Lecture 13 03/03/20

Last time: chain maps $f_*: (B_*, d_B) \to (C_*, d_C)$

chain homotopy $H_*: (B_*, d_B) \to (C_{*+1}, d_C)$
between two chain maps

Today: more "homological algebra"

Q. if $X = A \cup B$, how are $H_*(X)$, $H_*(A)$, $H_*(B)$, and $H_*(A \cap B)$ related?

- is there van Kampen type theorem for homology?

- First do $A \subset X$ and X/A $(= B/A\cap B)$

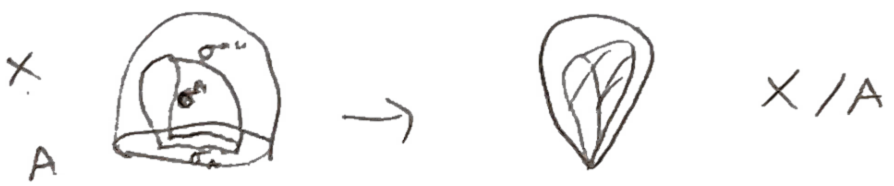
- $i: A \to X$, $q: X \to X/A$

  $i_*: H_*(A) \to H_*(X)$, $q_*: H_*(X) \to H_*(X/A)$

- $q_* \circ i_*: H_*(A) \to H_*(X/A)$ is zero map for $* > 0$

$\Longrightarrow$ $\operatorname{Im} i_* \subset \operatorname{Ker} q_*$

- $\operatorname{Ker} q_*$ "=" $\left\{ \begin{array}{l} \sigma^n: \Delta^n \to X \text{ so that exists} \\ \sigma^{n+1}: \Delta^{n+1} \to X/A, \ \partial\sigma^{n+1} = q \circ \sigma^n \end{array} \right.$



$\Longrightarrow$ $\sigma^n - \sigma^n_A = \partial \sigma^{n+1} \Longrightarrow [\sigma^n] = [\sigma^n_A]$ in $H_n(X)$

and $\sigma_A \in \mathrm{Im}\, i_*$

$\Rightarrow \ker q_* \subset \mathrm{Im}\, i_* \Rightarrow \ker q_* = \mathrm{Im}\, i_*$

so $H_*(A) \xrightarrow{i_*} H_*(X) \xrightarrow{q_*} H_*(X/A)$ is <u>exact</u>

- $\ker i_* = \left\{ \begin{array}{l} \sigma^n : \Delta^n \to A \quad \text{so that exists} \\ \sigma^{n+1} : \Delta^{n+1} \to X, \quad \partial \sigma^{n+1} = i \circ \sigma^n \end{array} \right.$

so $\partial \sigma^{n+1} = 0$ in X/A, $[\sigma^{n+1}] \in H_{n+1}(X/A)$

- "∂" : $H_{n+1}(X/A) \to H_n(A)$

well-defined:

and Im "∂" $= \ker i_*$

- finally, $\ker$ "∂" $= \mathrm{Im}\, q_*$

<u>Thm</u> there is a "long" exact sequence

$$\tilde{H}_{n+1}(X/A) \xrightarrow{\partial} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(X/A) \to$$

<u>Rmk</u> need $A \subset X$ closed, and exists open $U \supset A$ so that U deformation retracts to A; (X, A) called <u>good pair</u>

Cor. $\tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{otherwise} \end{cases}$ $\left(H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{otherwise} \end{cases} \right)$

- $H_n(S^n) = \mathbb{Z}$ generated by $\Delta_1 - \Delta_2$

where $S^n = \Delta_1 \sqcup \Delta_2 \big/ \partial_i \Delta_1 \sim \partial_i \Delta_2$

Pf. $S^{n-1} \xhookrightarrow{\text{boundy}} D^n \xrightarrow{q} D^n / S^{n-1} \simeq S^n$

- $\tilde{H}_*(D^n) = 0 \Rightarrow$ so have exact sequences

$$0 \xrightarrow{q_*} \tilde{H}_i(S^n) \xrightarrow[\sim]{\partial} \tilde{H}_{i-1}(S^{n-1}) \xrightarrow{i_*} 0$$

$\Rightarrow \operatorname{Ker} \partial = 0, \ \operatorname{Im} \partial = \operatorname{Ker} i_* = H_{i-1}(S^{n-1})$

$\Rightarrow \partial$ isomorphism

- $\tilde{H}_i(S^n) \simeq \tilde{H}_{i-n}(S^0) = \begin{cases} \mathbb{Z}, & i - n = 0 \\ 0 & \text{otherwise} \end{cases}$ ∎

- $\sqcup p_\alpha \xhookrightarrow{i} \sqcup X_\alpha \xrightarrow{q} \vee_\alpha X_\alpha$

Cor $q : \tilde{H}_*(\sqcup X_\alpha) \xrightarrow{\sim} \tilde{H}_*(\vee_\alpha X_\alpha)$ for $* \geq 1$

$$\tilde{H}_*(\sqcup X_\alpha) = H_*(\sqcup X_\alpha) \cong \bigoplus_\alpha H_*(X_\alpha)$$

Relative homology

- $i : A \hookrightarrow X$, $C_*(X, A) := C_*(X)/C_*(A)$
- $\partial : C_*(A) \to C_{*-1}(A)$

so get induced map $\partial : C_*(X)/C_*(A) \to C_{*-1}(X)/C_{*-1}(A)$

relative homology $H_*(X, A) \cong H_*(C_*(X, A), d)$

- elements of $H_n(X, A)$ represented by <u>relative</u> cycles
  i.e. $\alpha \in C_n(X)$ so that $\partial\alpha \in C_{n-1}(A)$

- relative cycle is trivial in $H_n(X, A)$ if it is relative boundary, $\alpha = \gamma + \partial\beta$,
  for $\gamma \in C_n(A)$, $\beta \in C_{n+1}(X)$,

<u>Prop.</u> if (X, A) good pair, $H_*(X, A) \cong \tilde{H}_*(X/A)$

chain map $j : C_*(X) \to C_*(X, A)$

<u>Thm</u>

$$
\begin{tikzcd}
H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "j_*"] & H_n(X, A) \arrow[dll, "\partial"'] \\
H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X, A) \arrow[r] & {}
\end{tikzcd}
$$

is exact

Def.

short exact sequence of chain complexes are chain maps $0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0$

so that $0 \to A_n \to B_n \to C_n \to 0$

is short exact sequence for all n

Thm for a short exact sequence of chain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$, there is a long exact sequence

$$\to H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to$$