Homework 4 Solution

Exercise 1.3.24. Given a covering space action of a group G on a path-connected, locally path-connected space X, each subgroup $H \subset G$ determines a composition of covering spaces $X \to X/H \to X/G$. Show that:

(a) Every path-connected covering space between X and X/G is isomorphic to X/H for some subgroup $H \subset G$.

(b) Two such covering spaces X/H_1 and X/H_2 of X/G are isomorphic iff H_1 and H_2 are conjugate subgroups of G.

(c) The covering space $X/H \to X/G$ is normal iff H is a normal subgroup of G, in which case the group of deck transformations of this cover is G/H.

Proof. (a) Given an intermediate covering space $X \xrightarrow{p} Y \to X/G$, let $H = G(X \to Y)$, the group of deck transformations of $X \to Y$. We then have a covering space $X \xrightarrow{p'} X/H$. Note that any such deck transformation also fixes X/G, hence $H \subset G(X \to X/G) = G$. We define a map $f : Y \to X/H$ sending $y \mapsto Hx$, where $x \in p^{-1}(y)$, and another map $g : X/H \to Y$ sending $Hx \mapsto p(x)$. We can check that f and g are well-defined, and are inverses of each other.

$$\begin{tikzcd}
X \arrow[d, "p"] \arrow[dr, "p'"] & \\
Y \arrow[r, "f"] \arrow[d] & X/H \arrow[dl] \\
X/G &
\end{tikzcd}$$

To show f is continuous, pick any open subset $U \subset X/H$, and consider $y \in f^{-1}(U)$. Then there is a neighborhood V around y that lifts to homeomorphic copies $\sqcup_\alpha V_\alpha$ of itself in X. Pick another neighborhood $U' \subset X/H$ around $f(y)$ that lifts to homeomorphic copies $\sqcup_\alpha U'_\alpha$ of itself in X. We can see that $p(U'_\alpha \cap V_\alpha)$ is an open neighborhood around y that is mapped by f to $p'(U'_\alpha \cap V_\alpha)$ that is an open neighborhood around $f(y)$ contained in U. Thus, f is continuous. A similar argument shows that g is also continuous.

(b) Assume $X/H_1 \xrightarrow{f} X/H_2$ is an isomorphism. Let $X/H_1 \xrightarrow{q_1} X/G$ and $X/H_2 \xrightarrow{q_2} X/G$ be the covering maps. Pick any $x_1 \in X$; we then have $f(H_1 x_1) = H_2 x_2$ for some $x_2 \in X$.

Projecting down to X/G gives us $q_1(H_1x_1) = q_2(H_2x_2) \iff Gx_1 = Gx_2$, hence there exists $g \in G$ such that $x_2 = gx_1$. We now show that $f(H_1x) = H_2gx$ for all $x \in X$, i.e: that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, dashed, "g"] \arrow[d, "p_1"] & X \arrow[d, "p_2"] \\
X/H_1 \arrow[r, "f"] \arrow[d, "q_1"] & X/H_2 \arrow[dl, "q_2"] \\
X/G &
\end{tikzcd}$$

For any $x \in X$, since X is path-connected, we can pick a path γ from x_1 to x, which gives a path $f \circ p_1 \circ \gamma$ from $f(H_1x_1) = H_2x_2$ to $f(H_1x)$ in X/H_2. We use the lifting property to find a path β from x_2 to some $x' \in p^{-1}(f(H_1x))$. Projecting down to X/G by the covering map $X \xrightarrow{p_G} X/G$ gives us:

$$p_G \circ \beta = q_2 \circ p_2 \circ \beta = q_2 \circ f \circ p_1 \circ \gamma = q_1 \circ p_1 \circ \gamma = p_G \circ \gamma$$

This means that both γ and β are lifts of $p_G \circ \gamma$, and hence by uniqueness of lifting, $\gamma = \beta$ if they start at the same point. We can conjugate γ to start at $gx_1 = x_2$, which means $g\gamma(1) = gx = \beta(1) = x'$, hence proving what we want.

From here, we can see from the diagram that given a deck transformation τ of $X \xrightarrow{p_1} X/H_1$, $g\tau g^{-1}$ is a deck transformation of $X \xrightarrow{p_2} X/H_2$, since:

$$p_2 \circ g \circ \tau \circ g^{-1} = f \circ p_1 \circ \tau \circ g^{-1} = f \circ p_1 \circ g^{-1} = p_2$$

with the last equality equivalent to $f(H_1g^{-1}x) = H_2x$. Thus, we get $gH_1g^{-1} \subset H_2$. Using f^{-1} would give us $gH_2g^{-1} \subset H_1$, hence H_1, H_2 are conjugates.

Conversely, given $H_1 = gH_2g^{-1}$ for some $g \in G$, we can define a map $f : X/H_1 \to X/H_2$ sending $H_1x \mapsto H_2gx$. We can prove that f is well-defined, and has inverse $g : X/H_2 \to X/H_1$ sending $H_2x \mapsto H_1g^{-1}x$ that is also well-defined. The two maps are continuous (proof similar to part a), hence X/H_1 is isomorphic as covering spaces to X/H_2.

(c) Assume $H \lhd G$. Then any deck transformation $x \mapsto gx$ of $X \to X/G$ descends to a deck transformation $Hx \mapsto gHx = Hgx$ of $X/H \to X/G$, as in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
X/H \arrow[r, "gH"] \arrow[d] & X/H \arrow[dl] \\
X/G &
\end{tikzcd}$$

Thus, given Hx, Hx' that maps to the same element $Gx \in X/G$, we must have $Gx' = Gx$, hence $x' = gx$ for some $g \in G$. The deck transformation $Hx \mapsto Hgx$ now maps $Hx \mapsto Hx'$, hence $X/H \to X/G$ is normal.

Conversely, assume $X/H \to X/G$ is normal. We want to show that $gHg^{-1} = H$ for any $g \in G$. Since $X/H \to X/G$ is normal, for a given $x_0 \in X$ and any $g \in G$, there is a deck transformation $\tau : X \to X$ of $X \to X/H$ that sends $Hx_0 \mapsto Hgx_0$. Similar to part (b), we can show that $\tau(Hx) = Hgx$ for all $x \in X$, hence τ fits in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d] & X \arrow[d] \\
X/H \arrow[r, "\tau"] \arrow[d] & X/H \arrow[dl] \\
X/G &
\end{tikzcd}$$

Therefore, we use the same argument in part (b) to show that if η is a deck transformation of $X \to X/H$, then $g\eta g^{-1}$ is also a deck transformation of $X \to X/H$. This implies $gHg^{-1} \subset H$, hence $gHg^{-1} = H$.

□

Exercise 1.3.25. Consider the action on $X = \mathbb{R}^2 \setminus \{0\}$ sending $(x, y) \mapsto (2x, y/2)$. Show that this generates a covering space action of $\mathbb{Z}$ on X, and that $X/\mathbb{Z}$ is not Hausdorff. Show that $X/\mathbb{Z}$ contains four subspaces homeomorphic to $S^1 \times \mathbb{R}$ coming from the complement of the x-axis and y-axis. Compute $\pi_1(X/\mathbb{Z})$.

Proof. Consider $(x, y) \in X$, and assume that $x \neq 0$. Then the neighborhood $U = (x - |x|/3, x + |x|/3) \times \mathbb{R}$ around (x, y) is sent to disjoint open neighborhoods under the action $(x, y) \mapsto (2x, y/2)$, hence this is a covering space action. If $y \neq 0$, then consider the neighborhood $\mathbb{R} \times (y - |y|/3, y + |y|/3)$ instead.

To show $X/\mathbb{Z}$ is not Hausdorff, consider the points $(1, 0)$ and $(0, 1)$ in $X/\mathbb{Z}$, and any open neighborhoods U_1, U_2 around $(1, 0)$, $(0, 1)$ respectively. Then we can find $k \in \mathbb{Z}^+$ such that $(1, 1/2^k) \in U_1$ and $(1/2^k, 1) \in U_2$. These points are the same in $X/\mathbb{Z}$, hence $U_1 \cap U_2 \neq \emptyset$, implying $X/\mathbb{Z}$ is not Hausdorff.

Consider the first quadrant $Q_1 = \{(x, y) : x, y > 0\} \subset X$; it consists of parabolas $xy = c$ for all $c > 0$. When taking quotient by covering space action $\mathbb{Z}$, each parabola is sent to S^1 as its universal cover; hence, $Q_1/\mathbb{Z} \simeq S^1 \times \mathbb{R}_{>0} \simeq S^1 \times \mathbb{R}$. The same argument applies to the other three quadrants.

Since X is path-connected and locally path-connected, by Proposition 1.40, we have $\mathbb{Z} \simeq \pi_1(X/\mathbb{Z})/p_*(\pi_1(X))$. Since p is a covering map, $p_*(\pi_1(X)) \simeq \pi_1(X) \simeq \mathbb{Z}$, hence $\pi_1(X/\mathbb{Z}) \simeq \mathbb{Z} \rtimes \mathbb{Z}$. We will show that this semidirect product is trivial by showing that the two generators commute. Note that the generators are the projection of the loop going once around the origin in X, and the projection of the path $(1,1) \to (2,1/2)$ going in the hyperbola $xy = 1$.

Consider the universal cover $\widetilde{X} = \mathbb{R} \times \mathbb{R}$ of $X \simeq S^1 \times \mathbb{R}$, with the covering map $q : \widetilde{X} \to X$ sending $(u,v) \mapsto (e^u, e^{iv})$. We can identify $\pi_1(X/\mathbb{Z})$ with the group of deck transformations of $\widetilde{X} \to X/\mathbb{Z}$, and show that the lift of the generators of $\pi_1(X/\mathbb{Z})$ commute in $\widetilde{X}$. The first generator lifts to a deck transformation α sending $(u,v) \mapsto (u, v+2\pi)$. The second generator lifts to a path $\gamma : I \to X$ with

$$\gamma(t) = \left((1+t)x, \frac{y}{1+t}\right).$$

Note that for any $(x,y) \neq (0,0)$, this path is in a neighborhood of X that is evenly covered by $q : \widetilde{X} \to X$ (for example, if (x,y) is in the first quadrant, then it is covered by $(0,R) \times (0,\pi/2)$ in polar coordinates for R large enough). Hence, γ is lifted identically to $\widetilde{\gamma} : I \to X$. We can write $\gamma : I \to X$ in polar coordinates as $\gamma(0) = (r,\varphi)$ and

$$\gamma(t) = \left(r\sqrt{(1+t)^2\cos^2(\varphi) + \frac{\sin^2(\varphi)}{(1+t)^2}}, \arctan\big((1+t)^2\tan(\varphi)\big)\right) = (rf_t(\varphi), g_t(\varphi))$$

where $f_t : [0,2\pi] \to \mathbb{R}$, $g_t : [0,2\pi] \to [0,2\pi]$ for all t.

For a given $(u,v) \in \widetilde{X}$ that projects to (r,φ), with $2k\pi \leq v \leq 2(k+1)\pi$ for some $k \in \mathbb{Z}$, we have $u = \log r$ and $v = \varphi + 2k\pi$. Therefore, if we consider the lift $\widetilde{\gamma} : I \to X$ starting at $(u,v) = (\log r, \varphi + 2k\pi)$, it will end at

$$\widetilde{\gamma}(1) = (u + \log f_1(v), g_1(v) + 2k\pi).$$

Similarly, if we consider the lift $\widetilde{\gamma}$ starting at $(u, v+2\pi)$, then it will end at

$$\widetilde{\gamma}(1) = (u + \log f_1(v), g_1(v) + 2(k+1)\pi).$$

Denote β to be the deck transformation corresponding to the second generator of $\pi_1(X/\mathbb{Z})$. We can reinterpret the above as saying that α commutes with β:

$$\begin{cases} \alpha(\beta(u,v)) = \alpha(u + \log f_1(v), g_1(v) + 2k\pi) = (u + \log f_1(v), g_1(v) + 2(k+1)\pi) \\ \beta(\alpha(u,v)) = \beta(u, v+2\pi) = (u + \log f_1(v), g_1(v) + 2(k+1)\pi) \end{cases}$$

Hence, $\pi_1(X/\mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$. □

Exercise 1.3.26. For a covering space $p : \widetilde{X} \to X$ with X connected, locally path-connected, and semilocally simply-connected, show that:

(a) The components of $\widetilde{X}$ are in one-to-one correspondence with the orbits of the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

(b) Under the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of $\widetilde{X}$ containing a given lift $\widetilde{x}_0$ of x_0 is the stabilizer of $\widetilde{x}_0$.

Proof. (a) Consider $x, x' \in p^{-1}(x_0)$ that lie in the same orbit of the action of $\pi_1(X, x_0)$. This implies that there is a loop $\gamma : S^1 \to X$ whose lift in $\widetilde{X}$ is from x to x'. Thus, x and x' are in the same connected component of $\widetilde{X}$.

Conversely, consider a connected component C of $\widetilde{X}$, and consider a point $x \in C \cap p^{-1}(x_0)$. If x' is another point in $C \cap p^{-1}(x_0)$, then since X is path-connected, $\widetilde{X}$ is also path-connected and we can find a path $\gamma : I \to \widetilde{X}$ going from x to x'. Its projection in X is a loop at x_0, hence x' is in the $\pi_1(X, x_0)$-orbit of x.

(b) We can restrict $\widetilde{X}$ to such a connected component, which is the same as assuming $\widetilde{X}$ is connected. If $\gamma \in \pi_1(X, x_0)$ stabilizes $\widetilde{x_0}$, then its lift $\widetilde{\gamma}$ is a loop at $\widetilde{x_0}$, hence $\widetilde{\gamma} \in \pi_1(\widetilde{X}, \widetilde{x}_0)$. Thus, $\gamma \in_* (\pi_1(\widetilde{X}, \widetilde{x}_0)$. Conversely, given $\gamma \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$, then its lift $\widetilde{\gamma}$ is a loop at $\widetilde{x}_0$, hence it acts on $\widetilde{x}_0$ by sending it to itself. This is the same as saying γ is in the stabilizer of $\widetilde{x}_0$).

□

Exercise 1.3.27. For a universal cover $p : \widetilde{X} \to X$, we have two actions of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$, namely the action given by lifting loops at x_0 and the action given by restricting deck transformations to the fiber. Are these two actions the same when $X = S^1 \vee S^1$ or $X = S^1 \times S^1$? Do the actions always agree when $\pi_1(X, x_0)$ is abelian?

Proof. Given $\widetilde{x} \in p^{-1}(x_0)$ and $\gamma \in \pi_1(X, x_0)$, the first action is defined (in p.69) by lifting γ to a path $\widetilde{\gamma}$ in $\widetilde{X}$ whose endpoint $\widetilde{\gamma}(1) = \widetilde{x}$, and then letting $\gamma \cdot \widetilde{x} = \widetilde{\gamma}(0)$. This is the same as lifting the inverse $\overline{\gamma}$ to a path $\widetilde{\overline{\gamma}}$ starting at $\widetilde{x}$, and mapping $\widetilde{x}$ to its endpoint $\widetilde{\overline{\gamma}}(1)$. On the other hand, Proposition 1.39 identifies $\pi(X, x_0)$ with the group of deck transformation $G(\widetilde{X})$ by sending γ to the deck transformation taking $\widetilde{\gamma}(0) = \widetilde{x}$ to $\widetilde{\gamma}(1)$. Thus, we can see that the two actions are inverses of each other, i.e: $\gamma \cdot_2 \widetilde{x} = \overline{\gamma} \cdot_1 \widetilde{x}$, where $\cdot_2$ is the second action, and $\cdot_1$ is the first action.

Thus, these actions are the same only when $\pi_1(X, x_0)$ is a group of exponent 2, i.e: $g^2 = 1$ for all $g \in \pi_1(X, x_0)$. In particular, these actions are not the same when $X = S^1 \vee S^1$ or $X = S^1 \times S^1$. □