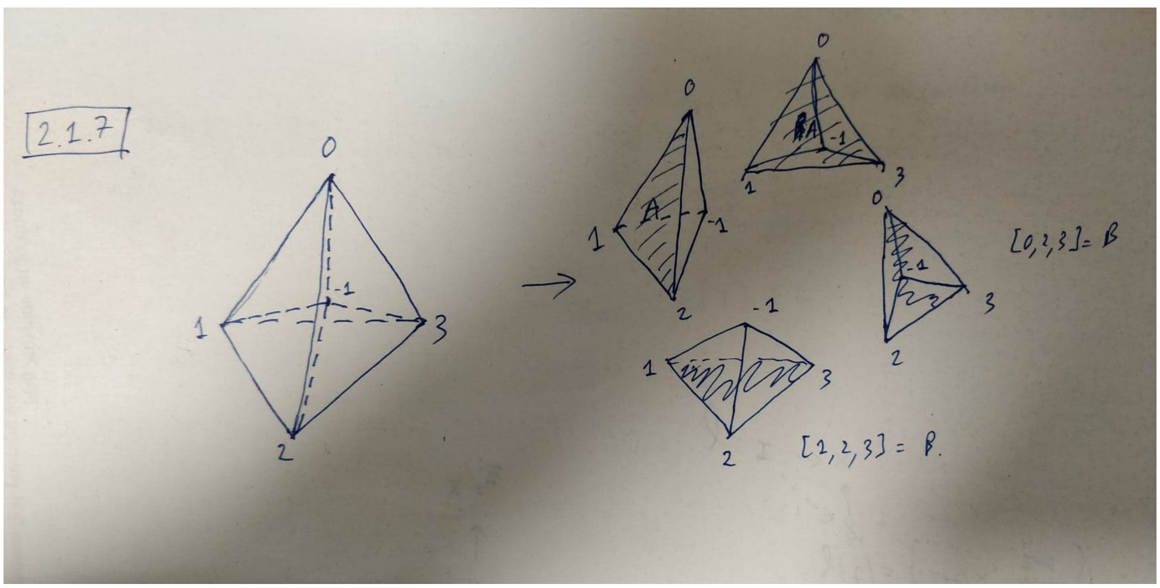
Homework 5 Solution

Exercise 2.1.7. Find a way of identifying pairs of faces of Δ^3 to produce a Δ-complex structure on S^3 having a single 3-simplex, and compute the simplicial homology group of this Δ-complex.

Proof. We identify faces $[0,1,2]$ with $[0,1,3]$ and $[0,2,3]$ with $[1,2,3]$ (see picture). To show that the resulting space is homeomorphic to S^3, it is helpful to consider the center -1 of the 3-simplex. We can see that two tetrahedra $[-1,0,1,2]$ and $[-1,0,1,3]$ are glued together along one pair of faces, hence creating a homeomorphic copy of D^3, and the same goes for gluing $[-1,0,2,3]$ with $[-1,1,2,3]$. Now, the two copies of D^3 are glued together via the faces $[-1,0,2]$, $[-1,0,3]$, $[-1,1,2]$, and $[-1,1,3]$, which is the same as gluing the boundary of one D^3 with the boundary of the other D^3 (in a consistent manner). This will create S^3.



We now compute the simplicial homology of this space. The chain complex is

$$0 \xrightarrow{\partial_4} \mathbb{Z}T \xrightarrow{\partial_3} \mathbb{Z}A \oplus \mathbb{Z}B \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{\partial_1} \mathbb{Z}x \oplus \mathbb{Z}y \xrightarrow{\partial_0} 0$$

where $A = [0,1,2] = [0,1,3]$, $B = [0,2,3] = [1,2,3]$ $a = [0,1]$, $b = [2,3]$, and $c = [1,2] = [1,3] = [0,2] = [0,3]$, and $x = 0 = 1$, $y = 2 = 3$. The boundary maps are

$$\begin{cases} \partial_3 T = A - A + B - B = 0 \\ \partial_2 A = a + c - c = a, \qquad \partial_2 B = b + c - c = b \\ \partial_1 a = \partial_1 b = 0, \qquad \partial_1 c = x - y \end{cases}$$

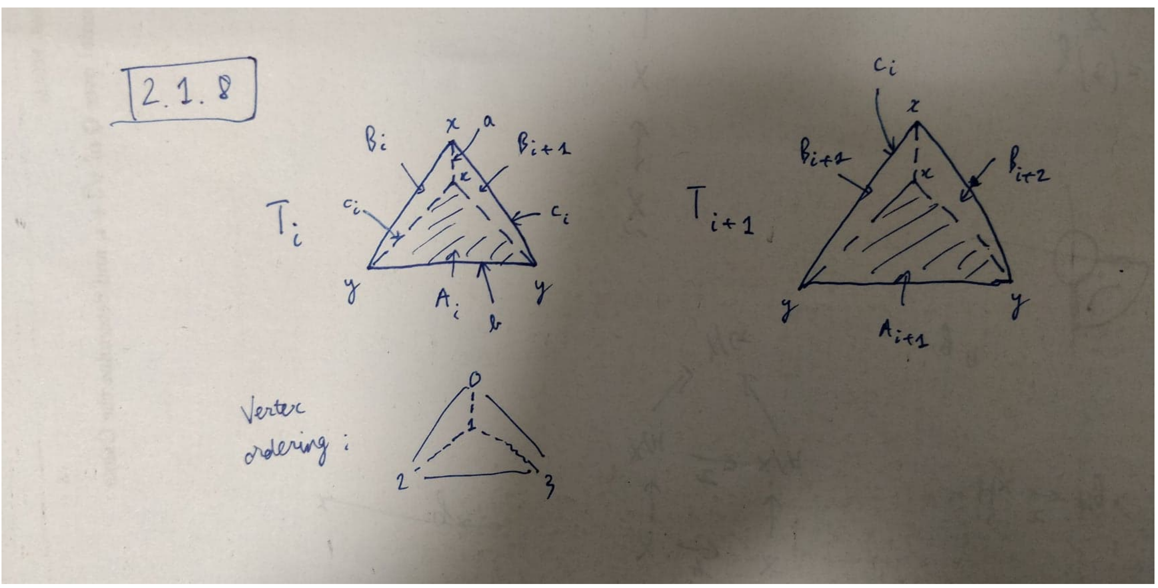
Therefore,

$$\begin{cases} H_0(X) = \operatorname{Ker}(\partial_0)/\operatorname{Im}(\partial_1) = \langle x, y\rangle/\langle x - y\rangle \simeq \mathbb{Z} \\ H_1(X) = \operatorname{Ker}(\partial_1)/\operatorname{Im}(\partial_2) = \langle a, b\rangle/\langle a, b\rangle = 0 \\ H_2(X) = \operatorname{Ker}(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 = 0 \\ H_3(X) = \operatorname{Ker}(\partial_3)/\operatorname{Im}(\partial_4) = \langle T\rangle \simeq \mathbb{Z} \end{cases}$$

and $H_n(X) = 0$ for $n \geq 4$. □

Exercise 2.1.8. Compute the simplicial homology of the space X as constructed in Hatcher p.131.

Proof. Note that the Δ-complex of X has two 0-simplices x (in the center) and y, $n+2$ 1-simplices $a, b, c_1, \ldots, c_n$, $2n$ 2-simplices $A_1, \ldots, A_n, B_1, \ldots, B_n$ and n 3-simplices $T_1, \ldots, T_n$ (see picture).



The chain complex is thus

$$0 \xrightarrow{\partial_4} \mathbb{Z}^n \xrightarrow{\partial_3} \mathbb{Z}^{2n} \xrightarrow{\partial_2} \mathbb{Z}^{n+2} \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0$$

The boundary maps are

$$\begin{cases} \partial_3 T_i = A_i - A_{i-1} + B_{i+1} - B_i \\ \partial_2 A_i = b - c_{i+1} + c_i, \qquad \partial_2 B_{i+1} = c_{i+1} - c_i + a \\ \partial_1 a = \partial_1 b = 0, \qquad \partial_1 c_i = x - y \end{cases}$$

Therefore,

$$\begin{cases} H_0(X) = \operatorname{Ker}(\partial_0)/\operatorname{Im}(\partial_1) = \langle x, y\rangle/\langle x-y\rangle \simeq \mathbb{Z} \\ H_1(X) = \operatorname{Ker}(\partial_1)/\operatorname{Im}(\partial_2) = \langle a, b, c_2-c_1, \dots, c_n - c_{n-1}\rangle/\langle a + c_{i+1} - c_i, b + c_i - c_{i+1}\rangle = \mathbb{Z}/n\mathbb{Z} \\ H_2(X) = \operatorname{Ker}(\partial_2)/\operatorname{Im}(\partial_3) = \langle A_i - A_{i-1} + B_{i+1} - B_i\rangle/\langle A_i - A_{i-1} + B_{i+1} - B_i\rangle = 0 \\ H_3(X) = \operatorname{Ker}(\partial_3)/\operatorname{Im}(\partial_4) = \langle T_1 + T_2 + \cdots + T_n\rangle \simeq \mathbb{Z} \end{cases}$$

and $H_n(X) = 0$ for $n \geq 4$. To explain the computation for $H_1(X)$, note that:

$$\langle a, b, c_2 - c_1, \dots, c_n - c_{n-1}\rangle = \langle a, a+b, a + c_2 - c_1, a + c_3 - c_2, \dots, a + c_n - c_{n-1}\rangle$$

since we can always subtract a from the other generators, and

$$\begin{aligned} \langle a + c_{i+1} - c_i, b + c_i - c_{i+1}\rangle &= \langle a + c_{i+1} - c_i, a + b\rangle \\ &= \langle a + c_2 - c_1, a + c_3 - c_2, \dots, a + c_n - c_{n-1}, na, a+b\rangle \end{aligned}$$

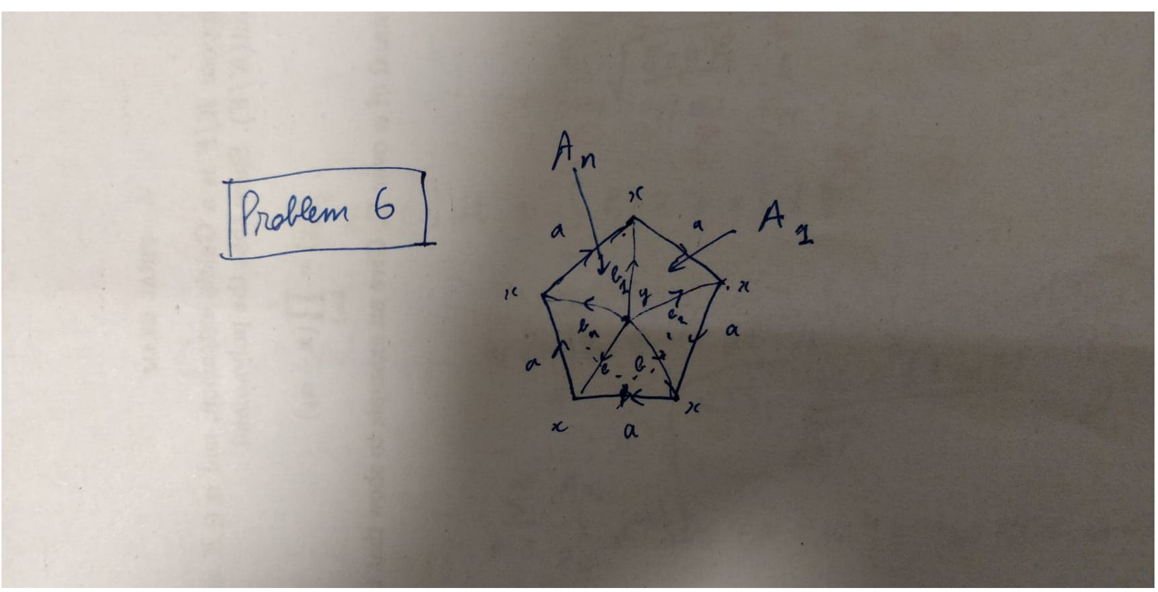
and hence $\operatorname{Ker}(\partial_1)$ quotient by $\operatorname{Im}(\partial_2)$ this group is $\langle a\rangle/\langle na\rangle \simeq \mathbb{Z}/n\mathbb{Z}$. □

Exercise 6. Find a Δ-complex that is homotopy equivalent to the CW complex

$$S^1 \cup D^2/\{x \sim \varphi(x), x \in \partial D^2\}$$

where $\varphi : \partial D^2 = S^1 \to S^1$ is the map that winds n times around S^1. Compute the simplicial homology of this space.

Proof. The CW complex structure is of a n-gon with all vertices identified, and all edge going counterclockwise identified. We give it a Δ-complex structure as follows:



The chain complex is thus:

$$0 \xrightarrow{\partial_3} \mathbb{Z}A_1 \oplus \dots \mathbb{Z}A_n \xrightarrow{\partial_2} \mathbb{Z}a \oplus \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_n \xrightarrow{\partial_1} \mathbb{Z}x \oplus \mathbb{Z}y \xrightarrow{\partial_0} 0$$

The boundary maps are

$$\begin{cases} \partial_2 A_i = a + b_i - b_{i+1} \\ \partial_1 a = 0, \qquad \partial_1 b_i = x - y \end{cases}$$

Therefore,

$$\begin{cases} H_0(X) = \operatorname{Ker}(\partial_0)/\operatorname{Im}(\partial_1) = \langle x, y\rangle / \langle x - y\rangle \simeq \mathbb{Z} \\ H_1(X) = \operatorname{Ker}(\partial_1)/\operatorname{Im}(\partial_2) = \langle a, b_2 - b_1, \dots, b_n - b_{n-1}\rangle / \langle a + b_i - b_{i+1}\rangle = \mathbb{Z}/n\mathbb{Z} \\ H_2(X) = \operatorname{Ker}(\partial_2)/\operatorname{Im}(\partial_3) = 0/0 = 0 \end{cases}$$

and $H_n(X) = 0$ for $n \geq 3$. Here, the computation for $H_1(X)$ is similar to the last problem; in particular, we have:

$$\langle a, b_2 - b_1, \dots, b_n - b_{n-1}\rangle = \langle a, a + b_2 - b_1, \dots, a + b_n - b_{n-1}\rangle$$

and

$$\langle a + b_i - b_{i+1}\rangle = \langle na, a + b_2 - b_1, \dots, a + b_n - b_{n-1}\rangle.$$

□