Homework 1 Solution

Exercise 0.16. Show that S^∞ is contractible.

Proof. Hatcher defines $S^\infty = \bigcup_n S^n$ to be the CW complex with two n-cells (corresponding to the northern and southern hemispheres of S^n) for each $n \geq 0$. We will show that S^∞ contracts into one of its 0-cells x_0. Note that for any $n \geq 1$, we can contract the Northern hemisphere of S^n to x_0, with the resulting space still being S^n but with the CW complex structure of an n-cell attached to x_0 (see picture).

Let $H^n : S^n \times I \to S^\infty$ be this homotopy. Since (S^∞, S^{n-1}) is a CW pair, it has the homotopy extension property by Proposition 0.16, hence we obtain a homotopy $\widetilde{H}^n : S^\infty \times I \to S^\infty$ for every $n \geq 1$ that contracts the Northern hemisphere of S^n to x_0. Note that $\widetilde{H}^n_0 : S^\infty \to S^\infty$ is the identity. Now define the homotopy $\widetilde{H} : S^\infty \times I \to S^\infty$ by

$$\widetilde{H}_t(x) = \begin{cases} \widetilde{H}^n_{2^n(t-1)+2} \circ \widetilde{H}^{n-1}_1 \circ \cdots \circ \widetilde{H}^1_1(x) & \text{for } t \in \left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right] \\ x_0 & \text{for } t = 1 \end{cases}$$

In other words, we perform the homotopy $\widetilde{H}^n$ during time $\left[1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}\right]$, and the final result is the constant map $\widetilde{H}_1(x) = x_0$. It remains to show $\widetilde{H}_t$ is continuous. Since S^∞ is given the weak topology, a map $\widetilde{H} : S^\infty \times I \to S^\infty$ is continuous if and only if its restriction to each n-skeleton $\widetilde{H}|_{S^n \times I}$ is continuous. However, this restriction is continuous because $\widetilde{H}$ restricted to $S^n \times I$ does the work of $\widetilde{H}^1, \ldots, \widetilde{H}^n$ during the t-interval $\left[0, 1 - \frac{1}{2^n}\right]$, and remains stationary during $\left[1 - \frac{1}{2^n}, 1\right]$. Since each of $\widetilde{H}^k$ is continuous, it follows that $\widetilde{H}|_{S^n \times I}$ is continuous as well. ☐

Remark 1. Some solutions use the definition of S^∞ as the unit ball of $\mathbb{R}^\infty$, where $\mathbb{R}^\infty = \{(x_1, \ldots, x_n, \ldots) \mid x_i \in \mathbb{R} \text{ for all } i, x_i \neq 0 \text{ for finitely many } i\} = \bigcup_n \mathbb{R}^n$, endowed with the weak topology (i.e. $U \subset \mathbb{R}^\infty$ is open iff $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all n). This definition turns out to be equivalent to the definition from Hatcher, but would take some effort to prove. In particular, you should mention this definition in your solution to get full credit.

Exercise 0.17. (a) Show that the mapping cylinder of every map $f : S^1 \to S^1$ is a CW complex.

(b) Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

Proof. (a) The CW complex structure has two 0-cells x, y, three 1-cells, and one 2-cell. The 1-cells consist of two circles a and b around x and y, and a segment c between x

and y. The 2-cell is now attached so that the top of the square is identified with a, the two sides identified with c, and the bottom glued to b based on the map $f : S^1 \to S^1$.

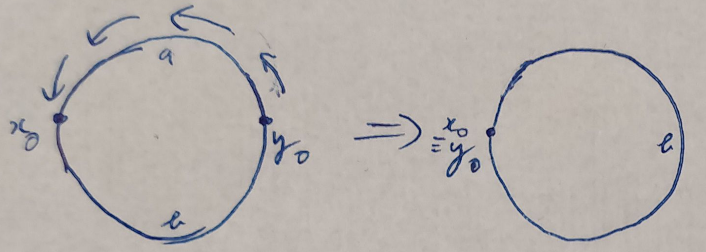
(b) Note that both the Mobius band M and the annulus A deformation retracts onto their middle circles. Hence, if we let X be the space obtained by gluing the Mobius band and the annulus along their middle circles, then X deformation retracts onto both M and A. X is given the structure of a CW complex according to the picture.

□

Exercise 0.19. Show that the space obtained from S^2 by attaching n 2-cells along any collection of n circles in S^2 is homotopy equivalent to the wedge sum of $n+1$ 2-spheres.

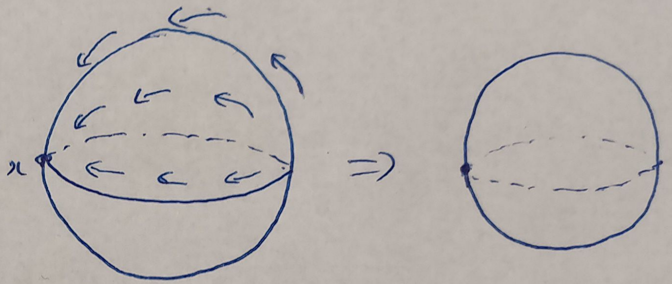
Proof. Pick a point $x_0 \in S^2$. We want to show that the space obtained by attaching a 2-cell D^2 along a circle in S^2 is homotopy equivalent to $S^2 \vee S^2$. Since S^2 is path-connected and $\pi_1(S^2) = 1$, the attaching map $f : \partial D^2 \to S^2$ is homotopic to the constant map $f_0 : \partial D^2 \to S^2$ sending $f_0(x) = x_0$. Thus, we can apply Proposition 0.18, noting that $(D^2, \partial D^2)$ is a CW pair and that $f \simeq f_0$, to obtain: $S^2 \sqcup_f D^2 \simeq S^2 \sqcup_{f_0} D^2 \simeq S^2 \vee S^2$. Similarly, attaching any number of 2-cells along circles in S^2 is equivalent to attaching them on x_0, hence if we attach n of them we get a wedge sum of $(n+1)$ 2-spheres. □

0.16.



$n = 1$

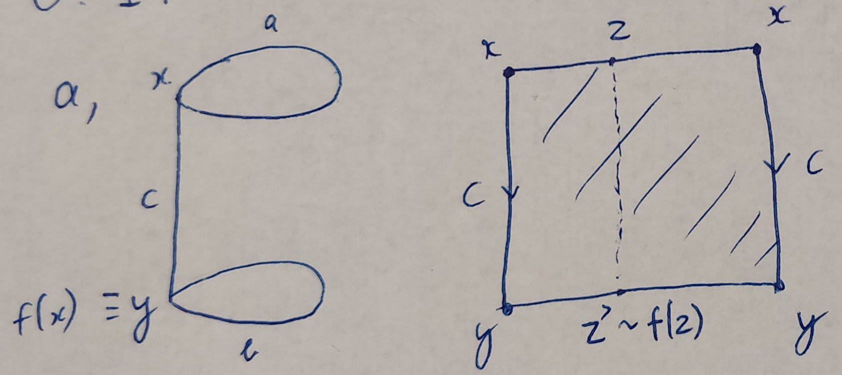
,



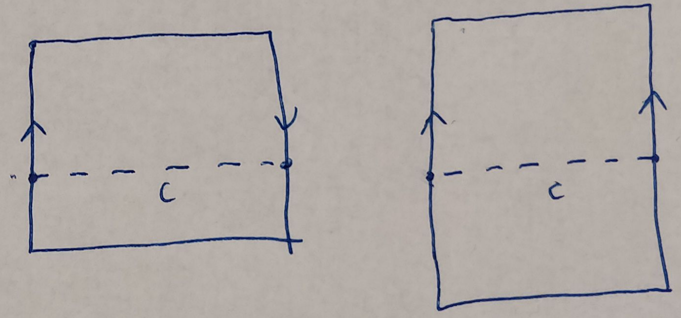
$n = 2$

0.17.

a,



b,



Glue along c.

Exercise 1.1.12 Show that all homomorphisms $\psi : \pi_1(S^1) \to \pi_1(S^1)$ can be realized as the induced homomorphism $\varphi_* : \pi_1(S^1) \to \pi_1(S^1)$ of a map $\varphi : S^1 \to S^1$.

Proof. Recall that $\pi_1(S^1) \cong \mathbb{Z}$ so we are considering group homomorphisms $\psi : \mathbb{Z} \to \mathbb{Z}$. The homomorphism is determined by $\psi(1)$ since if $\psi(1) = n$, then $\psi(k) = nk$ for all k. This map is induced by the map $w_n : S^1 \to S^1$ given by wrapping the circle around itself n times. $\square$

Problem 7. If $i : A \hookrightarrow X$ is the inclusion of a subspace $A \subset X$ and $r : X \to A$ is a retract, then the induced map $i_* : \pi_1(A) \to \pi_1(X)$ is injective and $r_* : \pi_1(X) \to \pi_1(A)$ is surjective.

Proof. Since r is a retract, we have $r \circ i = Id$ on A and so $r_* \circ i_* = Id$ on $\pi_1(A)$, as a group homomorphism. This implies that i_* is injective and r_* is surjective. $\square$

Exercise 1.1.16

a) The map $r_* : \pi_1(\mathbb{R}^3) \to \pi_1(S^1)$ is a map from 0 to $\mathbb{Z}$, which cannot be surjective.
b) The map $r_* : \pi_1(S^1 \times D^2) \to \pi_1(\partial(S^1 \times D^2)) = \pi_1(S^1 \times S^1)$ is a map from $\mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$, which cannot be surjective.
c) The map $i_* : \pi_1(A) \to \pi_1(S^1 \times D^2)$ is the zero map from $\mathbb{Z}$ to $\mathbb{Z}$ since the loop A is contractible in $S^1 \times D^2$, and so is not injective.
d) The map $i_* : \pi_1(S^1 \vee S^1) \to \pi_1(D^2 \vee D^2)$ is the zero map from $\mathbb{Z} * \mathbb{Z}$ to 0, which is not injective.
e) The map $i_* : \pi_1(S^1 \vee S^1) \to \pi_1(D^2 \vee S^1)$ is a map $\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$, which cannot be injective.
f) The map $i_* : \pi_1(\partial X) \to \pi_1(X)$ is a map $\mathbb{Z} \to \mathbb{Z}$ which sends 1 to 2 since the boundary of the Mobius band wraps twice around the core of the Mobius band. There is no map $r_* : \mathbb{Z} \to \mathbb{Z}$ so that $r_* \circ i_* = Id$; otherwise, $r_*(2) = 1$, which is not a group homomorphism.