Homework 2 Solution

Exercise 1.2.8. Compute the fundamental group of the space X obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Proof. Let T_1, T_2 be the two tori, and let U_1, U_2 be two open neighborhoods of $S^1 \times \{x_0\}$ in T_1, T_2 respectively that deformation retract onto $S^1 \times \{x_0\}$. This implies that $T_1 \cup U_1$ deformation retracts onto T_1, $T_2 \cup U_2$ deformation retracts onto T_2, and $U_1 \cup U_2$ deformation retracts onto $S^1 \times \{x_0\}$. Applying van Kampen's Theorem to $T_1 \cup U_1$ and $T_2 \cup U_2$, we get:

$$\pi_1(X) \simeq \pi_1(T_1) * \pi_1(T_2)/\pi_1(S^1)$$

where the identifications are $\pi_1(T_1) \simeq \langle a, b \mid [a, b] \rangle$, $\pi_1(T_2) \simeq \langle c, d \mid [c, d] \rangle$, and $\pi_1(S^1) = \langle a, c \mid ac^{-1} \rangle$. Thus: $\pi_1(X) \simeq \langle a, b, c, d \mid [a, b] = [c, d] = ac^{-1} = 1 \} \simeq (\mathbb{Z} * \mathbb{Z}) \times \mathbb{Z}$. □

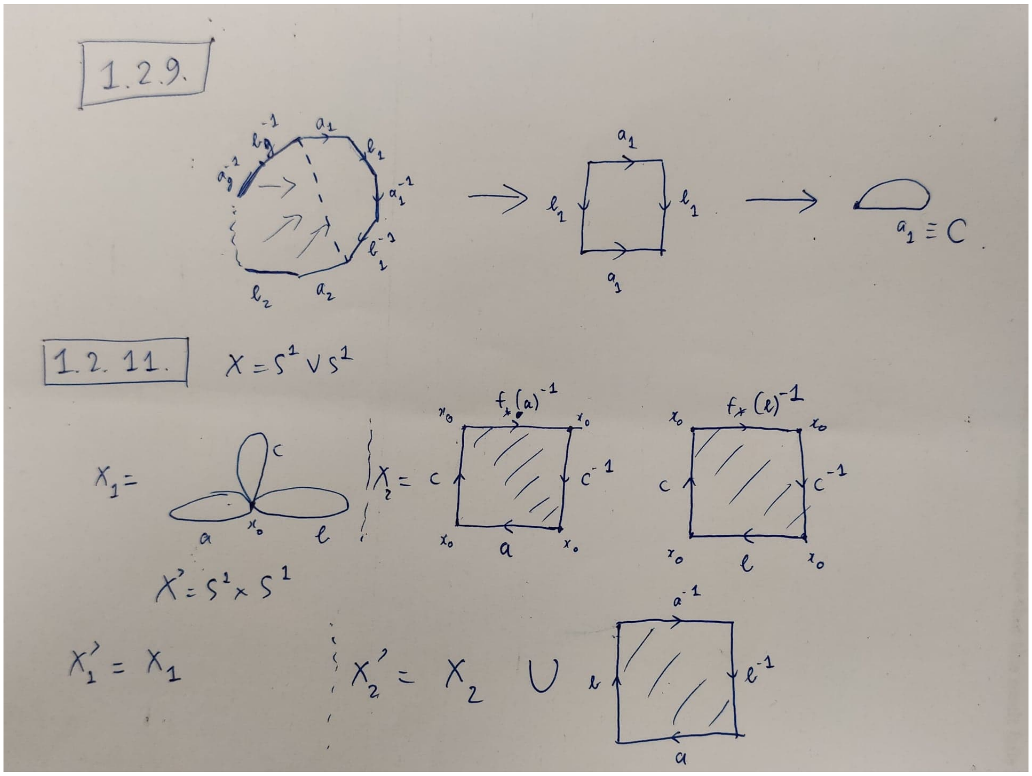
Exercise 1.2.9. In the surface M_g of genus g, let C be a circle that separates M_g into two compact sub-surfaces M'_h and M'_k obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M'_h does not retract onto its boundary circle C, and hence M_g does not retract onto C. But show that M_g does retract onto the non-separating circle C' in the figure.

Proof. Assume there is a retract $r : M'_h \to C$. We then have the induced homomorphisms on fundamental groups $r_* : \pi_1(M'_h) \to \pi_1(C)$ and $i_* : \pi_1(C) \to \pi_1(M'_h)$ with $r_* \circ i_* = 1$. Taking the abelianization gives us $r_*^{\text{ab}} \circ i_*^{\text{ab}} = 1$, hence $i_*^{\text{ab}} : \pi_1(C)^{\text{ab}} \to \pi_1(M'_h)^{\text{ab}}$ is injective. Since $\pi_1(C) = \mathbb{Z}$, it is the same as its abelianization, and $\pi_1(M'_h)$ is abelianized by modding the commutator $[a_1, b_1][a_2, b_2] \dots [a_h, b_h]$ with $a_1, b_1, \dots, a_h, b_h$ being the sides of M_h in its CW complex structure as a $2h$-gon. Note that the generator of $\pi_1(C)$ is sent to the commutator above, hence i_*^{ab} is actually trivial, a contradiction.

On the other hand, M_g retracts onto the torus M_1, which in turn retracts onto C' which is one of its 1-cells (see picture). □

Exercise 1.2.11. The **mapping torus** T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(X) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof. First, consider $X = S^1 \vee S^1$. Denote the two circles by a and b. Note that T_f has the CW structure of one 0-cell x_0, three 1-cells a, b, c attached to x_0, and two 2-cells attached



according to f. The first attachment goes in the order of a, c, $f_*(a)^{-1}$ and c^{-1}. Similarly, the second attachment goes in the order of b, c, $f_*(b)^{-1}$ and c^{-1}. Thus, we have the following presentation for $\pi_1(T_f)$:

$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1} \rangle.$$

For $X = S^1 \times S^1$, again let a and b denote the circles in the 1-skeleton of X. T_f now has the CW structure with the 2-skeleton as above, plus one more 2-cell corresponding to the torus (thus is attached along $aba^{-1}b^{-1}$), and one more 3-cell. Since attaching a 3-cell doesn't change π_1 by Proposition 1.26, we only care about the 2-skeleton of T_f in computing π_1. Similar to above, we obtain the following presentation of $\pi_1(T_f)$:

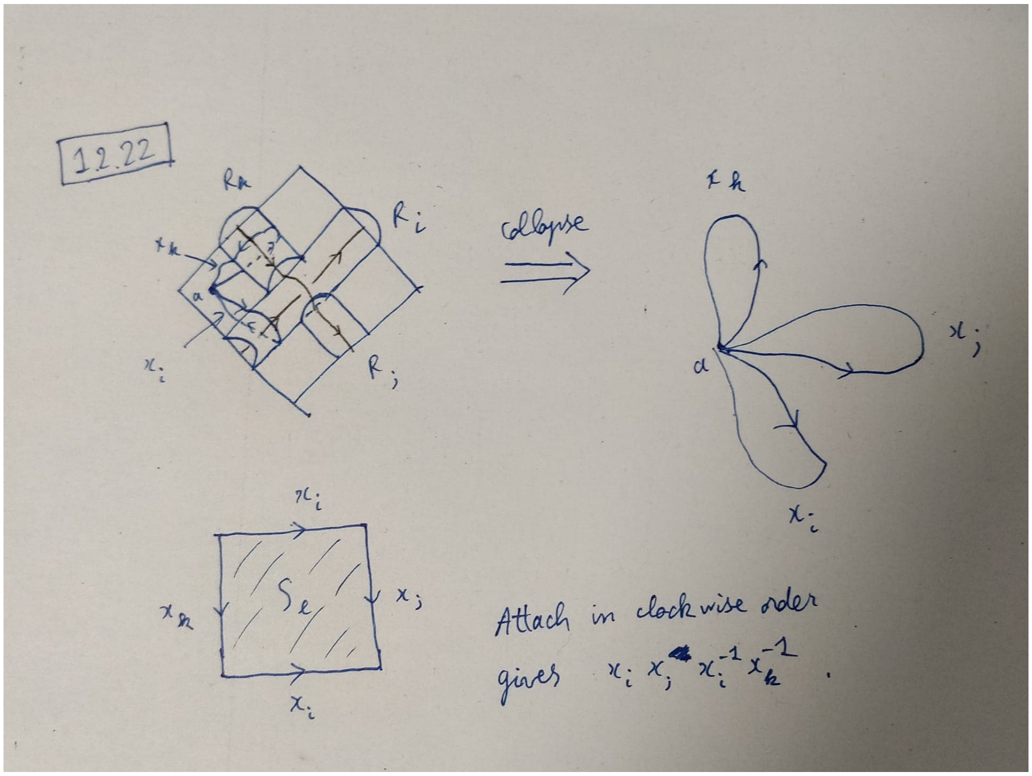
$$\pi_1(T_f) = \langle a, b, c \mid acf_*(a)^{-1}c^{-1}, bcf_*(b)^{-1}c^{-1}, [a, b] \rangle.$$

□

Exercise 1.2.22. Compute the *Wirtinger presentation* of a piecewise linear knot K in $\mathbb{R}^3$ according to Hatcher's instruction on p.55.

(a) Show that $\pi_1(\mathbb{R}^3 - K)$ has a presentation with one generator x_i for each strip R_i and one relation of the form $x_i x_j x_i^{-1} = x_k$ for each square S_l.

(b) Show that the abelianization of $\pi_1(\mathbb{R}^3 - K)$ is $\mathbb{Z}$.



(c) Compute $\pi_1(\mathbb{R}^3 - K)$ for the knots in the homework.

Proof. (a) We need to compute $\pi_1(X)$, where X is the space constructed in Hatcher. Pick an orientation of K and a point a in the plane. We construct a loop x_i based at a around each strip, and make it so that the direction of the loop is consistent with the orientation of K (say, by the right hand rule). Now, since the plane is contractible, we can contract the plane to the point a, and also contract each strip R_i longitudinally to a circle based at a (see picture).

The new space Y is homotopy equivalent to X, and has a CW complex structure of one 0-cell a, a 1-cell x_i for each strip R_i, and 2-cells corresponding to the squares S_l at each crossing. We also observe that the attaching map of each square S_l is of the form $x_i x_j x_i^{-1} x_k^{-1}$, so $\pi_1(Y) \simeq \pi_1(X)$ is indeed the desired group.

(b) From part (a), we have $\pi_1(\mathbb{R}^3 - K) = \langle x_1, x_2, \cdots \mid x_i x_j x_i^{-1} x_k^{-1} \text{ for each square } S_l \rangle$. Abelianizing $\pi_1(\mathbb{R}^3 - K)$ would imply that the relations become

$$x_i x_j x_i^{-1} x_k^{-1} = x_i x_i^{-1} x_j x_k^{-1} = x_j x_k^{-1},$$

which is the same as identifying generator x_j with x_k. Since K is homeomorphic to S^1, when we go around K once we would encounter all the strips, hence all generators are identified with that of a single strip x_1. Thus, $\pi_1(\mathbb{R}^3 - K) \simeq \langle x_1 \rangle \simeq \mathbb{Z}$.

(c) The first knot has one strand a and no crossing, hence $\pi_1(\mathbb{R}^3 - K) = \langle a \rangle \simeq \mathbb{Z}$.

The second knot has one strand a and one crossing, and the relation at the crossing is $aaa^{-1} = a$ which is trivial, hence $\pi_1(\mathbb{R}^3 - K) = \langle a \rangle \simeq \mathbb{Z}$.

The third knot has two strands a, b (with a the left strand and b the right strand), and two crossings. The relation at the left crossing is $aaa^{-1} = b$ and at the right crossing is $bbb^{-1} = a$. In either case, we get $a = b$, hence $\pi_1(\mathbb{R}^3 - K) \simeq \langle a \rangle \simeq \mathbb{Z}$.

$\square$