Lecture 11 02/25/20

Last time: Δ-complex X
$\rightsquigarrow$ Δ-chain complex $(C_*^\Delta(X), \partial)$, $\partial^2 = 0$
$\rightsquigarrow$ homology $H_*^\Delta(X) := \ker\partial / \operatorname{Im}\partial$

Review • Δ^n = standard n-simplex
= convex hull of $(1,0,\dots,0), (0,1,0,\dots), \dots, (0,\dots,0,1)$

• $v_0, \dots, v_n$ (affine independent points) in $\mathbb{R}^n$

$\rightsquigarrow$ $[v_0, \dots, v_n]$ and order gives a canonical homeomorphism to Δ^n, taking v_i to $(0, \dots, \underset{i}{1}, \dots, 0)$

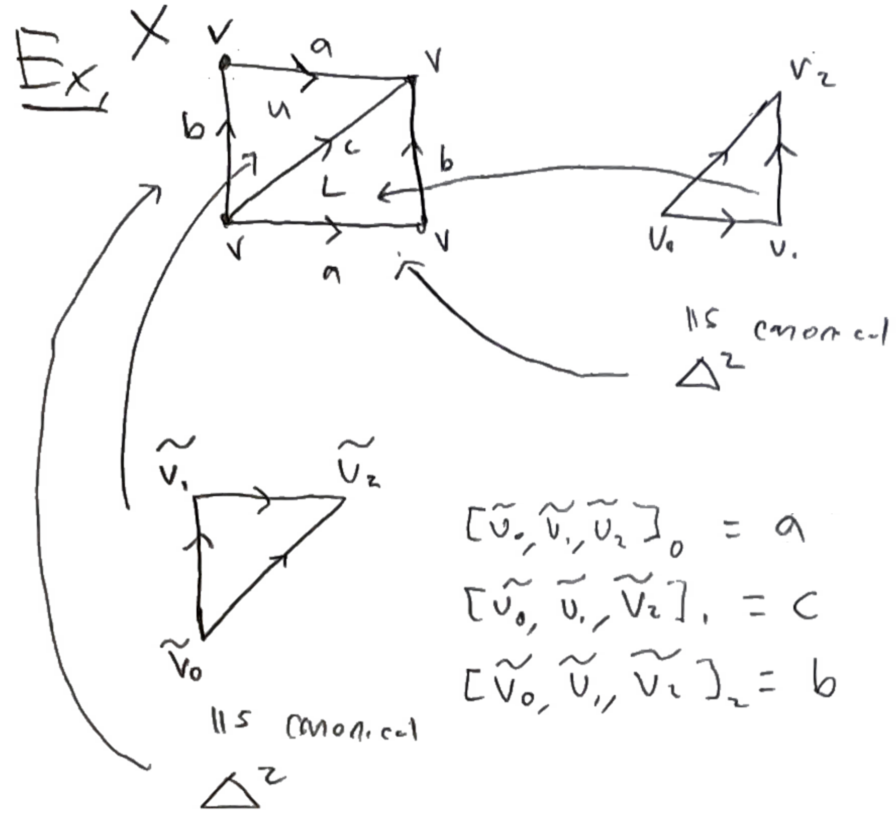
• ith face of $[v_0, \dots, v_n]$ is $[v_0, \dots, \hat{v}_i, \dots, v_n]$ is homeomorphic to Δ^{n-1}

Def. Δ-complex X with collection of maps $\sigma_\alpha : \Delta^n \to X$ so that 1) $\sigma_\alpha|_{\operatorname{Int}\Delta^n}$ is injective

2) $\sigma_\alpha|_{\underset{\Delta^{n-1}}{\underset{\text{ıs}}{\text{ith face of } \Delta^n}}} = \sigma_\beta : \Delta^{n-1} \to X$

for some β

Ex. X



$[v_0, v_1, v_2]_0 = [v_1, v_2] \to b$

$[v_0, v_1, v_2]_1 = [v_0, v_2] \to c$

$[v_0, v_1, v_2]_2 = [v_0, v_1] \to a$

$\cong$ canonical Δ^2

$[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]_0 = a$

$[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]_1 = c$

$[\tilde{v}_0, \tilde{v}_1, \tilde{v}_2]_2 = b$

$\cong$ canonical Δ^2

- so get two maps $\Delta^2 \to X$ $\quad U, L$
  three maps $\Delta^1 \to X$, $\quad a, b, c$

Def. n-chains $\Delta_n(X) =$ free abelian group generated by $\sigma_\alpha : \Delta^n \to X$

- boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$

$$\partial_n(\sigma_\alpha) = \sum (-1)^i \sigma_\alpha \big|_{i\text{th face of } \Delta_n}$$

use identification with Δ^{n-1}

Def. $\ker \partial_n = n$-cycles, $\operatorname{Im} \partial_{n+1} = n$-boundaries

Prop $\partial^2 = 0$

Ex. $\partial U = a - c + b$

$\partial L = b - c + a$

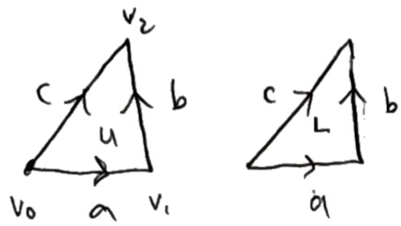
signs appear only from signs in boundary

$\Delta_*(X)$ is $\quad 0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^1 \xrightarrow{\partial_0} 0$

and last time $\quad H_*^\Delta(X) = \ker\partial / \mathrm{Im}\,\partial$

$$= \begin{cases} \mathbb{Z}^1 & i=0 & v \\ \mathbb{Z}^2 & i=1 & a,\ b \\ \mathbb{Z}^1 & i=2 & U-L \end{cases}$$

<u>Ex.</u> $\cdot\ S^2$



$$0 \to \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^3 \xrightarrow{0} 0$$

$U \longrightarrow b - c + a$

$L \longrightarrow b - c + a$

$a \longrightarrow v_1 - v_0$

$b \longrightarrow v_2 - v_1$

$c \longrightarrow v_2 - v_0$

- $\ker\partial_2 = \langle U - L \rangle \quad \Longrightarrow \quad H_2^\Delta(S^2) \cong \mathbb{Z}$
- $\mathrm{Im}\,\partial_2 = \langle b - c + a \rangle$
- $\ker\partial_1 = \ker\begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$= \left\langle \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\rangle = \langle a + b - c \rangle \Rightarrow H_1^\Delta(S^2) \cong 0$$

- $\mathrm{Im}\,\partial_1 = \mathbb{Z}^3 / \ker \partial_2 \cong \mathbb{Z}^2$
- $\ker \partial_0 = \mathbb{Z}^3$

$\Rightarrow \quad H_0^\Delta(S^2) \cong \mathbb{Z} \quad \square$

Think about: S^3 as two copies of 
all faces identified by identity

Singular homology

- X any topological space, $\sigma : \Delta^n \to X$ is singular n-simplex

Def. singular n-chains $C_n(X)$
= free abelian group generated by $\sigma : \Delta^n \to X$

- usually uncountable....
- boundary homomorphism $\partial_n : C_n(X) \to C_{n-1}(X)$, $\partial^2 = 0$

Def. singular homology $H_n(X) = \ker \partial_n / \mathrm{Im}\,\partial_{n+1}$

Prop. if X, Y homeomorphic, $H_*(X) \cong H_*(Y)$

Pf. • $f : X \to Y$, $\sigma : \Delta^n \to X \Rightarrow f \circ \sigma : \Delta^n \to X \to Y$
so get $f_* : C_n(X) \to C_n(Y)$ induced map

Key $f_* \partial_X = \partial_Y f_*$

$\Rightarrow$ $f_* : \ker \partial_X^n \longrightarrow \ker \partial_Y^n$

and $f_* : \mathrm{Im}\, \partial_X^{n+1} \longrightarrow \mathrm{Im}\, \partial_Y^{n+1}$

so get induced map $f_* : H_n(X) \longrightarrow H_n(Y)$

- if exists $g : Y \to X$ so that $g \circ f = Id_X$

Def: $\Phi : (B_*, d_B) \longrightarrow (C_*, d_C)$ is a chain map if $\phi \circ d_B = d_C \circ \phi$

$\Rightarrow$ induced map $H_\cdot(B) \longrightarrow H_\cdot(C)$

Later: if X is Δ-complex, $H_*^\Delta(X) \cong H_*(X)$

Geometric interpretation of $H(X)$

- write n-chain $\xi = \sum \varepsilon_\alpha \sigma_\alpha$, $\varepsilon_\alpha = \pm 1$
- $d\xi = 0 \Rightarrow$ cancelling pairs $\sigma_\alpha|_{i\text{th face}} = \sigma_\beta|_{i_\beta\text{th face}}$
- form $K_\xi = \sqcup \Delta^n_\alpha / \sim$ $\quad i_\alpha$ face of $\Delta_\alpha \sim i_\beta$ face of Δ_β
- get map $K_\xi \longrightarrow X$

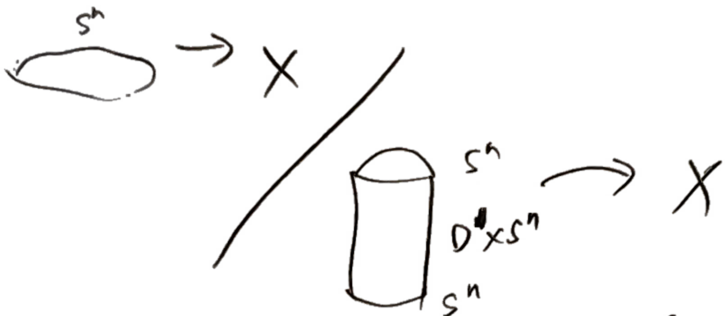
Rmk K^n_ξ is a manifold (locally homeo to $\mathbb{R}^n$) away from $(n-2)$-dimensional subcomplex (in fact $(n-3)$-subcomplex)

Ex.

not a mfld near point, codim 3

$$H_n(X) \cong \text{maps } K^n_{\mathbb{Z}} \to X \Big/ \begin{tikzcd} K^{n+1}_{\mathbb{Z}} \arrow[r] & X \\ K^n_{\mathbb{Z}} \arrow[u, "\partial"] \arrow[ur] & \end{tikzcd}$$

$\pi_n(X)$ = homotopy classes $S^n \to X$



<u>Ex.</u> $H_1(X) = \{\sqcup S^1 \to X\} \Big/$  $\to X$

no singularities since $2-3=-1$

genuine surface without singularities

$\pi_1(X) = \{S^1 \to X\} \Big/$  $\to X$

. there is a map $\pi_1(X) \twoheadrightarrow H_1(X)$, surjective and since $H_1(X)$ is abelian, there is an induced map $\pi_1(X)/[\pi_1(X), \pi_1(X)] \twoheadrightarrow H_1(X)$

<u>Prop.</u> $\pi_1(X)/[\pi_1(X), \pi_1(X)] \cong H_1(X)$

PE.

- kernel of $\pi_1(X) \longrightarrow H_1(X)$ consists of maps $S^1 \longrightarrow X$ that extend to

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, hook] & X \\ \Sigma_g \setminus D^2 \arrow[ur, dashed] & \end{tikzcd}$$

- recall that $\gamma \cong [a_1, b_1][a_2, b_2] \cdots [a_g, b_g]$

where

is basis for $\pi_1(\Sigma_g \setminus D^2)$

so $\phi \circ \gamma : S^1 \longrightarrow X$ is <u>homotopic</u>

to $\phi \circ ([a_1, b_1] \cdots [a_g, b_g]) = [\phi_* a_1, \phi_* b_1] \cdots [\phi_* a_g, \phi_* b_g]$

and so $\phi \circ \gamma \in [\pi_1(X), \pi_1(X)]$

<u>ie.</u> $\ker \cong [\pi_1(X), \pi_1(X)]$

and so $\pi_1(X) / [\pi_1(X), \pi_1(X)] \cong H_1(X)$