Lecture 4 01/30/20

Last time: free products of groups, van Kampen's theorem

Today: proof of van Kampen, applications

van Kampen Theorem: $X = \bigcup_\alpha A_\alpha$, A_α path-connected, $x_0 \in A_\alpha$

$i_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X)$

- if $A_\alpha \cap A_\beta$ path-connected for all α, β,

  then $*i_\alpha : * \pi_1(A_\alpha) \longrightarrow \pi_1(X)$ is surjective

- if $A_\alpha \cap A_\beta \cap A_\gamma$ path-connected for all α, β, γ

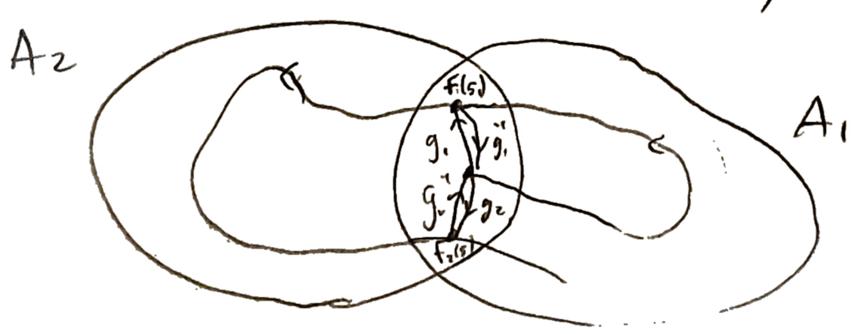
  then $\mathrm{Ker}(* i_\alpha)$ is normal subgroup generated by $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$,

  $$\omega \in \pi_1(A_\alpha \cap A_\beta) \xrightarrow{i_{\alpha\beta} * i_{\beta\alpha}^{-1}} \pi_1(A_\alpha) * \pi_1(A_\beta) \longrightarrow *_\alpha \pi_1(A_\alpha)$$

Pf. surjectivity

- $f \in \pi_1(X)$, $f : [0,1] \to X$, break up $[0,1]$ into $\bigcup_{i=0}^{m} [s_i, s_{i+1}]$

  so that $f([s_i, s_{i+1}]) \subset A_i$; call $f|_{[s_i, s_{i+1}]} = f_i$



- since $A_i \cap A_{i+1}$ path-connected, take path g_i in $A_i \cap A_{i+1}$
  from x_0 to $f_i(s_i)$
- then f homotopic (in X) to $\underbrace{f_1 g_1^{-1}}_{\in \pi_1(A_1)} \underbrace{g_1 f_2 g_2^{-1}}_{\pi_1(A_2)} \cdots \underbrace{g_{m-1} f_m g_m^{-1}}_{\pi_1(A_m)}$

  is in image of $*i_\alpha : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ ✓.

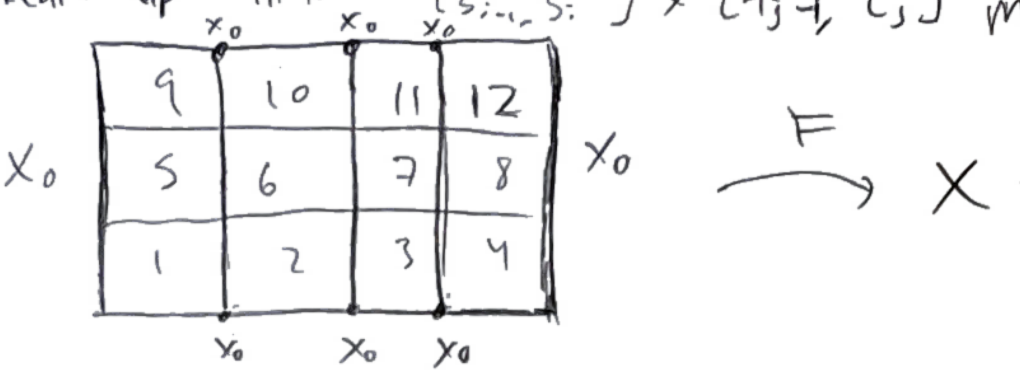
<u>injectivity</u>

- $f \in \pi_1(X) \rightsquigarrow$ factorization is $[f_1] \cdots [f_n]$
  element of $* \pi_1(A_\alpha)$ homotopic
  to f in X
- want to show any two factorizations of
  f are <u>equivalent</u>:
  - combine $[f_i][f_{i+1}]$ into $[f_i f_{i+1}]$ if
    $[f_i], [f_{i+1}] \in \pi_1(A_\alpha)$ same group
  - view $[f_i] \in \pi_1(A_\alpha)$ as in $\pi_1(A_\beta)$
    if $f_i \in \pi_1(A_\alpha \cap A_\beta)$

<u>Goal</u>: any two factorizations of
f are equivalent

$[f_1] \cdots [f_n]$, $[f_1'] \cdots [f_n']$ factorizations of f, ie, homotopic via $F: I \times I \to X$

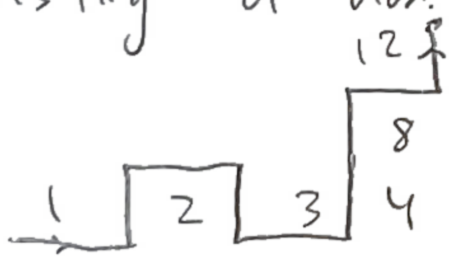
- break up into $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ mapping into A_α



- assume that each vertex maps via F to the basepoint x_0

- let γ be any path in $I \times I$ with endpoints on $0 \times I$, $1 \times I$ consisting of horizontal, vertical edges

Ex



- then $I \xrightarrow{\gamma} I \times I \xrightarrow{F} X$ is homotopic to f and hence is a factorization of f since all vertices map to x_0
- by sliding over squares, we get that the factorizations associated to any γ, γ' are equivalent

- in particular, the factorizations associated to the bottom horizontal, ie $[f_1] \cdots [f_n]$ is equivalent to top horizontal, ie, $[f_1'] \cdots [f_n']$

- Finally, given an arbitrary $F: I \times I \to X$ we can homotope it (by adding paths g_{ij} from x_0 to $F(v_{ij})$, the vertex v_{ij}) so that the vertices actually map to x_0 $\square$

uses fact that $A_i \cap A_j \cap A_k \cap A_l$ connected, which can be improved to just $A_i \cap A_j \cap A_n$ connected

Applications to CW complexes

- recall $X = \bigcup X^n$, where X^n obtained by attaching n-cells D^n_α to X^{n-1} via attaching maps $\partial D^n_\alpha = S^{n-1}_\alpha \xrightarrow{\phi_\alpha} X^{n-1}$

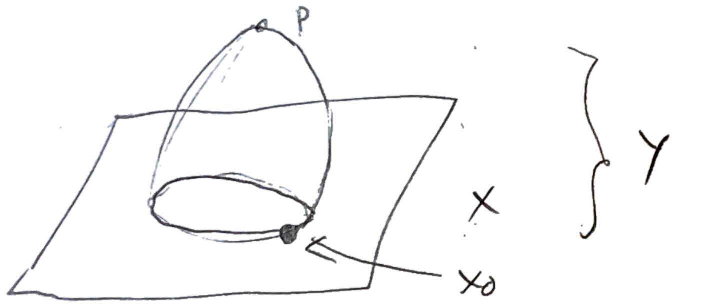
Q. if $Y = X \cup_\phi D^n = X \sqcup D^n / \underset{\text{for } x \in \partial D}{x \sim \phi(x)}$, $\phi : \partial D \to X$

how are $\pi_1(X)$, $\pi_1(Y)$ related?

$i : X \hookrightarrow Y$, so get $i_* : \pi_1(X) \to \pi_1(Y)$

Prop. if $n \geq 3$, $i_* : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$

Pf.



- cover Y by $Y \setminus p$ and $\operatorname{Int} D^n$ (open subsets)

- then $Y \setminus p \simeq X$

$\operatorname{Int} D^n \simeq pt$

$Y \setminus p \cap \operatorname{Int} D^n = S^{n-1} \times (0,1) \simeq S^{n-1}$ path-connect

von Kampen

$\Rightarrow$ $* i_* : \pi_1(X) * \pi_1(pt) \twoheadrightarrow \pi_1(Y)$ has kernel from $\pi_1(S^{n-1}) = 0$ if $n \geq 2$, injective $\square$

Cor. if $i: X^2 \hookrightarrow X$ is 2-skeleton of X

then $i: \pi_1(X^2) \xrightarrow{\sim} \pi_1(X)$

$\Rightarrow$ suffices to study 2-skeleton

- if $Y = X \cup_\phi D^2$, $\phi: \partial D^2 = S^1 \to X$

pick paths γ in X from x_0 to $\phi(0)$

$\Rightarrow \phi_\gamma = \gamma \phi \gamma^{-1} \in \pi_1(X)$

Prop. kernel of $i: \pi_1(X) \to \pi_1(Y)$
is normal subgroup generated by $\phi_\gamma \in \pi_1(X)$

Rmk if pick different γ', then

$$\phi_{\gamma'} = \gamma' \phi (\gamma')^{-1} = \gamma' \gamma^{-1} \gamma \phi \gamma^{-1} \gamma (\gamma'^{-1})^{-1}$$
$$= \alpha \phi_\gamma \alpha^{-1}, \text{ where } \alpha = \gamma' \gamma^{-1} \in \pi_1(X)$$

and normal subgroups generated by $\phi_\gamma, \alpha \phi_\gamma \alpha^{-1}$ are same

Pf again cover Y by $Y \setminus p$, $\mathrm{Int}\, D^2$

now $Y \setminus p \cap \mathrm{Int}\, D^2 \simeq S^1$,

main calculation

$$\mathbb{Z} \cong \pi_1(S^1) \to \pi_1(\mathrm{Int}\, D^2) = 0$$
$$\to \pi_1(Y \setminus p) \cong \pi_1(X)$$
$$1 \to 0$$
$$1 \to \phi, \text{ attaching map}$$

so Kernel of $\pi_1(X) * \pi_1(D^2) = \pi_1(X) \to \pi_1(Y)$
is $\phi \in \pi_1(X)$ ∎