Thoughts on Thermomechanics

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Abstract

First, I give a short description of the history of the interpretation of the second law of thermodynamics as a restriction on the kind of material properties that physical systems occurring in nature can have. Second, I describe the desirability of developing a framework, using the concept of a *state* of a material element, for developing a general thermomechanical theory of simple materials.Third, I give a taste of the mathematical infrastructure needed for such a theory. Finally, I give an outline of the paper [NS] by Brian Seguin and myself which may be a model for the first few chapters of future textbooks on continuum thermomechanics.

Key Words: *Thermomechanics, Second Law, Simple materials, mathematical infrastructure.*

1. History and Outlook

Until about 50 years ago, the overlap between thermodynamics and mechanics was very limited. It dealt mostly with the laws relating thermal energy and mechanical energy. However, the first and second laws of thermodynamics were not connected with continuum mechanics. Courses in thermodynamics, elasticity, and hydrodynamics had very little connection. The very term "thermomechanics" had not been invented. Even now the term is still not in the dictionary. Yet there are books with this title, and typing this term into Google yields 65 000 hits.

Of course, the development of statistical mechanics in the 19th century yielded an important connection between particle mechanics and thermodynamics, but this did not seem to have much relevance to continuum mechanics.

When working on *The Non-Linear Field Theories of Mechanics* [NLFT] with Clifford Truesdell in the early 1960's, I encountered the problem of justifying the formula relating the stress in elasticity to the gradient of the internal energy with respect to the transplacement gradient[1]. First, I naively conjectured that this formula can be derived from the law of conservation of energy. It cannot. What is needed is an appropriate interpretation of the second law of thermodynamics.

Here is the interpretation:

Dissipation principle: For all thermodynamic processes that are admissible for a given constitutive assumption, the entropy production must be positive or zero.

The decisive word in this postulate is the quantifier *all*. It makes the postulate a restrictive condition on the constitutive assumptions that can be imposed on systems of the type under consideration. Indeed, if constitutive assumptions are laid down at will and without restriction, the entropy production can be expected to be positive or zero only for some but not for all admissible processes. Thus, the second law is not a restriction on the kind of processes that can occur in nature, but a restriction on the kind of material properties that physical systems occurring in nature can have. This is not made clear in most explanations of the second law in the textbook literature on thermodynamics. An exception is the *Lectures on Physics* by R. Feynman. He writes: "So we see that a substance's properties must be limited in a certain way; one cannot make up anything he wants .... This principle, this limitation, is the only real rule that comes out of thermodynamics." (See pp. 44-6 and 44-7 of reference [F].)

What happened next is described by the following excerpt from Bernard Coleman's *Memories of Clifford Truesdell* [C]:

"In the early 1960's Walter Noll put to me the idea, as if it should be obvious to everyone, that the inequality is a restriction on all processes that are admissible in a material of which the body is composed, and, because one defines each material by giving a set of constitutive relations, the Clausius-Duhem inequality, as it must hold for all processes compatible with these relations, becomes a *restriction on constitutive relations.* In another act of great generosity, Walter suggested that we develop the idea together. It took a while to sort the argument out and to present it in a way that would convince the wary."

The result of our efforts was the paper *The*

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[1]In 1958, I introduced the unfortunate terms "configuration" and "deformation" for what are now called "placement" and "transplacement". I apologize. Since these old terms were used in [NLFT], they were widely accepted, and I am now in the ironic position to fight against something that I started.

Thermodynamics of Elastic Materials with Heat Conduction and Viscosity [CN]. There we gave rigorous mathematical proofs that the second law implies, among other things, that the heat conduction coefficients and the viscosities in fluids must be positive or zero. This paper became a citation classic, and the procedure used in this paper has been dubbed the "Coleman-Noll procedure". Typing this term into Google yields about 3500 hits.

In *The Non-Linear Field Theories of Mechanics*, published in 1965, Clifford Truesdell and I presented what we called "The general theory of material behavior" (Chapter C of [NLFT]). About seven years later, in about 1971, I realized that this "general theory" cannot be the final word. (See the *Preface to the Second Edition*, published in 1992.) What was needed is a general framework for the formulation of constitutive laws, using the concept of a *state* of a material element. I presented such a framework in the paper *A New Mathematical Theory of Simple Materials*, 50 pages long, published in 1972 [N1]. This framework dealt only with purely mechanical phenomena. However, in the end (Sect.21), I proposed that this framework should be generalized to become a general thermomechanical theory. It took 35 years before somebody started to develop such a generalization. It was my doctoral student Brian Seguin, now at the end of his fourth year as a graduate student at CMU. His work could be of groundbreaking significance.

I was disappointed that my 1972 paper [N1] mentioned above generated very little resonance. The reason is that it required familiarity with a mathematical infrastructure that was and still is foreign to most people interested in thermomechanics. Here is an example: A letter from the editor of the *Reviews of Modern Physics*, written in 1988, informed me that a 1961 paper by Bernard Coleman and me had become a citation classic and that they would welcome receiving other papers from me. In 1995 I submitted a paper entitled *On Material Frame Indifference* to this journal. I thought that this paper should be of interest to an audience wider than just those interested only in the mathematics of continuum physics. However, my paper was rejected even though the editor conceded that "the article is clearly written". Here are some quotes from the reviewer:

"I enjoyed reading this paper and very much would like to see it published. I am afraid, however, that the Reviews of Modern Physics is not the appropriate place. I believe that the overwhelming majority of the readers of the journal will consider the paper unreadable. Not because the material presented is intrinsically difficult, but rather because the author's individual form of the 'Bourbakian' style is far removed from anything that physicists are willing to digest. .... Professor Noll is highly respected in the mathematical community and has more than once proved himself to be ahead of his time. ..."

The paper is now part 2 of my *Five Contributions to Natural Philosophy* [FC].

In a recent exchange of e-mails, another scientist also told me that he has "reservations for the Boubakization of mechanics". Here is my answer to him:

"My more recent work is almost entirely expressed in terms of the stage 3 infrastructure of mathematics. This is not just a matter of notation or style. The language of stage 2 mathematics is simply insufficient to describe my ideas. I am convinced that this stage 3 infrastructure will eventually prevail, even among physicists and engineering scientist, but it may take another 40 years. To call it 'Bourbakization' is a gross mischaracterization."

The stage 2 mathematical infrastructure is based on the concepts of variables, constants, and parameters. The stage 3 mathematical infrastructure is based on the concepts of sets and mappings. For a detailed explanation, see my essay entitled *The Conceptual Infrastructure of Mathematics* [N2].

One of the issues that has created much confusion in the literature on thermomechanics is the *Principle of Material Frame Indifference*. It is often confused with material symmetry. I deal with this issue in part 2 of the *Five Contributions to Natural Philosophy* [FC] mentioned above. The best way to avoid using this Principle and not confuse it with material symmetry is to formulate constitutive laws without using a frame of reference. This has been done in [N1], [FC], [N3], and also in Brian Seguin's work to be described in a later presentation.

I am tired of being ahead of my time and hope to induce some people to become familiar with some of the stage 3 mathematical infrastructure that will be needed, for example, to understand the paper *Basic Concepts of Thermomechanics* [NS1] by Brian Seguin and me outlined in Section 3 below. A detailed treatment of some of this infrastructure is presented in my textbook [FDS], now available free of charge on the internet. The following gives a taste of it.

2. Mathematical infrastructure

2.1. Sets and Mappings

The set of real numbers will be denoted by $\mathbb{R}$. The set of positive numbers (including zero) will be denoted by $\mathbb{P}$ while the set of strictly positive numbers (excluding zero) will be denoted by $\mathbb{P}^{\times}$.

In order to specify a **mapping** $f : \mathcal{A} \longrightarrow \mathcal{B}$, one first has to prescribe two sets, $\mathcal{A}$ and $\mathcal{B}$, and then a definite procedure, called the **evaluation rule** of f, which assigns to each element $a \in \mathcal{A}$ exactly one element $f(a) \in \mathcal{B}$. The set $\mathcal{A}$ is called the **domain** of f and the set $\mathcal{B}$ is called the **codomain** of f. We say that f is a mapping from $\mathcal{A}$ to $\mathcal{B}$ or maps $\mathcal{A}$ to $\mathcal{B}$.

For every set $\mathcal{A}$ we have the **identity mapping** $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ of $\mathcal{A}$, defined by $1_{\mathcal{A}}(a) := a$ for all $a \in \mathcal{A}$.

The **composite** $g \circ f : \mathcal{C} \longrightarrow \mathcal{A}$ of two mappings $f : \mathcal{A} \longrightarrow \mathcal{B}$ and $g : \mathcal{B} \longrightarrow \mathcal{C}$ is defined by

$$(g \circ f)(a) := g(f(a)) \quad \text{for all} \quad a \in \mathcal{A}.$$

Now let a **mapping** $f : \mathcal{A} \longrightarrow \mathcal{B}$ be given. We say that f is **injective** if, for every $b \in \mathcal{B}$, there is *at most one* $a \in \mathcal{A}$ such that $f(a) = b$. We say that f is **surjective** if, for every $b \in \mathcal{B}$, there is *at least one* $a \in \mathcal{A}$ such that $f(a) = b$. The mapping f is both injective and surjective if and only if, for every $b \in \mathcal{B}$, there is *exactly one* $a \in \mathcal{A}$ such that $f(a) = b$. In that case, we say that f is **invertible** and we define the **inverse** $f^{\leftarrow} : \mathcal{B} \longrightarrow \mathcal{A}$ by the procedure which associates with each $b \in \mathcal{B}$ the only $a := f^{\leftarrow}(b) \in \mathcal{A}$ which satisfies $f(a) = b$. We then have

$$f \circ f^{\leftarrow} = 1_{\mathcal{B}} \quad \text{and} \quad f^{\leftarrow} \circ f = 1_{\mathcal{A}}.$$

The set

$$\text{Rng } f := \{f(a) \in \mathcal{B} \mid a \in \mathcal{A}\}$$

is called the **range** of f. We have $\text{Rng } f = \mathcal{B}$ if and only if f is surjective.

The mapping f induces a mapping $f_{>} : \text{Sub } \mathcal{A} \longrightarrow \text{Sub } \mathcal{B}$, from the set $\text{Sub } \mathcal{A}$ of all subsets of $\mathcal{A}$ to the set $\text{Sub } \mathcal{B}$ of all subsets of $\mathcal{B}$. It is defined by

$$f_{>}(A) := \{f(a) \in \mathcal{B} \mid a \in \mathcal{A}\} \quad \text{for all} \quad A \in \text{Sub } \mathcal{A}$$

and called the **image mapping** of f.

Given two sets, $\mathcal{A}_1$ and $\mathcal{A}_2$, one can form the **set-product** $\mathcal{A}_1 \times \mathcal{A}_2$ of $\mathcal{A}_1$ and $\mathcal{A}_2$. It is defined by

$$\mathcal{A}_1 \times \mathcal{A}_2 := \{(a_1, a_2) \mid a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2\}.$$

Given $h_1 : \mathcal{D} \longrightarrow \mathcal{A}_1$ and $h_2 : \mathcal{D} \longrightarrow \mathcal{A}_2$ one can construct the **term-wise evaluation mapping** $(h_1, h_2) : \mathcal{D} \longrightarrow \mathcal{A}_1 \times \mathcal{A}_2$ by

$$(h_1, h_2)(d) := (h_1(d), h_2(d)) \quad \text{for all} \quad d \in \mathcal{D}.$$

Conversely, given $h : \mathcal{D} \longrightarrow \mathcal{A}_1 \times \mathcal{A}_2$ then there are mappings $h_1 : \mathcal{D} \longrightarrow \mathcal{A}_1$ and $h_2 : \mathcal{D} \longrightarrow \mathcal{A}_2$ such that $h = (h_1, h_2)$. The mappings h_1 and h_2 are called the **component mappings** of h. A similar result holds when the codomain of h is the product of more the two sets.

2.2. Linear Algebra

Here we deal only with finite-dimensional real linear spaces. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be such linear spaces. We use the notation $\text{Lin}(\mathcal{T}_1, \mathcal{T}_2)$ for the set of all linear mappings from $\mathcal{T}_1$ to $\mathcal{T}_2$. This set also has the structure of a linear space and $\dim \text{Lin}(\mathcal{T}_1, \mathcal{T}_2) = \dim \mathcal{T}_1 \times \dim \mathcal{T}_2$. Given $\mathbf{L} \in \text{Lin}(\mathcal{T}_1, \mathcal{T}_2)$ and $\mathbf{v} \in \mathcal{T}_1$ we denote by $\mathbf{Lv}$ the element of $\mathcal{T}_2$ that $\mathbf{L}$ assigns to $\mathbf{v}$.

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are both linear mappings such that the composite $\mathbf{L}_1 \circ \mathbf{L}_2$ is meaningful then we will denote this composite simply by $\mathbf{L}_1 \mathbf{L}_2$. If a linear mapping $\mathbf{L}$ is invertible, we denote its inverse by $\mathbf{L}^{-1}$. We denote by $\text{Lis}(\mathcal{T}_1, \mathcal{T}_2)$ the set of all invertible linear mappings, i.e. **linear isomorphisms**, from $\mathcal{T}_1$ to $\mathcal{T}_2$. This set is non-empty if and only if $\dim \mathcal{T}_1 = \dim \mathcal{T}_2$. We use the abbreviations

$$\text{Lin } \mathcal{T} := \text{Lin}(\mathcal{T}, \mathcal{T}) \quad \text{and} \quad \text{Lis } \mathcal{T} := \text{Lis}(\mathcal{T}, \mathcal{T}).$$

The second of these sets forms a group with respect to composition, called the **linear group** of $\mathcal{T}$.

The **dual** of a linear space $\mathcal{T}$ is defined by

$$\mathcal{T}^* := \text{Lin}(\mathcal{T}, \mathbb{R}).$$

In accordance with the general rule of denoting the evaluation of linear mappings, the value of $\boldsymbol{\lambda} \in \mathcal{T}^*$ at $\mathbf{v} \in \mathcal{T}$ will be be denoted simply by $\boldsymbol{\lambda}\mathbf{v}$. The dual $\mathcal{T}^{**}$ of the dual space $\mathcal{T}^*$ will be identified with $\mathcal{T}$ in such a way that the value at $\boldsymbol{\lambda} \in \mathcal{T}^*$ of the element of $\mathcal{T}^{**}$ identified with $\mathbf{v} \in \mathcal{T}$ is $\mathbf{v}\boldsymbol{\lambda} := \boldsymbol{\lambda}\mathbf{v}$. We have $\dim \mathcal{T}^* = \dim \mathcal{T}$.

Given $\mathbf{v} \in \mathcal{T}_2$ and $\boldsymbol{\lambda} \in \mathcal{T}_1^*$ we define the **tensor product** $\mathbf{v} \otimes \boldsymbol{\lambda} \in \text{Lin}(\mathcal{T}_1, \mathcal{T}_2)$ of $\mathbf{v}$ and $\boldsymbol{\lambda}$ by

$$(\mathbf{v} \otimes \boldsymbol{\lambda})\mathbf{u} := (\boldsymbol{\lambda}\mathbf{u})\mathbf{v} \quad \text{for all} \quad \mathbf{u} \in \mathcal{T}_1.$$

The dual of $\text{Lin } \mathcal{T}$ contains a special element $\text{tr} \in (\text{Lin } \mathcal{T})^*$ called the **trace** which is characterized by the property

$$\text{tr}(\mathbf{v} \otimes \boldsymbol{\lambda}) = \boldsymbol{\lambda}\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathcal{T}, \boldsymbol{\lambda} \in \mathcal{T}^*.$$

Another mapping of interest is the **determinant** $\det : \text{Lin } \mathcal{T} \longrightarrow \mathbb{R}$.[2] We use the notation

$$\text{Unim } \mathcal{T} := \{\mathbf{L} \in \text{Lis } \mathcal{V} \mid |\det \mathbf{L}| = 1\}$$

for the **unimodular group**, which is a subgroup of $\text{Lis } \mathcal{T}$, and the notation

$$\text{Unim}^+ \mathcal{T} := \{\mathbf{L} \in \text{Lis } \mathcal{V} \mid \det \mathbf{L} = 1\}$$

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[2]For a matrix-free definition see Sect.14 of [FDS], Vol.II

for the **proper unimodular group**, which is a subgroup of Unim $\mathcal{T}$.

To every $\mathbf{L} \in \mathrm{Lin}\,(\mathcal{T}_1, \mathcal{T}_2)$ one can associate exactly one element $\mathbf{L}^\top \in \mathrm{Lin}\,(\mathcal{T}_2^*, \mathcal{T}_1^*)$, called the **transpose** of $\mathbf{L}$, characterized by the condition that

$$\boldsymbol{\lambda}(\mathbf{L}\mathbf{v}) = (\mathbf{L}^\top \boldsymbol{\lambda})\mathbf{v} \quad \text{for all} \ \ \mathbf{v} \in \mathcal{T}_1, \ \boldsymbol{\lambda} \in \mathcal{T}_2^*.$$

The space $\mathrm{Lin}\,(\mathcal{T}, \mathcal{T}^*)$ will be identified with the space of all bilinear forms on $\mathcal{T}$. The subspace

$$\mathrm{Sym}\,(\mathcal{T}, \mathcal{T}^*) := \{\mathbf{L} \in \mathrm{Lin}\,(\mathcal{T}, \mathcal{T}^*) \mid \mathbf{L}^\top = \mathbf{L}\}$$

of $\mathrm{Lin}\,(\mathcal{T}, \mathcal{T}^*)$ will be identified with the space of all symmetric bilinear forms. The subset

$$\mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*) := \{\mathbf{G} \in \mathrm{Sym}\,(\mathcal{T}, \mathcal{T}^*) \mid (\mathbf{G}\mathbf{v})\mathbf{v} > 0$$
$$\text{for all} \ \ \mathbf{v} \in \mathcal{T} \ \text{with} \ \mathbf{v} \neq \mathbf{0}\}$$

of $\mathrm{Sym}\,(\mathcal{T}, \mathcal{T}^*)$ will be identified with the set of all strictly positive bilinear forms. It is an open subset and a **linear cone** in $\mathrm{Sym}\,(\mathcal{T}, \mathcal{T}^*)$, but not a subspace. We note that $\mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*) \subset \mathrm{Lis}\,(\mathcal{T}, \mathcal{T}^*)$.

A (genuine) **inner-product space** $\mathcal{V}$ is a linear space endowed with additional structure by singling out a specific element $\mathrm{ip} \in \mathrm{Pos}^+(\mathcal{V}, \mathcal{V}^*)$, called the **inner-product**. The inner-product is used to identify the linear space $\mathcal{V}$ with its dual $\mathcal{V}^*$. It is customary to use the notation

$$\mathbf{v} \cdot \mathbf{u} := (\mathrm{ip}\,\mathbf{v})\mathbf{u} \quad \text{for all} \ \ \mathbf{v}, \mathbf{u} \in \mathcal{V}.$$

The **magnitude** $|\mathbf{u}|$ of an element $\mathbf{u} \in \mathcal{V}$ is defined by $|\mathbf{u}| := \sqrt{\mathbf{u} \cdot \mathbf{u}}$.

If $\mathcal{T}$ is just a linear space without inner product then the entire theory of inner-product spaces can be applied relative to any $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$.[3] For example, for each $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ one can define

$$\mathrm{Orth}\,\mathbf{G} := \{\mathbf{A} \in \mathrm{Lis}\,\mathcal{T} \mid \mathbf{A}^\top \mathbf{G} \mathbf{A} = \mathbf{G}\},$$

$$\mathrm{Orth}^+\,\mathbf{G} := \{\mathbf{A} \in \mathrm{Orth}\,\mathbf{G} \mid \det \mathbf{A} = 1\}.$$

The first of these is the **orthogonal group** of $\mathbf{G}$, a subgroup of Unim $\mathcal{T}$, and the second of these is the **proper orthogonal group** of $\mathbf{G}$, a subgroup of $\mathrm{Unim}^+\,\mathcal{T}$. The following two facts about orthogonal groups are of interest:
(1) For all $\mathbf{G}_1, \mathbf{G}_2 \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ we have

$$\mathrm{Orth}\,\mathbf{G}_2 = \mathrm{Orth}\,\mathbf{G}_2 \iff \mathbf{G}_1 = c\mathbf{G}_2 \ \text{for some} \ c \in \mathbb{P}^\times.$$

(2) The groups $\mathrm{Orth}\,\mathbf{G}$ are all maximal subgroups of Unim $\mathcal{T}$.[4]

In the applications to thermomechanics, a three-dimensional linear space $\mathcal{T}$ represents an *infinitesimal element* of a continuous body. The elements $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$ represent the **configurations** of the element. A **deformation** is simple a change of configuration, and a **deformation process** is a mapping $\mathbf{P} : I \longrightarrow \mathrm{Pos}^+(\mathcal{T}, \mathcal{T}^*)$, where I is a non-empty real interval.

3. Basic Thermomechanics

3.1. Description

Here is an outline of the paper entitled *Basic Concepts of Thermomechanics* [NS1] by Brian Seguin and me, which is intended to serve as a model for the first few chapters of future textbooks on continuum mechanics and continuum thermomechanics. It differs from most existing such textbooks in several important respects:

1) It uses the stage 3 mathematical infrastructure mentioned above.

2) It is completely coordinate-free and $\mathbb{R}^n$-free when dealing with basic concepts.

3) It does not use a fixed *physical space*. Rather, it employs an infinite variety of *frames of reference*, each of which is a Euclidean space. The motivation for avoiding physical space can be found in Part 1, entitled *On the Illusion of Physical Space*, of [FC]. Here, the basic laws are formulated without the use of a physical space or any external frame of reference. The only frames of reference used there are internal ones, generated by the configurations of the body systems used.

4) It considers inertia as only one of many external forces and does not confine itself to using only inertial frames of reference. Hence kinetic energy, which is a potential for inertial forces, does not appear separately in the energy balance equation. In particle mechanics, inertia plays a fundamental role and the subject would collapse if it is neglected. Not so in continuum mechanics, where it is often appropriate to neglect inertia, for example when analyzing the motion of toothpaste when it is extruded slowly from a tube.

3.2. Physical Systems

In the mid 1950's I regularly taught courses for engineering students with the titles *Statics* and *Dynamics*. In *Statics*, the students were asked to consider some system (a building, a bridge, or a machine), draw *free-body diagrams*, and apply to each of these the balance of forces and torques. This often gave enough linear equations to determine the stresses in each of the pieces of the system. In *Dynamics*, the students were asked to apply the same procedure as in Statics, except that inertial forces are

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[3]This insight, to the best of my knowledge, was first employed in [N1].
[4]The proof is given in [N5].

taken into account. This often led to linear differential equations. I then wondered what the underlying conceptual background of all this was. Here is the result:

Definition 1. *An ordered set Ω with order $\prec$ is said to be* **materially ordered** *if the following axioms are satisfied:*

(MO1) Ω has a maximum ma *and a minimum* mn.

(MO2) Every doubleton has an infimum.

(MO3) For every $p \in \Omega$ there is exactly one member of Ω denoted by p^{rem}, such that $\inf\{p, p^{\mathrm{rem}}\} = $ mn and $\sup\{p, p^{\mathrm{rem}}\} = $ ma.

(MO4) $(\inf\{p, q^{\mathrm{rem}}\} = $ mn$)$
$\implies p \prec q$ *for all $p, q \in M$.*

The mapping $\mathrm{rem} := (p \mapsto p^{\mathrm{rem}}) : \Omega \longrightarrow \Omega$ *is called the* **remainder mapping** *in Ω.*

Here Ω is considered to consist of the whole system and all of its parts. Given $a, b \in \Omega$, $a \prec b$ is read "a is a part of b". The maximum ma is the "material all", i.e. the whole system, and the minimum mn is the "material nothing". The $\inf\{a, b\}$ is the overlap of a and b, and a^{rem} is the part of the whole system ma that remains after a has been removed. With this in mind, the two conditions (MO3) and (MO4) above are very natural.

Theorem 1: *Let Ω be a materially ordered set. Then Ω has the structure of a Boolean algebra with*

$$p \wedge q := \inf\{p, q\} \text{ and } p \vee q := \sup\{p, q\}$$

$$\text{for all } p, q \in \Omega, \tag{1}$$

The proof is highly non-trivial. The best version is given in [NS2].

3.3 Additive Mappings a and Interactions

Let Ω be a materially ordered set and $\mathcal{W}$ a linear space. We say that the parts p and q are **separate** if $p \wedge q = $ mn. We use the notation

$$(\Omega^2)_{\mathrm{sep}} := \{(p, q) \in \Omega^2 \mid p \wedge q = \mathrm{mn}\} . \tag{2}$$

Definition 2. *A function $\mathbf{H} : \Omega \longrightarrow \mathcal{W}$ is said to be* **additive** *if*

$$\mathbf{H}(p \vee q) = \mathbf{H}(p) + \mathbf{H}(q)$$

$$\text{for all } (p, q) \in (\Omega^2)_{\mathrm{sep}} . \tag{3}$$

For every $p \in \Omega$ we put $\Omega_p := \{q \in \Omega \mid q \prec p\}$, which again has the structu of a materially ordered set.

Definition 3. *A function $\mathbf{I} : (\Omega^2)_{\mathrm{sep}} \longrightarrow \mathcal{W}$ is said to be an* **interaction** *if, for all $p \in \Omega$, both*

$$\mathbf{I}(\,\cdot\,, p^{\mathrm{rem}}) : \Omega_p \longrightarrow \mathcal{W} \text{ and } \mathbf{I}(p^{\mathrm{rem}}, \,\cdot\,) : \Omega_p \longrightarrow \mathcal{W}$$

are additive.

The **resultant** $\mathrm{Res}_{\mathbf{I}} : \Omega \to \mathcal{W}$ *of a given interaction I is defined by*

$$\mathrm{Res}_{\mathbf{I}}(p) := \mathbf{I}(p, p^{\mathrm{rem}}) \text{ for all } p \in \Omega . \tag{4}$$

We say that a given interaction is **skew** *if*

$$\mathbf{I}(q, p) = -\mathbf{I}(p, q) \text{ for all } (p, q) \in (\Omega^2)_{\mathrm{sep}} . \tag{5}$$

Theorem 2: *An interaction is skew if and only if its resultant is additive.*

The proof of this result is fairly easy but not entirely trivial.

Remark 1: The concept of an interaction is an abstraction. Its values may have the interpretation of forces, torques, or heat transfers. In most of the past literature these cases were treated separately even though much of the underlying mathematics is the same for all. Thus, this abstraction, like most others, is a labor saving device. ∎

3.4. Continuous Bodies

A continuous body system is a set $\mathcal{B}$ of **material points**, endowed with structure by the specification of a non-empty set $\mathrm{Conf}\,\mathcal{B}$, whose elements are called **configurations** of $\mathcal{B}$. These configurations are Euclidean metrics and the set $\mathrm{Conf}\,\mathcal{B}$ is subject to several conditions. In order to state these, one needs several concepts:

1) A class Fr of **fit regions**, which are subsets Euclidean spaces. These regions are those that a body system can occupy. Intuitively, the term "body" suggest that the regions it can occupy are connected. We do not assume this but, for simplicity, we will use the term "body" rather than "body system" from now on.

2) A class $\mathrm{Pl}\,\mathcal{B}$ of **placements** of the body, which are invertible mappings from the body $\mathcal{B}$ to a fit region in some Euclidean space, called the **frame-space** of the placement.

3) A class Tr of **transplacements**, which are invertible mappings from a fit region in some frame-space to a fit region in the same or another frame-space.

The precise definitions and conditions that these concept must satisfy are, to some extent, quite technical and can be found in [NS1].

Each configuration $\delta \in \mathrm{Conf}\,\mathcal{B}$ can be used to construct a Euclidean space $\mathcal{E}_\delta$ and and an **imbedding** imb_δ, which is a special placement of the body in the frame-space $\mathcal{E}_\delta$. These special frame-spaces are internal, because they constructed from configurations; they are not external, i.e., introduced from the

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outside. In general, placements are in frame-spaces introduced from the outside.

Every placement μ induces a configuration δ_μ, but infinitely many placements all induce the same configuration.

Given a continuous body $\mathcal{B}$, consider the set $\Omega_\mathcal{B}$ defined by

$$\Omega_\mathcal{B} := \{\mathcal{P} \in \mathrm{Sub}\mathcal{B} \mid \mathrm{imb}_{\delta>}(\mathcal{P}) \in \mathrm{Fr}$$

$$\text{for every } \delta \in \mathrm{Conf}\mathcal{B}\} . \qquad (6)$$

In (3.6) "every" can be replaced by "some" without change of meaning.

The members of $\Omega_\mathcal{B}$ are called **parts** or **subbodies** of $\mathcal{B}$.

$\Omega_\mathcal{B}$ is *materially ordered*, as defined in Sect.3.1, by inclusion. It has the structure of a Boolean algebra with

$$\mathcal{P} \wedge \mathcal{Q} := \mathcal{P} \cap \mathcal{Q} , \qquad (7)$$

$$\mathcal{P} \vee \mathcal{Q} := \mathrm{Int} \ \mathrm{Clo}(\mathcal{P} \cup \mathcal{Q}) , \qquad (8)$$

$$\mathcal{P}^{\mathrm{rem}} := \mathrm{Int}(\mathcal{B}\backslash\mathcal{P}) , \qquad (9)$$

The proof of this highly non-trivial result can be found in [NV].

3.5. Densities and Contactors

We assume that a continuous body $\mathcal{B}$ is given. For each part $\mathcal{P} \in \Omega_\mathcal{B}$ and each placement $\mu \in \mathrm{Pl}\,\mathcal{B}$ we use the notation $\mathcal{P}_\mu := \mu_>(\mathcal{P})$ and, in particular. $\mathcal{B}_\mu := \mathrm{Rng}\,\mu$. We say that a part $\mathcal{P} \in \Omega_\mathcal{B}$ is **internal** if, for every placement $\mu \in \mathrm{Pl}\,\mathcal{B}$, we have $\mathrm{Clo}\mathcal{P}_\mu \subset \mathcal{B}_\mu$. We denote the set of all internal parts by $\Omega_\mathcal{B}^{\mathrm{int}}$.

Definition 4. *An additive mapping* $\mathbf{H} : \Omega_\mathcal{B} \longrightarrow \mathcal{W}$ *is said to have* **densities** *if, for every* $\mu \in \mathrm{Pl}\,\mathcal{B}$*, there is a continuous mapping* $\mathbf{h}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{W}$ *such that*

$$\mathbf{H}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \mathbf{h}_\mu \quad \text{for all} \ \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}} . \qquad (10)$$

We call $\mathbf{h}_\mu$ *the* **density** *of* $\mathbf{H}$ *in the placement* μ.

Definition 5. *We say that an interaction* $\mathbf{I} : (\Omega_\mathcal{B})_{\mathrm{sep}}^2 \longrightarrow \mathcal{W}$ *has* **contactors** *if, for every placement* μ*, there is a* C^1 *mapping* $\mathbf{C}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Lin}\,(\mathcal{V}_\mu, \mathcal{W})$ *such that*

$$\mathbf{I}(\mathcal{P}, \mathcal{Q}) = \int_{\mathrm{Rct}_\mu(\mathcal{P},\mathcal{Q})} \mathbf{C}_\mu \mathbf{n}_{\mathcal{P}_\mu}$$

$$\text{for all} \ \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_\mathcal{B})_{\mathrm{sep}}^2 \ \text{with} \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}} . \qquad (11)$$

We call $\mathbf{C}_\mu$ *the* **contactor** *of* $\mathbf{I}$ *in the placement* μ.

In both these definitions, "every" can be replaced by "some" without change of meaning.

Theorem 3: *Given an interaction* $\mathbf{I} : (\Omega_\mathcal{B})_{\mathrm{sep}}^2 \longrightarrow \mathcal{W}$ *with contactors and an additive mapping* $\mathbf{H} : \Omega_\mathcal{B} \longrightarrow \mathcal{W}$ *with densities, the following three conditions are equivalent:*

1) We have

$$\mathrm{Res}_\mathbf{I}(\mathcal{P}) + \mathbf{H}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \ \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}} . \qquad (12)$$

2) For every placement $\mu \in \mathrm{Pl}\,\mathcal{B}$*, we have*

$$\mathrm{div}\mathbf{C}_\mu + \mathbf{h}_\mu = \mathbf{0} . \qquad (13)$$

where $\mathbf{h}_\mu$ *is the density of* $\mathbf{H}$ *in the placement* μ*, and* $\mathbf{C}_\mu$ *is the* **contactor** *of* $\mathbf{I}$ *in the placement* μ.

3) Condition 2) holds with with "every" replaced by "some".

3.6. Balance of Forces and Torques

It is often useful to fix a Euclidean space $\mathcal{E}$, with translation space $\mathcal{V}$, and confine one's attention to placements whose range space is $\mathcal{E}$. It is then useful to consider force systems with values in $\mathcal{V}$, independent of the choice of a configuration, as follows:

Definition 6. *A* **force system** *in the space* $\mathcal{V}$ *is a pair* $(\mathbf{F}^\mathrm{i}, \mathbf{F}^\mathrm{e})$*, where* $\mathbf{F}^\mathrm{i} : (\Omega_\mathcal{B})_{\mathrm{sep}}^2 \longrightarrow \mathcal{V}$ *is an interaction and* $\mathbf{F}^\mathrm{e} : \Omega_\mathcal{B} \longrightarrow \mathcal{V}$ *is additive. The mapping* $\mathbf{F}^\mathrm{i}$ *is called the* **internal force system** *in* $\mathcal{V}$ *and* $\mathbf{F}^\mathrm{e}$ *is called the* **external force system** *in* $\mathcal{V}$.

Let a force system $(\mathbf{F}^\mathrm{i}, \mathbf{F}^\mathrm{e})$ in $\mathcal{V}$ be a given. The first fundamental law of mechanics, called the **Balance of Forces**, says:

$$\mathrm{Res}_{\mathbf{F}^\mathrm{i}}(\mathcal{P}) + \mathbf{F}^\mathrm{e}(\mathcal{P}) = \mathbf{0} \quad \text{for all} \ \ \mathcal{P} \in \Omega_\mathcal{B} . \qquad (14)$$

We say that the system $(\mathbf{F}^\mathrm{i}, \mathbf{F}^\mathrm{e})$ is **force-balanced** if (14) holds.

Since $\mathbf{F}^\mathrm{e}$ is additive, the following **Law of Action and Reaction** is an immediate consequence of (14) and Theorem 2:

Every balanced balanced internal force system is skew, i. e. ,

$$\mathbf{F}^\mathrm{i}(\mathcal{P}, \mathcal{Q}) = -\mathbf{F}^\mathrm{i}(\mathcal{Q}, \mathcal{P}) \quad \text{for all} \ \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_\mathcal{B})_{\mathrm{sep}}^2 . \qquad (15)$$

Remark 2: The law of action and reaction is often referred to as *Newton's Third Law*. Thus, if one assumes the balance law (3.14), one can prove Newton's Third Law instead of assuming it a priori, as Newton and many physics textbooks since Newton have done. The balance of forces has been understood by engineers, if only implicitly, since antiquity. ∎

We now assume now that $\mathbf{F}^\mathrm{i}$ has contactors and $\mathbf{F}^\mathrm{e}$ has densities..

Let μ be a placement of the body in $\mathcal{E}$ and put $\mathcal{B}_\mu := \mu_>(\mathcal{B})$. Let $\mathbf{T}_\mu : \mathcal{B}_\mu :\longrightarrow \mathrm{Lin}\,\mathcal{V}$ denote the contactor for $\mathbf{F}^\mathrm{i}$ and let $\mathbf{b}_\mu : \mathcal{B}_\mu :\longrightarrow \mathcal{V}$ denote the

density of $\mathbf{F}^{\mathrm{e}}$ in the given the placement μ. It follows from Theorem 3 that the force balance, restricted to internal parts $\mathcal{P}$, is equivalent to

$$\mathrm{div}\,\mathbf{T}_\mu + \mathbf{b}_\mu = \mathbf{0} \ . \qquad (16)$$

It is usual to assume that the force system is not only force-balanced but also **torque-balanced**. We will not give a precise definition here but note that the condition

$$\mathrm{Rng}\,\mathbf{T}_\mu \subset \mathrm{Sym}\ \mathcal{V} \qquad (17)$$

is equivalent to balance of torques for internal parts.

From here on we will assume that (16) and (17) are valid. We adjust the codomain of $\mathbf{T}_\mu$ to $\mathrm{Sym}\ \mathcal{V}$ without change of notation and call $\mathbf{T}_\mu : \mathcal{B}_\mu :\longrightarrow \mathrm{Sym}\,\mathcal{V}$ the **Cauchy stress** of the force system in the placement μ and the mapping $\mathbf{b}_\mu : \mathcal{B}_\mu \longrightarrow \mathcal{V}$ the **external body-force density** in the placement μ.

Let a configuration $\delta \in \mathrm{Conf}\mathcal{B}$ be given. As in Section 3.4, we denote the imbedding space for δ by $\mathcal{E}_\delta$ and its translation space by $\mathcal{V}_\delta$, and we use the results above in the case when $\mu := \mathrm{imb}_\delta$ and write, for simplicity, δ rather than imb_δ as a subscript.

Definition 7. *A **force system in the configuration** δ is a pair $(\mathbf{F}_\delta^{\mathrm{i}}, \mathbf{F}_\delta^{\mathrm{e}})$ which is a force system, both force-balanced and torque-balanced, in the space $\mathcal{V}_\delta$ in the sense of Definition 6.*

Let such a force system $(\mathbf{F}_\delta^{\mathrm{i}}, \mathbf{F}_\delta^{\mathrm{e}})$ in $\mathcal{V}_\delta$ be given. We assume that $\mathbf{F}_\delta^{\mathrm{i}}$ has contactors and that $\mathbf{F}_\delta^{\mathrm{e}}$ has densities. The results (3.16), and (17) remain valid when the subscript δ is used instead of μ, when $\mathbf{T}_\delta$ is interpreted to be the contactor of $\mathbf{F}_\delta^{\mathrm{i}}$ in the placement imb_δ, and when $\mathbf{b}_\delta$ is interpreted to be the density of $\mathbf{F}_\delta^{\mathrm{e}}$ in the placement imb_δ. We may call $\mathbf{T}_\delta$ the **configurational stress** and imb_δ the **configurational body force density** for δ.

Definition 8. *A **mechanical process** is a time-family $((\bar\delta_t, \bar{\mathbf{F}}_t^{\mathrm{i}}, \bar{\mathbf{F}}_t^{\mathrm{e}}) \mid t \in I)$ of triples, where $(\bar\delta_t \mid t \in I)$ is a deformation process and, for every $t \in I$, $(\bar{\mathbf{F}}_t^{\mathrm{i}}, \bar{\mathbf{F}}_t^{\mathrm{e}})$ is a force system, both force-balanced and torque-balanced, in the configuration $\bar\delta_t$, as defined by Definition 7.*

3.7. Energy Balance

Energy and heat are fundamental concepts in thermodynamics.

Definition 9. *A **heat transfer system** is a pair $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$, where $Q^{\mathrm{i}} : (\Omega_\mathcal{B})_{\mathrm{sep}}^2 \longrightarrow \mathbb{P}^\times$ is an interaction and $Q^{\mathrm{e}} : \Omega_\mathcal{B} \longrightarrow \mathbb{P}^\times$ is additive. The mapping Q^{i} is called the **internal heat transfer** and Q^{e} is called the **external heat absorption**.*

We assume that Q^{e} has densities and Q^{i} has contactors. Hence, given any placement μ in a frame-space $\mathcal{E}$ with translation space $\mathcal{V}$, we obtain a density $r_\mu : \mathcal{B}_\mu \longrightarrow \mathbb{R}$ and a contactor $-\mathbf{q}_\mu : \mathcal{B}_\mu \longrightarrow \mathrm{Lin}\,(\mathcal{V}.\mathbb{R}) \cong \mathcal{V}$ such that

$$Q^{\mathrm{e}}(\mathcal{P}) = \int_{\mathcal{P}_\mu} r_\mu \quad \text{for all}\ \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}}\ . \qquad (18)$$

$$Q^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_\mu(\mathcal{P}, \mathcal{Q})} \mathbf{q}_\mu \cdot \mathbf{n}_{\mathcal{P}_\mu}$$

$$\text{for all}\ \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_\mathcal{B})_{\mathrm{sep}}^2\ \ \text{with}\ \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}}\ . \qquad (19)$$

r_μ is called the **heat absorption field** and $\mathbf{q}_\mu$ the **heat flux** of the given heat transfer system in the placement μ.

Definition 10 *An **energetic process** is a sextuple $((\bar\delta_t, \bar{\mathbf{F}}_t^{\mathrm{i}}, \bar{\mathbf{F}}_t^{\mathrm{e}}, \bar{Q}_t^{\mathrm{i}}, \bar{Q}_t^{\mathrm{e}}, \bar{E}_t) \mid t \in I)$ of time-families such that $((\bar\delta_t, \bar{\mathbf{F}}_t^{\mathrm{i}}, \bar{\mathbf{F}}_t^{\mathrm{e}}) \mid t \in I)$ is a mechanical process, $((\bar{Q}_t^{\mathrm{i}}, \bar{Q}_t^{\mathrm{e}}) \mid t \in I)$ is a time-family of heat transfer systems, and $(\bar{E}_t \mid t \in I)$ is an appropiatly defined differentiable time-family of additive mappings, called the **internal energy**.*

We say that a given energetic process $((\bar\delta_t, \bar{\mathbf{F}}_t^{\mathrm{i}}, \bar{\mathbf{F}}_t^{\mathrm{e}}, \bar{Q}_t^{\mathrm{i}}, \bar{Q}_t^{\mathrm{e}}, \bar{E}_t) \mid t \in I)$ is **energy-balanced** if

$$\bar{P}_t(\mathcal{P}) + \mathrm{Res}_{\bar{Q}_t^{\mathrm{i}}}(\mathcal{P}) + \bar{Q}_t^{\mathrm{e}}(\mathcal{P}) = \bar{E}_t^\bullet(\mathcal{P})$$

$$\text{for all}\quad \mathcal{P} \in \Omega_\mathcal{B}\ \text{and}\ t \in I\ , \qquad (20)$$

where $(\bar{P}_t \mid t \in I)$ is the power-family determined by the mechanical process $((\bar\delta_t, \bar{\mathbf{F}}_t^{\mathrm{i}}, \bar{\mathbf{F}}_t^{\mathrm{e}}) \mid t \in I)$.

Under suitable conditions, the energy balance law can be given a local form

3.8. Temperature and Entropy

Theorem 6. *Let a heat transfer system $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$, as defined by Definition 10, and a function $\theta : \mathcal{B} \longrightarrow \mathbb{P}^\times$ of class C^1, called **temperature**, be given, and assume that Q^{i} has contactors and Q^{e} has densities. Then there is a pair $(H^{\mathrm{i}}, H^{\mathrm{e}})$, where $H^{\mathrm{i}} : (\Omega_\mathcal{B})_{\mathrm{sep}}^2 \longrightarrow \mathbb{R}$ is an interaction and $H^{\mathrm{e}} : \Omega_\mathcal{B} \longrightarrow \mathbb{R}$ is the additive function such that, for every placement μ we have*

$$H^{\mathrm{i}}(\mathcal{P}, \mathcal{Q}) = -\int_{\mathrm{Rct}_\mu(\mathcal{P}, \mathcal{Q})} \frac{\mathbf{q}_\mu}{\theta_\mu} \cdot \mathbf{n}_{\mathcal{P}_\mu}$$

$$\text{for all}\ \ (\mathcal{P}, \mathcal{Q}) \in (\Omega_\mathcal{B})_{\mathrm{sep}}^2\ \ \text{with}\ \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}} \qquad (21)$$

and

$$H^{\mathrm{e}}(\mathcal{P}) = \int_{\mathcal{P}_\mu} \frac{r_\mu}{\theta_\mu} \quad \text{for all}\ \ \mathcal{P} \in \Omega_\mathcal{B}^{\mathrm{int}}\ , \qquad (22)$$

*where $\mathbf{q}_\mu$ is the heat flux in the placement μ, r_μ is the external heat absorption density in μ and $\theta_\mu := \theta \circ \mu^\leftarrow$. The pair $(H^{\mathrm{i}}, H^{\mathrm{e}})$ is called the **entropy***

transfer system *generated by the heat transfer system* $(Q^{\mathrm{i}}, Q^{\mathrm{e}})$ *and the temperature* θ. H^{i} *is called the* **internal entropy transfer**, *and* H^{e} *is called the* **external entropy absorption**.

The proof is not very difficult.

Definition 11. *Let by a time-famiy* $((\bar{Q}_t^{\mathrm{i}}, \bar{Q}_t^{\mathrm{e}}) \mid t \in I)$ *of heat transfer systems and a time-family of temperatures* $(\bar{\theta}_t \mid t \in I)$ *be given, and let* $((\bar{H}_t^{\mathrm{i}}, \bar{H}_t^{\mathrm{e}}) \mid t \in I)$ *be the resulting entropy transfer system as described in Theorem 6. Let* $(\bar{N}_t : \Omega_{\mathcal{B}} \longrightarrow \mathbb{R} \mid t \in I)$ *be a differentiable time-family of additive mappings, called the* **internal entropy**.

We say that $((\bar{H}_t^{\mathrm{i}}, \bar{H}_t^{\mathrm{e}}, \bar{N}_t) \mid t \in I)$ *is a* **dissipative entropical process** *if*

$$\bar{N}_t^{\bullet}(\mathcal{P}) \geq \mathrm{Res}_{\bar{H}_t^{\mathrm{i}}}(\mathcal{P}) + \bar{H}_t^{\mathrm{i}}(\mathcal{P})$$

$$\text{for all } \mathcal{P} \in \Omega_{\mathcal{B}}^{\mathrm{int}} \text{ and } t \in I . \qquad (23)$$

Under suitable conditions, the energy balance law can be given a local form.

Using the local form of the energy balance law and the local form of the dissipation inequality, one can obtain the following **intrinsic reduced dissipation inequality**:

$$\bar{\rho}_{\mathrm{m}}(\bar{\theta}\bar{\eta}^{\bullet} - \bar{\epsilon}^{\bullet}) + \frac{1}{2}\mathrm{tr}(\bar{\mathbf{S}}\bar{\mathbf{G}}^{\bullet}) - \frac{1}{\bar{\theta}}\bar{\gamma}\bar{\mathbf{h}} \geq 0 , \qquad (24)$$

where $\bar{\gamma}(X,t) := \nabla_X \bar{\theta}(\cdot, t) \in \mathcal{T}_X^*$ for all $X \in \mathcal{B}$, $t \in I$, and where the **intrinsic stress** $\bar{\mathbf{S}}$ and the **intrinsic heat flux** $\bar{\mathbf{h}}$ are related to the Cauchy-stress $\bar{\mathbf{T}}$ and the heat flux $\bar{\mathbf{q}}$ by

$$\bar{\mathbf{M}}\bar{\mathbf{S}}\bar{\mathbf{M}}^{\top} = \bar{\mathbf{T}}_{\mathrm{m}}, \quad \bar{\mathbf{M}}\mathbf{h} = \bar{\mathbf{q}}_{\mathrm{m}} , \qquad (25)$$

respectively, with $\bar{\mathbf{M}}(X,t) := \nabla_X \mu_t \in \mathrm{Lis}(\mathcal{T}_X, \mathcal{V})$ for all $X \in \mathcal{B}$, $t \in I$.

The inequality (24) involves only the (internal) **thermomechanical process**

$$(\bar{\delta}, \bar{\theta}, \bar{\mathbf{F}}^{\mathrm{i}}, \bar{Q}^{\mathrm{i}}, \bar{N}, \bar{E})$$

and not the external forces, nor the external heat transfer, nor any external frame of reference. It is (24) that is used to determine restrictions on frame-free formulated constitutive laws, as will be explained in a later presentation by Brian Seguin.

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