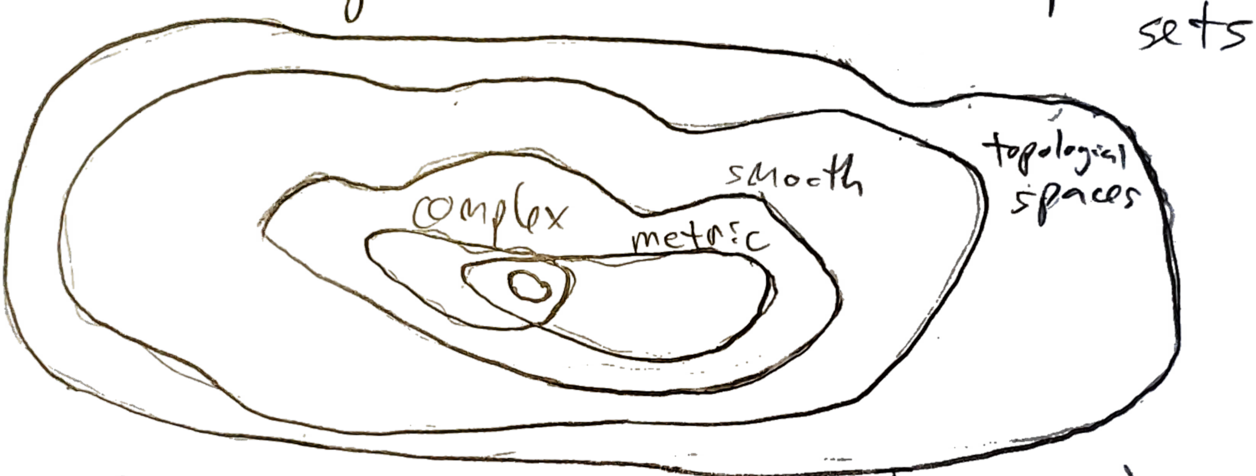
·math.columbia.edu/~olazarev/algebraic_topology

Lecture 1 01/21/20

Goal: understand topological spaces (and continuous maps between them)

via algebra invariants (and maps)



Equivalences: (all maps continuous)

1) X is homeomorphic to Y

if exists $f: X \to Y$ and $g: Y \to X$

so that $f \circ g = Id_Y$, $g \circ f = Id_X$

2) X is homotopy equivalent to Y

if exists $f: X \to Y$, $g: Y \to X$

so that $f \circ g \simeq Id_Y$, $g \circ f \simeq Id_X$

homotopic

Def. $X \underset{f'}{\overset{f}{\rightrightarrows}} Y$ are <u>homotopic</u> if exists

family $f_t : X \to Y$ for $t \in [0,1]$ so that

- $f_0 = f$, $f_1 = f'$
- $F : X \times [0,1] \to Y$ is continuous

  $F(x,t) = f_t(x)$

Lem X homeo to $Y \Rightarrow X$ homotopy equiv to Y

$X \cong Y$ $\qquad\qquad$ $X \simeq Y$

Ex. $\mathbb{R}^n \not\cong \{0\}$ but $\mathbb{R}^n \simeq \{0\}$ all n

$\mathbb{R}^n \underset{i}{\overset{\pi}{\rightleftarrows}} \{0\}$, $\qquad \pi \circ i = id_{\{0\}}$ ✓

$i \circ \pi \neq id_{\mathbb{R}^n}$ but $\simeq id_{\mathbb{R}^n}$

- $f_t(x) = tx$ is homotopy bet $i \circ \pi$ and $id_{\mathbb{R}^n}$

Def. $A \subset X$ is a deformation retract if exists a homotopy $F : X \times [0,1] \to X$ so that

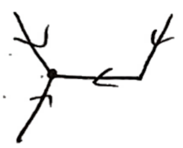
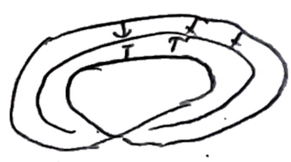
$F|_{A \times [0,1]} = Id$

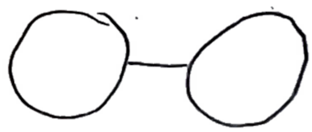
Def. $A \subset X$ is a <u>deformation retract</u> if exists a homotopy $F: X \times [0,1] \to X$ so that

- $F|_{A \times \{t\}} = Id_A$, ie. <u>relative</u> to A
- $F|_{X \times \{0\}} = Id_X$
- $F(X \times \{1\}) \subset A$

Lem A def retract of $X \Rightarrow A \simeq X$

(**Pf.** $r = F|_{X \times \{1\}}$, $X \underset{i}{\overset{r}{\rightleftarrows}} A$, $r \circ i = Id|_A$, $i \circ r \simeq Id|_X$ via F)

Ex. 1)  , 2)  $\simeq$ 

3)  $\simeq$ 

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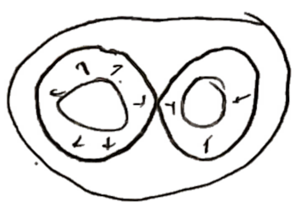
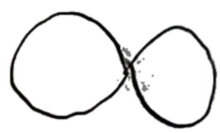
 $\simeq$ 

topological space $\rightsquigarrow$ algebraic object, integer, group, ring

<u>Fundamental group</u>: $(X, x_0) \rightsquigarrow$ group $\pi_1(X, x_0)$

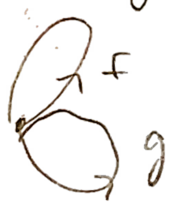
- $\pi_1(X, x_0) =$ homotopy classes of paths based at x_0
  $f: [0,1] \to X$, $f(0) = f(1) = x_0$

- $f \sim g$ in $\pi_1(X, x_0)$ if exists homotopy $f_t : [0,1] \to X$, $f_0 = f$, $f_1 = g$ and $f_t(0) = f_t(1) = x_0$

<u>Product</u> concatenation of paths

$$[f] \cdot [g] = [f * g]$$

$$f * g(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$



<u>Prop.</u> $\pi_1(X, x_0)$ is a group

(need to show product well-defined, associative and has inverses)

<u>Prop.</u>
- if X path-connected, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$
- if $X \simeq Y$ then $\pi_1(X) \cong \pi_1(Y)$

<u>Ex.</u>
- $\pi_1(\mathbb{R}^n) \cong 0$

<u>Prop.</u>
- $\pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(S^n, q) \cong 0$ for $n \neq 2 \Rightarrow S^n \not\simeq S^1$ if $n \neq 1$

<u>Pf sketch:</u> first homotope $f : [0,1] \to S^n$ so that misses some point $p \neq q$

•  → 

• then $f:[0,1] \to S^n \setminus p \cong \mathbb{R}^n$

• since $\pi_1(\mathbb{R}^n, q) \cong 0$, f homotopic to q □

Cell complexes or CW complexes

defined inductively by attaching cells,

1) X^0 = discrete set of points   n-cell = n-disk D^n

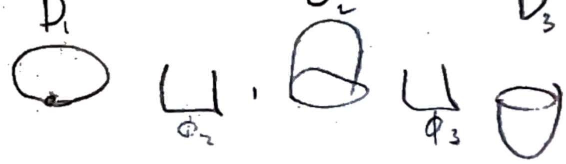
inductively

2) $\phi_\alpha : \partial D^n_\alpha \to X^{n-1}$ attaching maps,

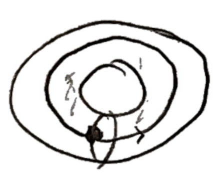
$\rightsquigarrow X^n = X^{n-1} \sqcup_\alpha D^n_\alpha / x \sim \phi_\alpha(x)$ if $x \in \partial D^n_\alpha$ $= X^{n-1} \cup_{\phi_\alpha} D^n$

3) $X = \cup X^n$ with weak topology ($A \subset X$ open if $A \cap X^n$ open for all n)

Ex. ① S^n =  D^n $= \cdot \cup_\phi D^n$, $\phi : \partial D^n \to p$

② S^2 =  $= \cdot \sqcup_{\phi_1} D^1_1 \sqcup_{\phi_2} D^2_2 \sqcup_{\phi_3} D^2_3$ 

$S^1 \vee S^1 = pt \cup D^1 \sqcup D^1$

③ $S^1 \times S^1 = T^2$  =  $\sqcup D^2$

$\phi : \partial D^2 = S^1 \to S^1 \vee S^1$

$a b a^{-1} b^{-1}$

④ real projective space $\mathbb{R}P^n \cong S^n / x \sim -x \cong D^n / x \sim -x$ for $x \in \partial D^n$

$= \mathbb{R}P^{n-1} \cup_\phi D^n$

where $\underset{S^{n-1}}{\partial D^n} \xrightarrow{\ \phi\ } \underset{S^{n-1}/x\sim -x}{\mathbb{R}P^{n-1}}$ is quotient map