Lecture 10 : Homology

- fundamental group not good for detecting high-dimensional space

  $\pi_1(S^n) \simeq 0$ for $n > 1$

  $\pi_1(X) \simeq \pi_1(X^2)$ for any CW complex
  (X^2 ← 2 skeleton)

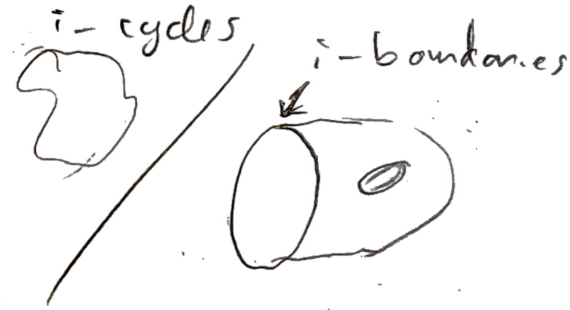
- $\pi_n(X)$ = homotopy classes of maps $(D^n, \partial D^n) \to (X, x_0)$

  very hard to compute to compute $\pi_k(S^n)$, $k > n$

- topological space ⟹ abelian group $H_i(X)$, $i \geq 0$
  maps bet spaces ⟹ group homomorphisms

- very computable and $H_i(X) \cong H_i(X^{i+1})$

- very roughly $H_i(X) \cong$



<u>Def.</u> a chain complex (C_*, ∂_*) is a sequence

$$\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

where
- each C_i is an abelian group (i-cells)
- each ∂_i is group homomorphism (boundary operator)
- $\partial_n \circ \partial_{n+1} = 0$ for all

abelian ← unlike $\pi_1(X)$

quotient group

Def. $H_n(C_*, \partial_*) := \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$ is nth homology group of (C_*, ∂)

= i-cycles / boundaries of i+1-cells

Simplicial homology

Def standard n-simplex is

$\Delta^n = \{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1,\ t_i \geq 0 \text{ all } i \}$

$\Delta^0 = \cdot$

$\Delta^1 =$ 

$\Delta^2 =$ 

- more generally, given points $v_0, \ldots, v_n \subset \mathbb{R}^m$ that are affine linearly independent (ie. $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent)

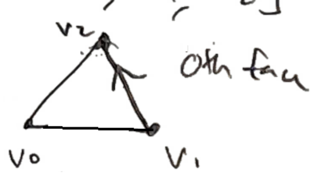
  $[v_0, \ldots, v_n]$ = convex hull of $v_0, \ldots, v_n$

  $\cong \Delta^n$

  ↖ homeomorphic

$[v_0, v_1, v_2]$ $[v_0, v_2, v_1]$

$[\hat{v}_0, v_1, v_2]$



Def. $[v_0, \ldots, v_n] \to$ ith face $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$

Δ^n $d_i \Delta^n$

Def. a Δ-complex str on a topological space X is a collection of maps $\sigma_\alpha^n : \Delta^n \to X$ s.t.

1) $\sigma_\alpha^n |_{\text{Int}\,\Delta^n}$ is injective and each $p \in X$ is in image of exactly one $\sigma_\alpha^n |_{\text{Int}\,\Delta^n}$

2) $[v_0, \ldots, \hat{v}_i, \ldots, v_n] = \Delta^{n-1}$

$$\downarrow \sigma_\alpha^n$$

$$X$$

σ_β^{n-1} ← another one of the maps in Δ-structure

3) $A \subset X$ open $\iff$ $\sigma_\alpha^{-1}(A)$ open in Δ^n for each σ_α

• so this is a CW structure so that

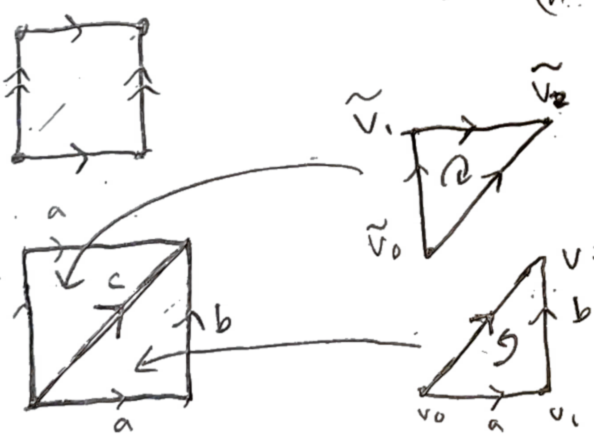
$$X = \coprod_n \coprod_\alpha \Delta_\alpha^n \Big/ d_i \Delta_\alpha^n \sim \Delta_\beta^{n-1}$$

where β s.t. $\sigma_\beta : \Delta_\beta^{n-1} \to X$ is $(n-1)$-simplex $\sigma_\alpha|_{i\text{th-face of } \Delta_\alpha^n}$

Ex.

CW-complex

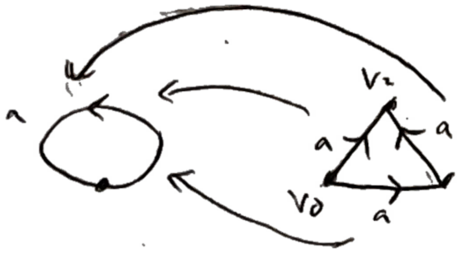
Δ-complex



$[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]$ 2-face
$[\tilde{v}_0, \tilde{v}_1] \to b$ 0-face
$[\tilde{v}_1, \tilde{v}_2] \to a$ 0-face
$[\tilde{v}_0, \tilde{v}_2] \to c$ 1-face

$[v_0, v_1, v_2]$ 2-face
$[v_0, v_1] \to a$ 2-face
$[v_1, v_2] \to b$ 0-face
$[v_0, v_2] \to c$ 1-face

Ex. 1 Δ^0, 1 Δ^1, 1 Δ^2 simplex



$[v_0, v_1, v_2]$

$[v_0, v_1] \to a$ 2 - face

$[v_1, v_2] \to a$ 0 - face

$[v_0, v_2] \to a$ 1 - face

- so attaching map is homotopic to $Id : S^1 \to S^1$
  $\Longrightarrow X$ homotopic to $D^2 \simeq pt$

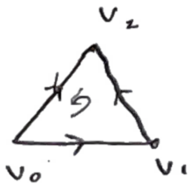
- X is Δ-simplex
- let $S_n(X)$ = set of n-simplices $\{\sigma_\alpha^n : \Delta^n \to X\}$

Def $\Delta_n(X)$ = free abelian group generated by $S_n(X)$

i.e. $\sum_{\sigma_\alpha \in S_n(X)} a_\alpha \sigma_\alpha^n$, $a_\alpha \in \mathbb{Z}$

- $\alpha_\alpha \neq 0$ for only finitely many α

called n-chains on X



$\partial \sigma = [v_0, v_1] + [v_1, v_2] - [v_0, v_2]$

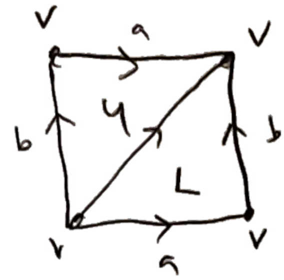
$= \sigma|_{[v_1, v_2]} - \sigma|_{[v_0, v_2]} + \sigma|_{[v_0, v_1]}$

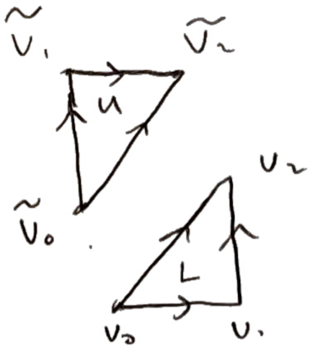
- $\partial(\sigma_\alpha^n) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

Boundary map: $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$

$\partial_n(\sum a_\alpha \sigma_\alpha^n) = \sum a_\alpha \partial_n \sigma_\alpha^n$

Ex,





$[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]$

$[v_0, v_1, v_2]$

$$\begin{array}{ccccccccc} & \Delta_2(X) & & \Delta_1(X) & & \Delta_0(X) & & \\ 0 \to & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\partial_2} & \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow[0]{\partial_1} & \mathbb{Z} & \xrightarrow{\partial_0} 0 \\ & \uparrow \quad \uparrow & & \uparrow \quad \uparrow \quad \uparrow & & \uparrow & \\ & L \quad U & & a \quad b \quad c & & v & \end{array}$$

$\partial a = v - v = 0$

$\partial b = 0 \quad \Rightarrow \quad \partial_1 = 0$

$\partial c = 0$

$\partial U = U|_{[\tilde{v}_0, \tilde{v}_2]} - U|_{[\tilde{v}_0, \tilde{v}_2]} + U|_{[\tilde{v}_0, \tilde{v}_1]} = a - c + b$

$\partial L = L|_{[v_1, v_2]} - L|_{[v_0, v_2]} + L|_{[v_0, v_1]} = b - c + a$

so $\partial_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix} : \mathbb{Z}^2 \to \mathbb{Z}^3$

Lemma $\partial_{n-1} \circ \partial_n = 0$

Pf, $\partial_n \sigma = \sum (-1)^i \sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_n]}$

extra negative sign since $j > i$

$\partial_{n-1} \partial_n \sigma = \sum (-1)^i \left(\sum_{j<i} (-1)^j \sigma|_{[v_0, \ldots, \hat{v}_j, \ldots, \hat{v}_i, \ldots v_n]} + \sum_{j>i} (-1)^{j-1} \right)$

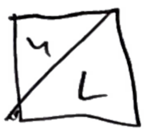
$= \sum_{j<i} (-1)^{i+j} + \sum_{j>i} (-1)^{i+j-1} \quad = 0$

Cor for all n, $\text{Im}\,\partial_n \subset \ker \partial_{n-1}$

Def. $\ker \partial_n$ = n-cycles Z_n^Δ

$\text{Im}\,\partial_{n+1}$ = n-boundaries B_n^Δ

n-simplicial homology group $H_n^\Delta(X) = Z_n^\Delta / B_n^\Delta$

Ex. 

- $\ker \partial_2 \cong \mathbb{Z}$, $\partial_3 = 0$

  ↗ generated by $U - L$

$\Longrightarrow H_2^\Delta(X) \cong \mathbb{Z}$

- $\ker \partial_1 \cong \mathbb{Z}^3$
- $\text{Im}\,\partial_2 \cong \mathbb{Z}$

  generated by $a + b - c$ $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\Longrightarrow H_1^\Delta(X) = \ker \partial_1 / \text{Im}\,\partial_2 \cong \mathbb{Z}^3 / \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cong \mathbb{Z}^2$

↗ generated by classes $[a], [b]$

- $\ker \partial_0 = \mathbb{Z}$, $\text{Im}\,\partial_1 = 0$

$\Longrightarrow H_0^\Delta(X) \cong \mathbb{Z}$

Summary:

$H_0 \cong \mathbb{Z}$

$H_1 \cong \mathbb{Z}^2$

$H_2 \cong \mathbb{Z}$