Lecture 2 01/23/20

Fundamental group $\pi_1(X, x_0)$:

= <u>homotopy</u> classes of paths in X based at x_0

$f : [0,1] \to X, \ f(0) = f(1) = x_0$

<u>Prop</u> $\pi_1(X, x_0)$ is a group with product path concatenation $f * g(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$

<u>Rmk</u> well-defined $f \simeq f',\ g \simeq g' \Rightarrow f * g \simeq f' * g'$

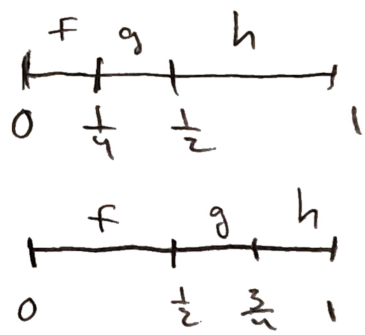
<u>Pf.</u> • inverses: $f(1-t)$ is inverse to $f(t)$

so $f(1-t) * f(t) \simeq x_0$ constant map

• associativity: $(f * g) * h$

is

$f * (g * h)$



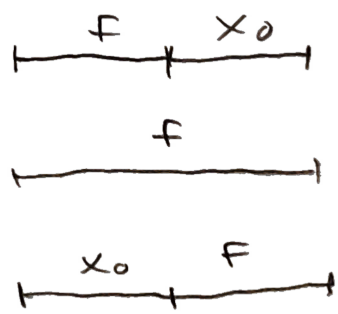
• constant x_0 is identity element : $f * x_0$

is

f

is

$x_0 * f$



□

Induced maps $\phi : (X, x_0) \longrightarrow (Y, y_0)$ $f(x_0) = y_0$

induces $\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

$\phi_* f = [\phi \circ f]$ homotopy class

Prop ϕ_* well-defined and a group homomorphism

and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$; $(Id_X)_* = Id$ on $\pi_1(X, x_0)$

Fancy language: π_1 is a functor from spaces to groups

Cor if $\phi : X \to Y$ homeomorphism, $\phi_* : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$

exists $\psi : Y \to X$ so that $\psi \circ \phi = Id$, $\phi \circ \psi = Id$

$\Rightarrow$ $\psi_* \circ \phi_* = (\psi \circ \phi)_* = Id_* = Id$

and $\phi_* \circ \psi_* = Id$ similarly ∎

Prop $\phi : X \xrightarrow{\sim} Y$ homotopy equiv, then $\phi_* : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$

Thm $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1)$, $n \to \omega_n$

where $\omega_n(s) = (\cos(2\pi n s), \sin(2\pi n s))$  n times

is a group isomorphism

Pf. • group homomorphism: $\underset{\omega_{a+b}}{\underset{\parallel}{\Phi(a+b)}} = \underset{\omega_a * \omega_b}{\underset{\parallel}{\Phi(a) + \Phi(b)}}$

• isomorphism (injective/surjective):

covering spaces: $p : \mathbb{R} \to S^1$

$s \to (\cos(2\pi s), \sin(2\pi s))$

• note that $\omega_n = p \circ \tilde{\omega}_n$, $\tilde{\omega}_n : [0,1] \to \mathbb{R}$
$s \to ns$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\omega}_n"] \arrow[r, "\omega_n"] & S^1 \end{tikzcd}$$

n times

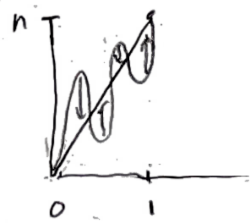
i.e. $\tilde{\omega}_n$ is a <u>lift</u> of ω_n to $\mathbb{R}$

<u>claim 1</u> any $f : ([0,1], 0) \to (S^1, 0)$ has a unique lift $\tilde{f} : ([0,1], 0) \to (\mathbb{R}, 0)$ ie $f = p \circ \tilde{f}$

$\Longrightarrow$ Φ surjective.

• f has lift $\tilde{f}$, $\tilde{f}(0) = 0$, $p\tilde{f}(1) = 0$
so $\tilde{f}(1) \in p^{-1}(0) = \mathbb{Z}$ $\rightsquigarrow$ $\tilde{f}(1) = n$ some n

• $\tilde{f}(s)$ homotopic to $\tilde{\omega}_n(s)$ rel $0, n \in \mathbb{R}$
via linear homotopy $\tilde{f}_t(s)$



• then $p \circ \tilde{f}_t$ homotopy between f and ω_n

<u>claim 2</u>: any homotopy $f_t : ([0,1], 0) \to (S^1, 0)$ has a unique lift $\tilde{f}_t : ([0,1], 0) \to (\mathbb{R}, 0)$ (homotopy lifting property)

$\Longrightarrow$ Φ injective

• if f_t homotopy bet ω_n, ω_m lifts to homotopy $\tilde{f}_t$ and by uniqueness from claim 1), $\tilde{f}_0 = \tilde{\omega}_n$, $\tilde{f}_1 = \tilde{\omega}_m$

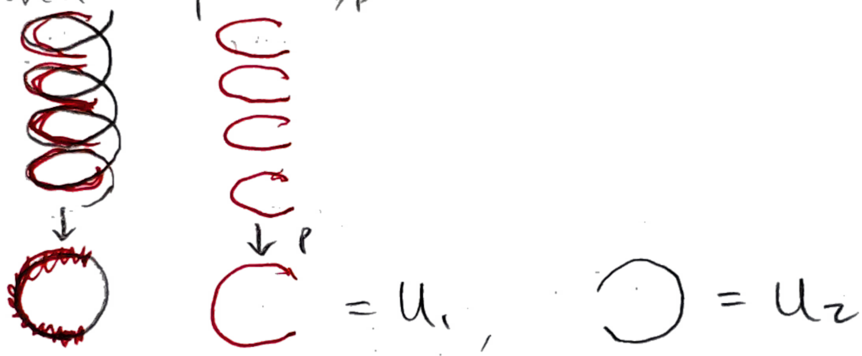
• $p \circ \tilde{f}_t(1) = 0 \Rightarrow \tilde{f}_t(1) \in p^{-1}(0) = \mathbb{Z}$ so constant

$\Longrightarrow$ $n = \tilde{f}_0(1) = \tilde{f}_1(1) = m$ $\square$
$\uparrow$ constant in t

Key fact for proof of claims 1, 2

- exists a cover of S^1 by open subsets U_α so that $p^{-1}(U_\alpha) = \coprod_\beta V_{\alpha,\beta}$ disjoint union

and $p : V_{\alpha,\beta} \to U_\alpha$ is a homeomorphism



Rmks 1) can't take $U = S^1$ since $p : \mathbb{R} \to S^1$ not homeo

2) this is general definition of covering space

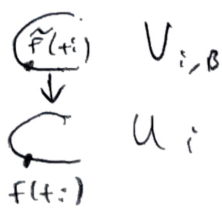
- existence: to lift $f : [0,1] \to (S^1, 0)$ to $\tilde{f}$

break up $[0,1]$ into $\bigcup_i [t_i, t_{i+1}]$ so that

$f([t_i, t_{i+1}]) \subset U_i$ some i

- assume have constructed $\tilde{f}$ on $[0, t_i]$, $t_i \in U_i$

and $\tilde{f}(t_i) \subset V_{i,\beta}$



- define $\tilde{f}$ on $[t_i, t_{i+1}]$ by $[t_i, t_{i+1}] \xrightarrow{f} U_i \xrightarrow{p^{-1}} V_{i,\beta}$

which is continuous on $[0, t_{i+1}]$ and lift f

- uniqueness if $\tilde{f}, \tilde{f}'$ lift f and $\tilde{f}(0) = \tilde{f}'(0) \in V_{1,\beta}$

then agree on $[0, t_1]$ since $\tilde{f}([0,t_1])$, $\tilde{f}'([0,t_1])$

connected, so contained in $V_{1,\beta}$ and $p : V_{1,\beta} \xrightarrow{\sim} U_1$ injects