Lecture 3 01/28/20

- HW due Thursday start of class
- my office hours today, 4:30 – 6:30, Room 307A
- TA's office hours Monday, Wednesday 12–1, Help Room 406

TA: Quang Dao.

Last time: $\pi_1(S^1) \cong \mathbb{Z}$
$\qquad\qquad w_n \leftarrow n$

w_n wraps n times.

Thm every polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$, $n \geq 1$ has a zero in $\mathbb{C}$

Pf sketch: • if $p(z) \neq 0$ all z, then $f_r(s) = \dfrac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$ is loop in S^1 based at 1

- $f_0(s) = 1$ constant loop ω_0
- For $|z| >> 1$, $p(z) \approx z^n$ and so for $r >> 1$, $f_r(s) \simeq (e^{2\pi i s})^n = \omega_n$

so w_0 homotopic to w_n by f_r, contradiction □

Today compute π_1 for any CW complex using just $\pi_1(S^1) \cong \mathbb{Z}$!

Goal: if $j_\alpha : A_\alpha \hookrightarrow X$ open subsets covering X, want to compute $\pi_1(X)$ from $(j_\alpha)_* : \pi_1(A_\alpha) \to \pi_1(X)$ (assume $x_0 \in X$ basepoint and $x_0 \in A_i$)

- get words of elements of $\pi_1(A_\alpha)$
$\gamma_1\, \gamma_2\, \gamma_3\, \gamma_4\, \gamma_5 \ldots$

Group Theory

- let G_α be a (possibly infinite) set of groups

Def. • the free product $\ast_\alpha G_\alpha$ of G_α is set of

- finite length, reduced words $g_1 \cdots g_m$, $g_i \in G_{\alpha_i}$

  reduced: g_i, g_{i+1} belong to different G_α, $\alpha_i \neq \alpha_{i+1}$ and $g_i \neq Id$
- product is concatenation

  $(g_1 \cdots g_m) * (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$ and reduce
- empty word is identity

Prop. product associative

⇝ all reductions of a word are the same

- note that $i_\alpha : G_\alpha \hookrightarrow \ast_\alpha G_\alpha$ is a subgroup

Universal property for any homomorphisms $G_1 \xrightarrow{\Phi} H$, $G_2 \xrightarrow{\Phi_2} H$

exists a unique hom. $\Phi_1 * \Phi_2 : G_1 * G_2 \to H$

$\Phi_1 * \Phi_2(g_1 g_2) = \Phi_1(g_1)\Phi_2(g_2)$

every diagram

$$\begin{tikzcd}
& G_2 \arrow[d] \arrow[ddr, bend left, "\Phi_2"] & \\
G_1 \arrow[r] \arrow[drr, bend right, "\Phi_1"'] & G_1 * G_2 & \\
& & H
\end{tikzcd}$$

can be factored as

$$\begin{tikzcd}
& G_2 \arrow[d] \arrow[ddr, bend left, "\Phi_2"] & \\
G_1 \arrow[r] \arrow[drr, bend right, "\Phi_1"'] & G_1 * G_2 \arrow[dr, dashed] & \\
& & H
\end{tikzcd}$$

Rmk any other Q satisfying this universal property

has $Q \cong G_1 \times G_2$ $\left(\begin{tikzcd}[cramped, sep=small] & & G_2 \arrow[dl] \arrow[dd] \\ G_1 \arrow[r] \arrow[drr] & Q \arrow[dr, dashed] & \\ & & H \end{tikzcd} \right)$

Return to topology X covered by A_α

- $j_\alpha : \pi_1(A_\alpha) \to \pi_1(X)$ induces $\ast_\alpha \pi_1(A_\alpha) \to \pi_1(X)$
- when surjective, injective?
- $$\begin{tikzcd}
A_\alpha \cap A_\beta \arrow[r, "j_{\alpha\beta}"] \arrow[dr, "j_{\beta\alpha}"'] & A_\alpha \arrow[r, "j_\alpha"] & X \\
& A_\beta \arrow[ur, "j_\beta"'] &
\end{tikzcd}$$
  and so $j_\alpha \circ j_{\alpha\beta} = j_\beta \circ j_{\beta\alpha}$

  as maps $\pi_1(A_\alpha \cap A_\beta) \to \pi_1(X)$

so for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$
is in kernel of $\pi_1(A_\alpha) * \pi_1(A_\beta) \to \pi_1(X)$

<u>van Kampen Theorem</u> $X = \bigcup_\alpha A_\alpha$, A_α path-connected
and $x_0 \in A_\alpha$ for all α

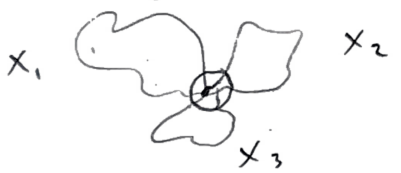
- if $A_\alpha \cap A_\beta$ path-connected for all α, β, then

$*i_\alpha : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is <u>surjective</u>

- if $A_\alpha \cap A_\beta \cap A_\gamma$ path-connected, kernel N is (normal subgroup generated by) $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$

so $*_\alpha \pi_1(A_\alpha) / N \cong \pi_1(X)$

<u>Ex. 1</u> wedge sums $\bigvee_\alpha X_\alpha = \bigsqcup X_\alpha / x_0 \sim x_\alpha$ identify basepoints



- cover by $A_\alpha := X_\alpha \cup U(x_0)$, neighborhood of x_0

$A_\alpha \simeq X_\alpha$ and $A_\alpha \cap A_\beta = U(x_0) \simeq x_0$, $\pi_1 \simeq 0$

and $A_\alpha \cap A_\beta \cap A_\gamma = U(x_0)$ path-connected

$\Rightarrow \pi_1(X) \cong *_\alpha \pi_1(A_\alpha)$

- $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \Rightarrow$ all loops homotopic to unique (reduced) word $\omega_{n_1} * \omega_{n_2} * \dots * \omega_{n_n}$

∴ not abelian!

Ex. 2 

$\pi_1(A_1) = \pi_1(A_2) = 0$

but $\pi_1(A_1 \cup A_2) = \pi_1(S^1) \cong \mathbb{Z}$

so $\overset{0}{\overset{\shortparallel}{\pi_1(A_1) * \pi_1(A_2)}} \to \overset{\mathbb{Z}}{\overset{\shortparallel}{\pi_1(S^1)}}$ not surjective

if $A_1 \cap A_2$ not path-connected

Ex 3 

cover  $\simeq S^1 \vee S^1$

$A_i \cap A_j \simeq x_0$, $A_1 \cap A_2 \cap A_3 =$  not connected

$$\Rightarrow \quad \underset{\mathbb{Z} * \mathbb{Z} * \mathbb{Z}}{\underset{\shortparallel}{\pi_1(A_1) * \pi_1(A_2) * \pi_1(A_3)}} \twoheadrightarrow \underset{\mathbb{Z} * \mathbb{Z}}{\underset{\shortparallel}{\pi_1(S^1 \vee S^1)}}$$

not injective (abelianization $\mathbb{Z}^3 \to \mathbb{Z}^2$)

Universal property $\Phi_1 : K \to G_1$, $\Phi_2 : K \to G_2$, exists

$G_1, G_2 \exists G_1 *_K G$ satisfying universal property:

any commutative diagram can be factored as

$$\begin{tikzcd} K \arrow[r] \arrow[d] & G_2 \arrow[d] \\ G_2 \arrow[r] & H \end{tikzcd} \rightsquigarrow \begin{tikzcd} K \arrow[r] \arrow[d] & G_2 \arrow[d] \arrow[ddr, bend left] & \\ G_1 \arrow[r] \arrow[drr, bend right] & G_1 *_H G_2 \arrow[dr, dashed] & \\ & & H \end{tikzcd}$$

- $G_1 *_H G_2 = G_1 * G_2 / \Phi_1(K)\Phi_2(K)^{-1}$

  is the pushout of
$$\begin{tikzcd} K \arrow[r] \arrow[d] & G_1 \\ G_2 & \end{tikzcd}$$

- if X covered by A_1, A_2 then X is

  pushout of
$$\begin{tikzcd} A_1 \cap A_2 \arrow[r] \arrow[d] & A_2 \\ A_1 & \end{tikzcd}$$
(ie. satisfies universal property for this diagram)

π_1 : spaces $\longrightarrow$ groups
continuous maps $\longrightarrow$ group homomorphisms

Van Kampen: "pushouts of spaces $\longrightarrow$ pushouts of groups"