Homework 6 Solution

Exercise 2.1.14. Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

Proof. We choose to embed $\mathbb{Z}_4$ into $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ by sending the generator of $\mathbb{Z}_4$ to $(2, 1)$. The quotient $(\mathbb{Z}_8 \oplus \mathbb{Z}_2)/\mathbb{Z}_4$ is a group of order 4, and we can check that $(1, 0)$ has order 4 in the quotient group, hence it is isomorphic to $\mathbb{Z}_4$. This gives our short exact sequence.

In the more general case:

$$0 \to \mathbb{Z}_{p^m} \xrightarrow{f} A \xrightarrow{g} \mathbb{Z}_{p^n} \to 0$$

we must have $|A| = p^{m+n}$, hence $A \simeq \mathbb{Z}_{p^{m_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{m_k}}$ for some $m_1, \ldots, m_k \geq 0$ such that $m_1 + \cdots + m_k = m + n$. Since $\mathbb{Z}_{p^m}$ embeds into A, A will have an element of order p^m, hence $\min\{m_1, \ldots, m_k\} \geq m$. Also, note that A is generated by two elements, $f(1)$ and any x such that $g(x) = 1$. Projecting A onto $\mathbb{Z}_p^k$ via $\mathbb{Z}_{p^{m_i}} \to \mathbb{Z}_p$ for all i will give us 2 generators for the vector space $\mathbb{Z}_p^k$, which means $k = 2$.

From these structural results, we see that A has the form $\mathbb{Z}_{p^{m+t}} \oplus \mathbb{Z}_{p^{n-t}}$ for $0 \leq t \leq n$. We will show that any of these groups fit into the long exact sequence. Indeed, for a given t, we can send the generator of $\mathbb{Z}_{p^m}$ to $(p^t, 1)$; then the quotient $A/\mathbb{Z}_{p^m}$ will be generated by $(1, 0)$, hence is cyclic and is isomorphic to $\mathbb{Z}_{p^n}$.

Finally, if A fits into $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$, then A has an element of infinite order. By the classification of finitely generated abelian groups, A is isomorphic to $\mathbb{Z}^l \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$ for $l \geq 1$ and n_i divides n_{i+1} for $i = 1, \ldots, k-1$. Pick a prime p of n_1, hence of all n_i's; we can then project $\mathbb{Z}_{n_i} \to \mathbb{Z}_p$ for all i and $\mathbb{Z}^l \to \mathbb{Z}_p^l$, hence $\mathbb{Z}_p^{l+k}$ is generated by two elements. Since this is a vector space, we have $l + k = 2$, and since $l \geq 1$, we have $l = k = 1$. In other words, we obtain $A \simeq \mathbb{Z} \oplus \mathbb{Z}_d$. Assume that $\mathbb{Z} \hookrightarrow A$ sends $1 \mapsto (k, l)$. Since

$$\mathbb{Z}_n \simeq (\mathbb{Z} \oplus \mathbb{Z}_d)/\mathbb{Z} \qquad \text{and} \qquad \mathbb{Z} \oplus \mathbb{Z}_d \simeq \mathbb{Z}a \oplus \mathbb{Z}b/\langle db \rangle,$$

$\mathbb{Z}_n$ can be written as the quotient $\mathbb{Z}a \oplus \mathbb{Z}b/\langle ka + lb, db \rangle$. The presentation matrix for this is $\begin{pmatrix} k & l \\ 0 & d \end{pmatrix}$ which must have determinant n, hence $kd = n$, meaning $d \mid n$.

For $d \mid n$, we will show that $A \simeq \mathbb{Z} \oplus \mathbb{Z}_d$ fits into the short exact sequence. Let $n = qd$. Then we send the generator of $\mathbb{Z}$ to $(q, 1)$, and projecting from A to $\mathbb{Z}_n$ via $(u, v) \mapsto \overline{u - qv} \in \mathbb{Z}_n$. It is then a simple exercise to check that these maps make the sequence exact. $\square$

Exercise 2.1.20. Show that $\widetilde{H}_n(X) \simeq \widetilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of

X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases attached.

Proof. We will prove generally that $\widetilde{H}_{n+1}(S_kX) \simeq (\widetilde{H}_n(X))^{k-1}$ for all $k \geq 2$, where S_kX is formed by attaching k cones $(CX)_1, \dots, (CX)_k$ to the same base $X \times \{0\}$. Let $(CX)_{\varepsilon,i}$ be a small neighborhood $X \times [0, \varepsilon)$ in the i-th cone $(CX)_i$. Consider the pair $A = (CX)_1 \cup \dots \cup (CX)_{k-1} \cup (CX)_{\varepsilon,k}$ and $B = (CX)_{\varepsilon,1} \cup \dots \cup (CX)_{\varepsilon,k-1} \cup (CX)_k$. Then A deformation retracts onto $A' = (CX)_1 \cup \dots \cup (CX)_{k-1}$, B deformation retracts onto $B' = (CX)_k$, and $A \cap B$ deformation retracts onto X. Since the union of the interiors of A, B covers S_kX, we use excision to get:

$$\widetilde{H}_n(S_kX, B') \simeq \widetilde{H}_n(S_kX, B) \simeq \widetilde{H}_n(A, A \cap B) \simeq \widetilde{H}_n(A', X)$$

Since B' is contractible, the long exact sequence of homology gives $\widetilde{H}_n(S_kX) \simeq \widetilde{H}_n(S_kX, B')$. Since (A', X) is a good pair, we have

$$\widetilde{H}_n(A', X) \simeq \widetilde{H}_n(A'/X) \simeq \widetilde{H}_n(SX \vee \dots \vee SX) \simeq \bigoplus_{i=1}^{k-1} \widetilde{H}_n(SX) = (\widetilde{H}_{n-1}(X))^{k-1}.$$

Putting it all together, we get: $\widetilde{H}_n(S_kX) \simeq (\widetilde{H}_{n-1}(X))^{k-1}$, and shifting index $n \mapsto n+1$ gives the desired conclusion.

$\square$

Exercise 7. (a) Is every chain map $B_* \to C_*$ such that $B_k \to C_k$ is surjective for all k induces a surjective map $H_k(B) \to H_k(C)$ on homology? If not, give a counterexample.

(b) Is every chain map $B_* \to C_*$ such that $B_k \to C_k$ is injective for all k induces a injective map $H_k(B) \to H_k(C)$ on homology? If not, give a counterexample.

Proof. (a) Consider the following chain map:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{Z}^2 & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

where i is the inclusion in the first component, and p, q are the projections in the second component. We can check that this is a chain map $B_* \to C_*$ such that $B_k \to C_k$ is surjective for all k, but the induced homology map $q_* : H_2(B) = 0 \to H_2(C) = \mathbb{Z}$ is not surjective.

(b) Similar to the above, we consider the following chain map:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow j & & \downarrow & & \downarrow \\
\dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{Z}^2 & \xrightarrow{p} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

where i is the inclusion in the first component, j is the inclusion in the second component, and p is the projection in the second component. We can check that this is a chain map $B_* \to C_*$ such that $B_k \to C_k$ is injective for all k, but the induced homology map $j_* : H_2(B) = \mathbb{Z} \to H_2(C) = 0$ is not injective.

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