Lecture 12 02/27/20

Last time singular homology

Def. singular n-simplex in X is $\sigma : \Delta^n \to X$

$\rightsquigarrow$ singular n-chains
$C_n(X)$ = free abelian group generated by σ
$\sum a_i \sigma_i, \quad a_i \in \mathbb{Z}$

- boundary $\partial_n : C_n(X) \to C_{n-1}(X)$

$\rightsquigarrow$ singular cohomology $H_n(X) = \underbrace{\ker \partial_n}_{n\text{-cycles}} / \underbrace{\operatorname{Im} \partial_{n+1}}_{n\text{-boundaries}}$

Prop. if $X = \coprod X_\alpha$, then $H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha)$

Pf. $C_n(X_\alpha) \subset C_n(X)$ subgroup, $C_n(X) \cong \bigoplus_\alpha C_n(X_\alpha)$

and $\partial_n : C_n(X_\alpha) \to C_{n-1}(X_\alpha)$ call $\partial_{n,\alpha}$

$\Rightarrow \ker \partial_n = \bigoplus_\alpha \ker \partial_{n,\alpha}$
$\operatorname{Im} \partial_n = \bigoplus_\alpha \operatorname{Im} \partial_{n+1,\alpha}$ $\square$

Prop. if X path-connected, $H_0(X) \cong \mathbb{Z}$

Pf. $H_0(X) = C_0(X) / \operatorname{Im} \partial_1$ (since $\partial_0 = 0$)

- define $\phi : C_0(X) \twoheadrightarrow \mathbb{Z}$ surjective group homomorphism
$\sum a_i \sigma_i \to \sum a_i$
counting number of points (with sign)

Claim: $\ker \Phi = \operatorname{Im} \partial_1$

$$\Rightarrow \quad H_0(X) = C_0(X) / \operatorname{Im}\partial_1 = C_0(X) / \ker \Phi$$

$$\cong \operatorname{Im} \Phi \cong \mathbb{Z}$$

$\nwarrow$ first isomorphism theorem for Φ

Pf of claim: $\operatorname{Im} \partial_1 \subset \ker \Phi$

$\sigma : \Delta^1 \to X, \quad \partial\sigma = \sigma|_1 - \sigma|_0$

$\Phi(\partial\sigma) = 1 - 1 = 0$ $\qquad -1 \quad +1$

- $\operatorname{Im}\partial_1 \supset \ker\Phi$
- if $\Phi(\sum a_i \sigma_i) = 0$, then $\sum a_i = 0 = \underbrace{\sum a_i^+}_{q} - \underbrace{\sum a_i^-}_{q}$
- $\Rightarrow \sum a_i \sigma_i = \sum_{i=1}^{q} \sigma_i - \sum_{i=1}^{q} \sigma_i'$

take path $\gamma_i : \Delta^1 \to X$, $\gamma_i|_1 = \sigma_i$, $\gamma_i|_0 = \sigma_i'$

$\Rightarrow \partial(\sum_{i=1}^{q} \gamma_i) = \sum \sigma_i - \sigma_i' = \sum a_i \sigma_i$ $\qquad 1 \quad 2 \quad -3$ $\qquad \square$

Cor. if $X = \sqcup X_\alpha$, $H_0(X) \cong \bigoplus_\alpha \mathbb{Z}$,

$\mathbb{Z}$ for each path-component

- reduced homl., $C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, $\quad H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$

Prop. if X is a point, $H_0(X) \cong \mathbb{Z}$, $H_n(X) = 0$ for $n > 0$

Pf unique n-simplex σ_n so $C_n(X) \cong \mathbb{Z}\langle \sigma_n \rangle$

$$\cdots \xrightarrow{\sim} \overset{3}{\mathbb{Z}} \xrightarrow[0]{\partial_3} \overset{2}{\mathbb{Z}} \xrightarrow[1]{\partial_2 \, \sim} \overset{1}{\mathbb{Z}} \xrightarrow[0]{\partial_1} \overset{0}{\mathbb{Z}} \to 0$$

- $\partial_n(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_n|_{i\text{th face of } \Delta^n} = \left(\sum_{i=0}^{n} (-1)^i\right) \sigma_{n-1}$

$\Rightarrow \partial_{\text{even}}$ is isomorphism $\qquad$ 1 if n even

$\Rightarrow H_n(X) \cong 0, \ n > 0$ $\qquad$ 0 if n odd

$$\pi_1(X, x_0) \longrightarrow H_1(X)$$

$$[0,1] \to (X, x_0) \to \sigma : \Delta^1 \longrightarrow X, \ \partial\sigma = 0$$

- $H_1(X)$ is abelian group

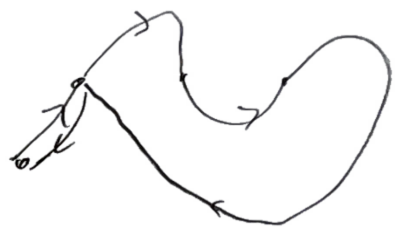
Prop. $\pi_1^{ab}(X) = \pi_1(X) / [\pi_1(X), \pi_1(X)] \xrightarrow{\sim} H_1(X)$

Pf surjective: $\partial(\sum a_i \sigma_i) = 0$

$\Longrightarrow$ pair off Δ^1's

get map $S^1 \sqcup S^1 \longrightarrow X$

- no base point,



same homology class, now element of $\pi_1(X)$

Maps on homology

- $f : X \to Y$, get $f_* : C_n(X) \longrightarrow C_n(Y)$

$$\sigma : \Delta \to X, \quad f_* \sigma = f \circ \sigma \in C_n(Y)$$

Key $f_* \circ \partial_X = \partial_Y \circ f_*$

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial_X"] \arrow[d, "f_*"'] & C_{n-1}(X) \arrow[d, "f_*"] \\ C_n(Y) \arrow[r, "\partial_Y"] & C_{n-1}(Y) \end{tikzcd}$$

$\Rightarrow$ $f_* : \ker \partial_X \to \ker \partial_Y$

$f_* : \operatorname{Im} \partial_X \longrightarrow \operatorname{Im} \partial_Y$

$\Rightarrow$ $f_* : H_n(X) \to H_n(Y)$ induced map

Def. $\phi : (B, d_B) \to (C, d_C)$ with $\phi \circ d_B = d_C \circ \phi$ is called a chain map

$\Rightarrow$ induced map $\phi_* : H_\bullet(B) \to H_\bullet(C_\bullet)$

Prop. if $f : X \to Y$ is a homeomorphism,

then $f_* : H_*(X) \xrightarrow{\sim} H_*(Y)$ isomorphism

Prop. if f homotopy equivalence, f_* also isomorphism

Prop : if $f, g : X \to Y$ are homotopic

then $f_* = g_* : H_*(X) \to H_*(Y)$

__Pf__ let $F : X \times I \to Y$ be a homotopy

between f, g, ie. $F|_{X\times 0} = f$, $F|_{X \times 1} = g$

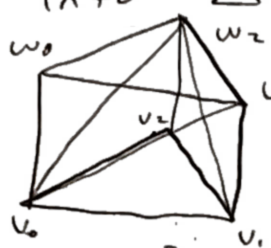
- $\sigma : \Delta^n \to X \rightsquigarrow F \circ (\sigma \times Id) : \Delta^n \times I' \to X \times I' \to Y$.
- $\Delta^n \times I'$ basically Δ^{n+1}, want $\Delta^{n+1} \to Y$
- " to get map $C_n(X) \to C_{n+1}(Y)$
- decompose $\Delta^n \times I$ into Δ^{n+1}

__Ex,__

Δ^1

$[v_0, v_1, w_1]$

$[v_0, w_0, w_1]$



Δ^2

$[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_1, w_2]$

$[v_0, w_0, w_1, w_2]$

- get $P : C_n(X) \to C_{n+1}(X)$

$$\sigma \longrightarrow \sum F \circ (\sigma \times Id)|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

- __Key__: $\partial_Y P = g_* - f_* - P\partial_X$

total boundary top boundary bottom side boundaries

- if $\alpha \in \operatorname{Ker} \partial_*$, then

$$g_*(\alpha) - f_*(\alpha) = \partial(P\alpha) + P\partial\alpha \overset{0}{} = \partial P \alpha \in \operatorname{Im} \partial_Y$$

$$\Rightarrow g_*(\alpha) = f_*(\alpha) \in H_n(Y)$$

chain maps

Def. $\Phi_1, \Phi_2 : (B, \partial_B) \to (C, d_C)$ are <u>chain homotopic</u> if exists

$P : B_* \to C_{*+1}$ so that $\partial_C P = \Phi_1 - \Phi_2 + P\partial_B$

Prop. chain homotopic maps induce same map on homology