fact implies that for every G the number of turns Breaker can delay Maker's victory in (K_N, G) is non-increasing in N. We also know that (K_N, G) cannot be won by Maker when N is smaller then the number of vertices in G, since the game hypergraph has no winning sets. However, Beck showed [3] that there exists an explicit winning strategy for Maker in (K_{2^n}, K_n). Since K_n contains every graph G of n vertices, we can apply the same strategy for winning (K_{2^n}, G). Hence, for every graph G there exists N_0 large enough such that Maker can win (K_{N_0}, G). We can therefore reduce our study to the following couple of questions:
Given a graph G,

- What is the minimal N such that (K_N, G) is won by Maker?
- For N for which Maker wins (K_N, G), for how many rounds can Breaker delay Maker's victory?

Since for every G the number of turns Breaker can delay Maker's victory in (K_N, G) is non-increasing in N we may wish to replace our second question with the following:

- For how many rounds can Breaker delay Maker's victory in (K_N, G) for all N?

In this paper we try to give bounds for both questions constructively; that is to say, through the description of a fast and explicit winning strategy for the first player, and through the description of a delay strategy for Breaker. In fact, we will mostly concentrate on the second question and derive bounds for the first one from results on the length of the game.

One can expect different graphs with the same number of vertices to differ greatly in the answer to both questions, and that denser graphs are harder (take more turns) to build. For example, due to Pekeč and to Beck [8], [3] we know that the minimal N such that the Clique Game (K_N, K_n) is won by Maker is $N = P(n)2^{n/2}$ where P is a polynomial[1], and that for N large enough Maker can win (K_N, K_n) in less than 2^{n+2} turns, but cannot win in less than $2^{n/2}$ turns. However, constructing

[1] Actually Beck describes the exact board-size in [3].

a star of order n can easily be done on K_{2n} as the board and in n rounds. These examples show us that the order of G is far from being enough to determine both board-size and game-length required for Maker to win.

Another class of graphs that has been treated in the literature in this regard is that of graphs with bounded degree. Beck proved in [2] that if $N \geq P(d)3^d \cdot n$, then in a Maker-Breaker game played on the edges of K_N Maker can create a universal graph for the class of graphs on n vertices with maximum degree d, i.e., a graph that contains all such graphs. This immediately implies that a board-size linear in n (with a coefficient of order 3^d) is sufficient to construct a graph on n vertices of maximum degree d. Yet, Beck's approach can only give an upper bound on the game-length which is quadratic in n. Our result yields an upper bound on game-length for such graphs, which is linear in n.

In this paper we show that the *degeneracy* of G is a key graph theoretic parameter when analyzing graph construction games.

Definition 1.2. (d-degenerate graph) *A graph G is called* d-degenerate *if every subgraph of G contains some vertex with degree d or less. We call the minimal d such that G is d-degenerate the* degeneracy *of G.*

The most important property of a d-degenerate graph is that it has a *d-degenerate ordering.*

Definition 1.3. (d-degenerate ordering) *Let $G = (V, E)$ be a graph. An ordering $\sigma = v_1, ..., v_n$ of V is called a* d-degenerate ordering *if every vertex v_i has at most d neighbors among $\{v_{i+1}, ..., v_n\}$.*

Note that if a graph has maximum degree at most d, then it is d-degenerate, and every ordering of its vertices is a d-degenerate ordering. We also define:

Definition 1.4. (degenerate indegree) *Let $G = (V, E)$ be a graph, let $\sigma = v_1, ..., v_n$ be a d-degenerate ordering of G, and let $v_i \in V$ be a vertex of G.*

We define $d^-(v_i)$ the indegree of v_i with respect to σ *to be the number of neighboring vertices v_i has among $v_1, ..., v_{i-1}$.*

In the rest of the paper we assume G is a d-degenerate graph of order n whose vertices are $\{1, 2, ..., n\}$, ordered according to some d-degenerate ordering.

In order to distinguish in our proof between vertices of G and vertices of K_N we mark the vertices of K_N with a star.

We also use $N(v)$ to denote the set of neighbors of a vertex v in G.

Our main result is that among graphs with bounded degeneracy, both the threshold for victory, and the length of the game depend linearly on the number of vertices:

Theorem 1. (Quick victory theorem) *Let G be a d-degenerate graph on n vertices. For every natural $N > d^{11}2^{2d+9}n$, Maker can win the game (K_N, G) in at most $d^{11}2^{2d+7}n$ rounds.*

We learn from this result that the order of the target graph has a relatively small impact on the length of the construction game. We will see that the reason for this fact is that graphs which are sparse (in the sense of degeneracy) can be built fast using a local strategy.

Clearly the bound stated in Theorem 1 applies to every graph G of maximum degree d on n vertices.

We also show a simple lower bound for the length of a sparse graph construction game:

Theorem 2. (Long game theorem) *Let G be a graph of order n with m edges. Let $k = |aut(G)|$ be the number of automorphisms of G. For every $N < (2^{m-1}k)^{1/n}$ the game (K_N, G) is won by Breaker. Also, the game (K_M, G) cannot be won by Maker in less than $\frac{1}{2}(2^{m-1}k)^{1/n}$ rounds, for every M.*

Using Theorem 2 we can give a lower bound for the board-size and the game-length required for Maker to win. Applying the bound to complete bipartite graphs, we obtain a family of d-degenerate graphs that cannot be constructed by Maker much faster than our strategy suggests:

Corollary 1. (The bounds are relatively close) *For every d and n such that $1 \leq d < n$, there exists a d-degenerate graph G on n vertices such that for $N < 2^{d-\frac{d^2}{n}-\frac{1}{n}}(d!(n-d)!)^{\frac{1}{n}}$, the game (K_N, G) is won by Breaker. Also, for every M the game (K_M, G) cannot be won by Maker in less than $2^{d-1-\frac{d^2}{n}-\frac{1}{n}}(d!(n-d)!)^{\frac{1}{n}}$ rounds.*

Observe that for a fixed d and a large n, the last quantity behaves as $c2^d n$ for an absolute constant $c > 0$, showing that the upper bound obtained in Theorem 1 cannot be improved much in the general d-degenerate case.

Our interest in these results stems in part from their relation to the famous Burr-Erdős conjecture [6], and the study of *Ramsey Numbers* of different graph families, in particular of d-degenerate graphs and graphs with bounded degree. The Ramsey Number of a graph G, denoted by $r(G)$ is the smallest integer n such that the edges of K_n can not be divided into two disjoint sets, neither of which contains a copy of G. We know that if $N \geq r(G)$ then a draw in (K_N, G) is impossible. We can then deduce using the Strategy Stealing argument that Maker has a winning strategy for (K_N, G) (though this approach does not reveal us how to describe that strategy explicitly). The Burr-Erdős conjecture asserts that for d-degenerate graphs, $r(G) < Cn$ where n is the number of vertices of G, and C is a constant depending only on the degeneracy d of G. So far this conjecture has been settled for several cases including graphs with bounded degree. Our results thus support this conjecture. The value of our proof in the case of graphs with bounded degrees, or in other cases where the Burr-Erdős conjecture is settled, is partly given by its constructive nature, unlike its Ramsey theory-based counterpart (which relies on the inherently non-constructive strategy stealing argument); it also deals with the dependency between C and the degeneracy d of G.

2 Playing d-degenerate graph construction games

In this section we prove Theorem 1 by describing a quick winning strategy for Maker. Let us first describe a sketch of the proof.

Maker sequentially produces for every vertex j in G several vertices in K_N, amongst which we will eventually pick one to play the role of j in Maker's claimed copy of G. We call such vertices j-candidates. Every j-candidate v_j^* in K_N will have the property that for every $i < j$, if vertices i and j of G have an edge between them, then there will be many i-candidates connected to it by some edge claimed by Maker. Moreover, we require, for every set $i < i_1 < i_2 < \ldots < i_k$ of k vertices neighboring i in G, and for every choice $v_{i_1}^*, v_{i_2}^*, \ldots, v_{i_k}^*$ of their respective candidates, to have at least one suitable candidate for i, connected to all of them. Having achieved this, we will be able to show that a copy of G exists in Maker's claimed graph, simply by picking its vertices one-at-a-time, in reverse order, meeting the requirements for edges between them.

In the sketch of the proof, we mentioned the concept of a *candidate vertex*; it is not surprising that the fact that a specific vertex in K_N functions as a candidate for some vertex in G depends on which vertices of K_N are the candidates for other vertices of G. Our strategy will advise Maker to construct candidate vertices one by one, starting from candidates for the first vertex in G and moving forward. As part of preparation for the proof of Theorem 1, we now give a definition for candidate vertices that relies only on candidates for preceding vertices in G. Moreover, since our construction treats every edge in G separately, it will be wise to start by defining first a *Vertex Candidate with respect to a Specific Edge.*

Definition 2.1. (Vertex candidate w.r. to a specific edge) *Let G be a d-degenerate graph of order n, and let $(i, m) \in E(G)$ with $i < m$. Let H^* be a position in (K_N, G), let b^* be a vertex of K_N and let $\mathcal{B}^* = (B_1^*, B_2^*, \ldots, B_{m-1}^*)$ be a family of non-empty pairwise disjoint vertex sets of $K_N - \{b^*\}$. Let $(v_1, \ldots, v_k)$ be the set of neighbors of i in G in the interval $\{i+1, ..., m = v_k\}$, ordered according to the ordering of the vertices of G. We say that b^* is a* candidate for m with respect to the edge (i, m) *and the candidate family $\mathcal{B}^*$, if for every choice of vertices $b_1^* \in B_{v_1}^*, b_2^* \in B_{v_2}^*, ..., b_{k-1}^* \in B_{v_{k-1}}^*$ one has:*

$$\frac{\left|\{u^* | u^* \in B_i^*, \textit{Maker claimed } (u^*, b^*), (u^*, b_1^*), \ldots, (u^*, b_{k-1}^*) \textit{ in } H^*\}\right|}{|B_i^*|} \geq \frac{1}{k2^k} \ .$$

Now that we have a definition for a candidate with respect to an edge, we can define a candidate with respect to a vertex of G, as a vertex which functions as a candidate with respect to every edge of G connecting that vertex to a vertex preceding it.

Definition 2.2. (Vertex candidate) *Let G be a d-degenerate graph of order n, let $m \in V(G)$ be a vertex of G, let H^* be a position in (K_N, G), let b^* be a vertex of K_N and let $\mathcal{B}^* = (B_1^*, B_2^*, \ldots, B_{m-1}^*)$ be a family of non-empty pairwise disjoint vertex sets of $K_N - \{b^*\}$. If for every $i \in N(m) \cap \{1, ..., m-$*

1}, b^* is a candidate for m with respect to (i,m) and $\mathcal{B}^*$, then b^* is a candidate for m with respect to $\mathcal{B}^*$.

Finally we would like to define a *candidate scheme* in a manner that will support our inductive construction.

Definition 2.3. (Candidate scheme) *Let G be a d-degenerate graph of order n, and let H^* be a position in (K_N, G). An ordered family $\mathcal{B}^* = (B_1^*, B_2^*, \dots, B_m^*)$ of non-empty pairwise disjoint vertex sets of K_N is called a* candidate scheme *for the first m vertices of G if for every $2 \leq i \leq m$ and for every $b_i^* \in B_i^*$, b_i^* is a candidate for i with respect to $(B_1^*, ..., B_{i-1}^*)$.*

Note that a candidate scheme is designed to allow construction of $\mathcal{B}^*$ sequentially according to the degenerate order of the graph. It is also clear that in a specific game, once a vertex of K_N becomes a candidate for some vertex in G with respect to a candidate scheme $\mathcal{B}^*$, it will maintain this property for the rest of the game.

We show next that if Maker was able to construct in K_N a candidate scheme containing enough candidate vertices for each vertex of G, then he has already claimed a copy of G in K_N.

Lemma 2.1. *Let G be a d-degenerate graph of order n, let H^* be a position in (K_N, G). If there exists a candidate scheme $\mathcal{B}^* = (B_1^*, B_2^*, ..., B_n^*)$ in G, such that $|B_i^*| \geq d2^d$ for every $i \in 1, .., n$, then there exists a copy of G in H^* entirely claimed by Maker.*

Proof. Let us define $M^* \subset H^*$ as the graph composed of Maker's claimed edges. We construct an embedding $\phi : G \longrightarrow M^*$ starting from $\phi(n)$ and moving down to $\phi(1)$ using induction.

First, we choose an arbitrary vertex $b^* \in B_n^*$ and a set $\phi(n) = b^*$.

Let us assume that we have already defined $\phi(n), ..., \phi(i+1)$ so that $\phi(j) \in B_j^*$, $i+1 \leq j \leq n$, and that they embed $n, ..., i+1$ properly. By the degeneracy of G we know that i has most d neighbors in $[i+1, \dots, n]$ in G, let us mark them as $v_1, \dots, v_t$ according to their order in G, where $t \leq d$. Since $\phi(v_t)$ is a candidate for v_t with respect to $(B_1^*, B_{i+2}^*, \dots, B_{v_{t-1}}^*)$ we know that :

$$|\{u^* | u^* \in B_i^*, \text{Maker claimed } (u^*, \phi(v_1)), \dots, (u^*, \phi(v_t)) \text{ in } H^*\}| > \frac{|B_i^*|}{t2^t} \geq \frac{d2^d}{t2^t} \geq 1 \ .$$

Let us choose one of these vertices as $\phi(i)$. From the way we chose $\phi(i)$ it is clear that $\phi(n), ..., \phi(i)$ is a proper embedding of the last $n - i + 1$ vertices of G.

We can continue this process until ϕ defines an embedding of G in M^*. The image of the embedding is a graph isomorphic to G, and entirely claimed by Maker. □

All that remains is to show that the conditions for applying Lemma 1 can be met. In order to provide with such a construction, we need the concept of an *untouched* vertex.

Definition 2.4. (Untouched vertex) *Let G be a graph and let H^* be a position in the game (K_N, G). A vertex $v \in K_N$ is called* untouched *in H^* if neither Maker nor Breaker claimed any of the edges containing it.*

The existence of an untouched vertex is certain as long as less than $N/4$ rounds are played, since at every round no more than four vertices can be touched. Our intention is to show that once a large candidate scheme of the first m vertices has been generated, Maker can quickly transform an untouched vertex b^* into a candidate for vertex $m+1$.

In order to create such a candidate quickly, Maker would like to make sure that if b^* is eventually chosen it would reduce every set of candidates for the neighbors of $m+1$ amongst $1, 2, ..., m$ to about half of its size; we would also like it to reduce the possibilities for every future choice of candidates in a balanced way. To do this we apply a version of a theorem by Alon, Krivelevich, Spencer and Szabó [1], extending a previous result by Székely [9]:

Theorem 2.1 (Alon-Krivelevich-Spencer-Szabó)**.** *Let H be a hypergraph with X hyperedges, whose smallest hyperedge contains at least x vertices. In a weak positional game on H Maker can claim at least $\frac{x}{2} - \sqrt{\frac{x \ln(2X)}{2}}$ vertices of each hyperedge.*

The next lemma makes use of this theorem to prove that candidate schemes can indeed be extended quickly.

Lemma 2.2. (Scheme extension) *Let G be a d-degenerately ordered graph of order n, let H^* be a position in (K_N, G) after T rounds, and let $d^-(m+1)$ be the degenerate indegree of $m+1$ in G. Assume that H^* contains a candidate scheme $\mathcal{B}^* = (B_1^*, B_2^*, ..., B_m^*)$ for the first m vertices of G, satisfying $|B_i^*| = d^5 2^{d+4}$ for every $i \in 1, .., m$. Maker can turn any untouched vertex into a candidate for $m+1$ in at most $d^-(m+1) d^5 2^{d+3}$ rounds.*

Proof. We prove the lemma by describing a constructive strategy for Maker. Denote the neighbors of $m+1$ preceding it by $v_1, v_2, ..., v_{d^-(m+1)}$. Let us pick some untouched vertex of K_N and call it b_{m+1}^*. Our strategy should transform b_{m+1}^* into a candidate for $m+1$ with respect to $\mathcal{B}^*$. In order to do so Maker must make sure that for every $i \in \{1, \ldots, d^-(m+1)\}$, the vertex b_{m+1}^* will transform into a candidate for $m+1$ with respect to the edge $(v_i, m+1)$ and the candidate family $\mathcal{B}^*$. Since we do not intend to claim any more edges between the vertices of $\mathcal{B}^*$, and since becoming a candidate with respect to the edge $(v_i, m+1)$ depends only on edges between the vertices of $\mathcal{B}^*$, and edges between B_i^* and b_{m+1}^*, our strategy would be to play on each of these edge sets separately. Whenever Breaker claims an edge between b_{m+1}^* and some vertex in B_i^*, Maker will claim an edge between b_{m+1}^* and B_i^*. (If Breaker claims an edge of a different form Maker will act as though Breaker claimed an arbitrarily chosen edge between b_{m+1}^* and some B_i^*.)

Next we would like to describe a specific strategy for Maker regarding the connection of b_{m+1}^* and some specific B_i^*. It should be such that when all the edges between b_{m+1}^* and B_i^* are claimed,

b^*_{m+1} will have already become a candidate for $m+1$ with respect to the edge $(v_i, m+1)$ and the candidate family $\mathcal{B}^*$. In order to apply Theorem 2.1 it will be convenient to treat each $B^*_i - b^*_{m+1}$ connection game as a separate weak game.

Let us define formally $H_i = (V_i, E_i)$, the $B^*_i - b^*_{m+1}$ connection game. Let $(u_1, \ldots, u_k)$ be the neighbors of $i \in G$ in the interval $\{i+1, ..., m+1 = u_k\}$ ordered according to their order in G; recall that since the order of the vertices of G is d-degenerate, i has at most d neighbors following it, and thus $k \leq d$. If $k = 1$, then Maker turns b^*_{m+1} into a candidate for $m+1$ with respect to the edge $(i, m+1)$ simply by connecting b^*_{m+1} to a half of the vertices of B^*_i; this can be done in $|B^*_i|/2 = d^5 2^{d+3}$ rounds. We thus assume that $k \geq 2$. The set V_i, the board of the game, consists of all $d^5 2^{d+4}$ edges between b^*_{m+1} and the vertices in B^*_i. As for E_i – for every choice of candidates $b^*_{u_1} \in B^*_{u_1}, b^*_{u_2} \in B^*_{u_2}, ..., b^*_{u_{k-1}} \in B^*_{u_{k-1}}$ there is a single hyperedge in E_i, consisting of the edges of K_N between b^*_{m+1} and every vertex of B^*_i connected to all $b^*_{u_1}, b^*_{u_2}, ..., b^*_{u_{k-1}}$. Note that now the property of b^*_{m+1} being a candidate for $m+1$ with respect to the edge $(i, m+1)$ and the candidate family $\mathcal{B}^*$ translates into the property of Maker having at least $\frac{|B^*_i|}{k 2^k} = \frac{d^5 2^{d-k+2}}{k}$ vertices in every hypergraph hyperedge in E_i. By the fact that $\mathcal{B}^*$ is a candidate scheme for the first m vertices we already know that for every $e \in E_i$,

$$\begin{aligned} |e| &= \left| \{u^* | u^* \in B^*_i, \text{Maker claimed } \{u^*, b^*_{u_1}\}, ..., \{u^*, b^*_{u_{k-1}}\}\} \text{ in } H^*\} \right| \\ &\geq \frac{|B^*_i|}{2^{k-1}(k-1)} . \end{aligned}$$

Using our assumption $|B^*_i| = d^5 2^{d+4}$ we get $|e| \geq \frac{d^5 2^{d-k+5}}{k}$. Also, since every hyperedge of E_i is determined by a choice of one element of every set $B^*_{u_1}, B^*_{u_2}, ..., B^*_{u_{k-1}}$, we know that $|E_i| \leq (d^5 2^{d+4})^{k-1}$ (This is not an equality since two choices of candidates in sets B^*_i may define the same hyperedge). Using Theorem 2.1 we can now be certain that a strategy for Maker exists, such that when all the edges between b^*_{m+1} and B_i are claimed, Maker will have no less than

$$\frac{d^5 2^{d-k+4}}{k-1} - \sqrt{\frac{d^5 2^{d-k+4}}{k-1} \ln(2(d^5 2^{d+4})^{k-1})}$$

claimed vertices on each hyperedge, and thus it remains to verify:

$$\frac{d^5 2^{d-k+4}}{k-1} - \sqrt{\frac{d^5 2^{d-k+4}}{k-1} \ln(2(d^5 2^{d+4})^{k-1})} > \frac{d^5 2^{d-k+4}}{k} .$$

The above is equivalent to:

$$\frac{d^5 2^{d-k+4}}{(k-1)k^2} > 5(k-1)\ln d + (d+4)(k-1)\ln 2 + \ln 2,$$

which is easily seen to hold for integers $1 < k \leq d$.

Hence, by the end of each game H_i, vertex b^*_{m+1} becomes a candidate for $m+1$ with respect to the edge $(i, m+1)$ and the candidate family $\mathcal{B}^*$. How many rounds does this strategy require? We play $d^-(m+1)$ parallel games, each on board of size $d^5 2^{d+4}$, hence we do not play more than $d^-(m+1)d^5 2^{d+3}$ rounds, as promised. □

Proof of Theorem 1.

Assume $N > d^{11}2^{2d+9}n$. Since G is a d-degenerate graph on n vertices, G has at most dn edges. For every vertex $i \in V(G)$ we denoted by $d^-(i)$ the degenerate indegree of i in G; clearly, $\sum_{i=1}^{n} d^-(i) = |E(G)| \leq dn$. Using Lemma 2.2, Maker can now construct $d^5 2^{d+4}$ candidates for every $i \in V(G)$, sequentially and according to the degenerate order of G, in at most $d^-(i)d^5 2^{d+3}$ rounds. Since

$$N > d^{11}2^{2d+9}n \geq 4\sum_{i=1}^{n}(d^-(i)d^5 2^{d+3})d^5 2^{d+4} \ ,$$

we are certain to have untouched vertices available throughout the construction. The construction is complete after at most

$$\sum_{i=1}^{n}(d^-(i)d^5 2^{d+3})d^5 2^{d+4} \leq d^{11}2^{2d+7}n$$

rounds with a candidate scheme containing $d^5 2^{d+4}$ candidates for every vertex of G. At this point we can apply Lemma 2.1 to be certain that the board contains a copy of G. □

3 Lower bound

In this section we prove a lower bound for the number of rounds needed for Maker to win in sparse graph construction games. Before addressing the calculation itself it is important to understand the relations between how quickly can Maker win, and what board-size does he require to do so.

Lemma 3.1. (Boardsize–delay relations) *Suppose the game (K_N, G) is Breaker's win. Then for all M, the game (K_M, G) cannot be won by Maker in less than $N/4$ rounds.*

Proof. Suppose to the contrary that for some $M > N$ Maker can win (K_M, G) in less than $N/4$ rounds. If Maker plays according to his winning strategy, then by the end of the game there will always be an untouched vertex in K_N. Hence (K_{M-1}, G) can also be won using the same strategy. Therefore (K_N, G) can be won by Maker which leads to a contradiction. □

We derive our lower bound using an important theorem of Erdős and Selfridge [7]. If every edge of a hypergraph contains exactly m vertices, the hypergraph is called m-uniform.

Theorem 3.1 (Erdős-Selfridge)**.** *If $\mathcal{F}$ is an m-uniform hypergraph and $|\mathcal{F}| < 2^{m-1}$ then Breaker can win a weak game on $\mathcal{F}$.*

Proof of Theorem 2.

Let $G = (V, E)$ be a graph, and suppose $|V| = n$ and $|E| = m$. Denote by $k = |aut(G)|$ the number of automorphisms of G. Let us examine the number of winning sets in (K_N, G). This number is equal to the number of copies of G in K_N – which is $\binom{N}{n}\frac{n!}{k}$. Each copy of G in K_N contains $|E| = m$ edges. Therefore the hypergraph of (K_N, G) is m-uniform. By the Erdős-Selfridge criterion (Theorem 3.1) we get that if $\binom{N}{n}\frac{n!}{k} < 2^{m-1}$, then Breaker wins (K_N, G). Since $\binom{N}{n}\frac{n!}{k} < \frac{N^n}{n!}\frac{n!}{k} = \frac{N^n}{k}$, the latter condition is satisfied when $N < (2^{m-1}k)^{1/n}$. Applying Lemma 3.1 we get that Maker cannot win (K_M, G) in less than $\frac{1}{2}(2^{m-1}k)^{1/n}$ rounds for all M. □

Corollary 1 is an immediate result, supplying a lower bound for d-degenerate graph construction games.

Proof. Let G be a complete bipartite graph with parts of size d and $n - d$. This graph has $d(n - d)$ edges, is clearly d-degenerate and has $k = d!(n - d)!$ automorphisms. Plugging all this into Theorem 2, we get the claimed result. □

4 Remarks and open questions

In this section we give several remarks about graph construction games. We also mention some open questions suggested by our present work.

Local versus Global. One may have noticed that the strategy we provided for the upper bound was a local strategy, leaning on local arguments; on the other hand our proof for the lower bound used only the Erdős-Selfridge theorem which is global in nature. What is the reason for this difference?

It appears that in graph construction games, where the target graph's structure is relatively regular (for example – if it has many automorphisms), a local construction, immediately designating every touched vertex to potentially play the role of a specific vertex in Maker's claimed copy of G, may be the fastest possible - or at least asymptotically the fastest. This speculation is supported by Corollary 1. However, when the graph is very irregular it might be possible for Maker to use a global strategy, and to decide which vertex in K_N shall function as which vertex in G very late in the game, making Breaker's life a bit more difficult. Whether this is true remains an open problem.

Problem 1. *For a fixed $d > 0$, are there infinitely many d-regular graphs G for which Maker can win the game (K_N, G) in at most $f(d)|V(G)|$ rounds for large-enough board-size N, where $f(d)$ is sub-exponential?*

Induced graphs versus copies. Note that in the proof of Theorem 1 it is clear that Maker's claimed graph not only contains a copy of G in K_N, but also contains an *induced copy* of it. This follows from the fact that we never claim an edge between two candidates for vertices which have no edge between them in G. The reason we chose not to define the graph construction game for induced copies of G is that such a game cannot be represented as a weak game on a hypergraph.

Disjoint cliques. A particular d-regular graph G, for which it would be quite interesting to analyze the construction game (K_N, G), is a collection of n/d disjoint cliques of size d each. According to the results of Pekeč [8] and of Beck [3], Maker can construct such G in $O(2^d n)$ rounds, for large enough N. The lower bound from Theorem 2 is significantly weaker.

The gap between bounds. Though in this paper we were able to show that for a fixed d, the length of the construction of a d-degenerate graph depends linearly on the number of vertices, it remains an open question to find how this ratio depends on d. There is a gap, exponential in d, between our lower bound of $C2^d n$ rounds and our upper bound of $Cd^{11}4^d n$ rounds. We suspect that the reason for this gap lies in the fact that our lower bound was proved using board-size arguments, while our upper bound comes from the game-length argument. We conjecture that by playing carefully with vertices which are not untouched, but have a small number of incident edges claimed by Breaker, it may be possible to reduce the board-size significantly leaving only a polynomial gap between bounds. Formally:

Conjecture 1. *Maker can win the game (K_N, G) for every d-degenerate graph G with n vertices if $N > 2^d P(d)n$ where $P(d)$ is some polynomial.*

The game-length gap remains an open question too.

Biased games. It may be of interest to extend our method to biased graph construction games. In these games, Maker claims one edge in each turn as before, but Breaker answers by taking $b \geq 1$ edges. The core of our proof can be used without any change. It will require, however, a version of Theorem 2.1 for biased games. Also, when trying to extend Lemma 2.2, we will no longer be able to play each B_i^*–b_{m+1}^* connection game separately, as we can no longer answer every Breaker's move in the same game.

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