MATH GU4053: INTRODUCTION TO ALGEBRAIC TOPOLOGY

Homework 2
Due: 02/06/20 beginning of class

(1) Hatcher 1.2.3 (p. 53)

(2) Hatcher 1.2.8 (p. 53)

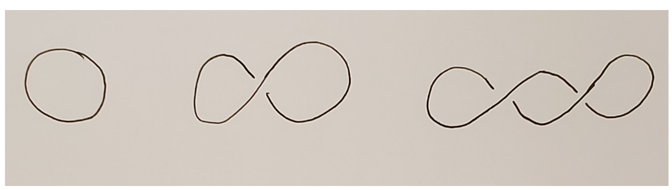
(3) Hatcher 1.2.9 (p.53)

(4) Hatcher 1.2.11 (p.53)

(5) Hatcher 1.2.16 (p.54)

(6) Hatcher 1.2.22 (p.55) Parts a) and b)

c) Use part a) to compute presentations for $\pi_1(\mathbb{R}\backslash K)$, where K is one of the following three knots:

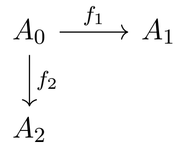


and use these presentations to show that $\pi_1(\mathbb{R}\backslash K)$ are all isomorphic

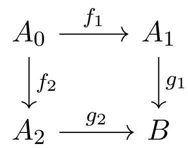
Remark: if K_1, K_2 are isotopic knots (i.e. there is a family of embeddings $f_t : S^1 \hookrightarrow \mathbb{R}^3$ interpolating between K_1, K_2), then $\pi_1(\mathbb{R}^3\backslash K_1), \pi_1(\mathbb{R}^3\backslash K_2)$ are isomorphic groups.

Optional:

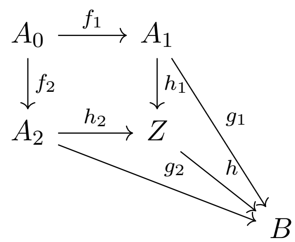
Let A_0, A_1, A_2 be topological spaces and $f_1 : A_0 \to A_1, f_2 : A_0 \to A_2$ be continuous maps. Then the *pushout* of the diagram



is a space Z (and maps $h_1 : A_1 \to Z, h_2 : A_2 \to Z$) if it satisfies the following "universal" property: for any other space B and commutative diagram



there exists a map $h : Z \to B$ so that the following diagram commutes:



7)

a) Show that Z, if it exists, is unique up to homeomorphism.

b) Show that $Z = A_1 \coprod A_2/(f_1(x) \sim f_2(x)$ for all x $\in A_0)$ satisfies this universal property. In particular, if a space X is covered by two subspaces A_1, A_2 and $A_0 = A_1 \cap A_2$ and $f_1 : A_0 \to A_1, f_2 : A_0 \to A_2$ are the inclusions, then X is the pushout of f_1, f_2. If A_0 is the empty set, then $Z = A_1 \coprod A_2$.

c) Now replace all the words 'space' with 'group' and 'continuous map' with 'group homomorphism.' Namely, suppose that A_0, A_1, A_2, B, Z are groups and the maps $f_1, f_2, h_1, h_2, g_1, g_2, h$ are group homomorphisms. Then show that if the pushout Z satisfying the universal property exists, it is unique up to group homomorphism.

d) Show that the group $Z = A_1 * A_2/N$, where N is the normal subgoup generated by words of the form $f_1(x) * f_2(x^{-1})$ for all $x \in A_0$, satisfies this universal property. If A_0 is the trivial group, then $Z = A_1 * A_2$.

Hence van Kampen's theorem can be reformulated as follows: the fundamental group of the topological pushout of $f_1 : A_0 \hookrightarrow A_1, f_2 : A_0 \hookrightarrow A_2$ (namely the space X covered by A_1, A_2) is the group-theoretic pushout of induced maps on fundamental groups $(f_1)_* : \pi_1(A_0) \to \pi_1(A_1), (f_2)_* : \pi_1(A_0) \to \pi_1(A_2)$.