Lecture 14 03/05/20

Last time:

- long exact sequence for $A \hookrightarrow X \xrightarrow{q} X/A$

$$\to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to$$

- relative chains $C_*(X, A) := C_*(X)/C_*(A)$

Thm there is a long exact sequence for $H(X,A)$
and $H(X,A) = H(X/A)$

Rmk. • $A \subset X$ closed $\rightsquigarrow$ point $A/A \subset X/A$ closed
(X,A) good pair if there is open $U \supset A$
so that U def retracts to A

• $C_\cdot(X,A)$ is a replacement for,
$G = \{\sum a_i \sigma_i \mid \sigma_i : \Delta^n \to X,\ \operatorname{Im} \sigma_i \not\subset A\} \subset C_\cdot(X)$

$G \xrightarrow{\sim} C_\cdot(X,A)$ but G is not invariant under ∂

X $\operatorname{Im} \sigma \not\subset A$, but $\operatorname{Im} \partial\sigma \subset A$

A

Short exact sequence of chain complexes

• recall $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a
short exact sequence if $\operatorname{Ker} \phi = 0$, $\operatorname{Im} \phi = \operatorname{Ker} \psi$,
and $\operatorname{Im} \psi = C$

- a short exact sequence of <u>chain complexes</u> are chain maps $(A, d_A) \xrightarrow{\phi_*} (B, d_B) \xrightarrow{\psi_*} (C, d_C)$ so that $0 \to A_n \to B_n \to C_n \to 0$ is SES for all n

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "\phi_{n+1}"] & A_n \arrow[r] \arrow[d, "\phi_n"] & A_{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"] & {} \\
{} \arrow[r] & B_{n+1} \arrow[r] \arrow[d, "\psi_{n+1}"] & B_n \arrow[r] \arrow[d, "\psi_n"] & B_{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d] & C_n \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

<u>Ex.</u> $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$ is SES of chain complexes

<u>Prop.</u> if $0 \to A_* \to B_* \to C_* \to 0$ is SES of chain complexes, there is a LES

$$\begin{tikzcd}
 & & H_{n+1}(C_*) \arrow[dll, "\partial"'] \\
H_n(A_*) \arrow[r, "\phi_*"] & H_n(B_*) \arrow[r, "\psi_*"] & H_n(C_*) \arrow[dll, "\partial"'] \\
H_{n-1}(A_*) \arrow[r] & \cdots &
\end{tikzcd}$$

<u>Pf.</u>

- First define $\partial : H_{n+1}(C_*) \to H_n(A_*)$

- take $c \in C_{n+1}$, $\partial c = 0$,
- then since ψ_{n+1} surjective, exists $b \in B_{n+1}$ so that $\psi_{n+1}(b) = c$
- $\psi_n \circ \partial (b) = \partial \circ \psi_{n+1}(b) = \partial c = 0$

$\Rightarrow \partial b \in \operatorname{Ker} \psi_{n+1} = \operatorname{Im} \phi_n$, i.e. $\partial b = \phi_n(a)$

define $\partial c = a \in A_n$

- $\phi_{n-1} \circ \partial a = \partial \circ \phi_n(a) = \partial \partial b = 0$

and since ϕ_{n-1} injective, $\partial a = 0$

so $[a] \in H_n(A)$

<u>claim</u>: $[a] \in H_n(A)$ is well-defined

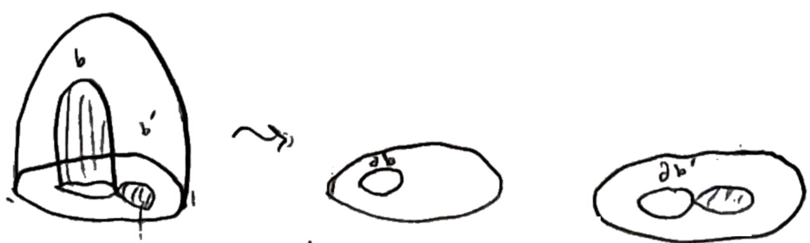
- a determined by ∂b since ϕ_n injective
- if take different b' so that $\psi_{n+1}(b') = \psi_{n+1}(b)$

then $\psi_{n+1}(b' - b) = 0$, so $b' - b \in \operatorname{Im} \phi_{n+1}$

$b' - b = \phi_{n+1}(\tilde{a}) \Rightarrow \partial b' - \partial b = \partial \phi_{n+1}(\tilde{a}) = \phi_n \partial \tilde{a}$

and so $\partial b' = \phi_n(a + \partial \tilde{a})$, i.e. $a' = a + \partial \tilde{a}$

so $[a] = [a'] \in H_n(A)$



$b - b' = 0$ in $C_{n+1}(X, A)$

$\Leftrightarrow b - b' \in C_{n+1}(A)$

• if take different c' so that $[c] = [c'] \in H_{n+1}(C)$,
then $c' - c = \partial \tilde{c}$, $\tilde{c} \in C_{n+2}$

• exists $\tilde{b} \in C_{n+2}$ so that $\psi_{n+2}(\tilde{b}) = \tilde{c}$

so $c' = c + \partial \tilde{c} = \psi_{n+1}(b) + \partial \psi_{n+2}(\tilde{b}) = \psi_{n+1}(b + \partial \tilde{b})$

and $\partial(b + \partial \tilde{b}) = \partial b \longrightarrow a$ unchanged

<u>Exactness</u>

• $\psi_* \circ \phi_* = 0 \implies \operatorname{Im} \phi_* \subset \ker \psi_*$

• <u>claim:</u> $\operatorname{Im} \phi_* \supset \ker \psi_*$

• $[b_n] \in \ker \psi_* \implies \psi_n(b_n) = \partial c_{n+1}$

• ψ_{n+1} surjective, so $\psi_{n+1}(b_{n+1}) = c_{n+1}$

$\psi_n(b_n - \partial b_{n+1}) = \partial c_{n+1} - \partial \psi_{n+1}(b_{n+1}) = 0$

$\implies b_n - \partial b_{n+1} = \phi_n(a_n)$, $\partial a_n = 0$ since $\partial b_n = 0$

• so $[b_n] = [\phi_n(a_n)] = \phi_*[a_n]$, i.e. $[b_n] \in \operatorname{Im} \phi_*$ □

map $(A,B) \to (C,D)$ induces by
$q : (X, A) \to (X/A, A/A) = (X/A, a_0)$

<u>Prop.</u> if (X, A) good pair,
$q_* : H_*(X, A) \xrightarrow{\sim} H_*(X/A, a_0) = \tilde{H}_*(X/A)$

<u>Thm</u> (excision) suppose $A, B \subset X$ whose interiors cover X. then $i : (B, A \cap B) \longrightarrow (X, A)$ induces isomorphism $i_* : H_*(B, A \cap B) \xrightarrow{\sim} H_*(X, A)$

excise $A \setminus A \cap B$

• alternately, $Z \subset A \subset X$, $\mathrm{closure}(Z) \subset \mathrm{Int}\, A$

$\rightsquigarrow \varepsilon_* : H_*(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_*(X, A)$

<u>Pf of Prop assuming excision</u>: $\underset{\text{excision}}{A \subset U}$, $(X,A) \xrightarrow{j} (X,U)$

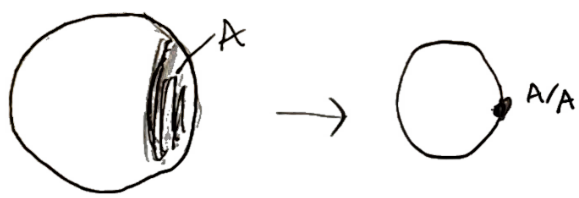
$$\begin{tikzcd}
H_n(X,A) \arrow[r, "j_*", "\sim"'] \arrow[d, "q_*"] & H_n(X,U) \arrow[d, "q_*"] & H_n(X \setminus A, U \setminus A) \arrow[l, "i_*"', "\sim"] \arrow[d, "q_*", "\cong"'] \\
H_n(X/A, A/A) \arrow[r, "j_*", "\sim"'] & H_n(X/A, U/A) & H_n(X/A \setminus A/A, U/A \setminus A/A) \arrow[l, "i_*"', "\sim"]
\end{tikzcd}$$

$H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$

homotopy invariance

• right vertical q_* is isomorphism since

$X \setminus A \to X/A \setminus A/A$ is homeomorphism



□

<u>Next time</u>: prove excision Thm