Lecture 5 02/04/20

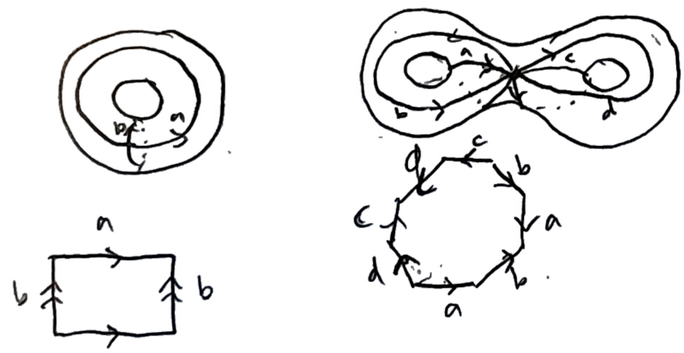
Last time: van Kampen proof, applications

Today: some more applications, covering spaces

- $Y = X \cup D^n / x \sim \phi(x)$, $x \in \partial D^n$, $\phi : \partial D^n \to X$

Prop.
- if $n \geq 3$, $i_* : \pi_1(X) \xrightarrow{\sim} \pi_1(Y)$
- if $n = 2$, i_* surjective with kernel generated by $\phi : \partial D^2 = S^1 \to X$ as element of $\pi_1(X)$

Ex. π_1 of surfaces (locally homeomorphic to $\mathbb{R}^2$)



4g polygon

- CW complex for Σ_g is
  - 1 0-cell
  - 2g 1-cells
  - 1 2-cell

- $X' = X^0 \cup^{2g} D^1/\sim = \bigvee^{2g} S^1$ wedge of 2g circles

$\Rightarrow \pi_1(X') \cong *^{2g} \pi_1(S^1) \cong *^{2g} \mathbb{Z} = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \rangle$

attaching map for 2-cell $\partial D^2 = S^1 \longrightarrow \bigvee^{2g} S^1$

is $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

$= [a_1, b_1] \cdots [a_g, b_g]$

$\rightsquigarrow \pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$

- abelianization $G^{ab} = G/[G,G]$

$\rightsquigarrow \pi_1^{ab}(\Sigma_g) \cong \mathbb{Z}^{2g}$

<u>Cor.</u> $\Sigma_g \not\cong \Sigma_{g'}$ if $g \neq g'$

<u>Prop.</u> For any group G, exists a 2-dimensional CW complex X_G with $\pi_1(X_G) \cong G$

<u>Pf.</u> $G = \langle g_\alpha \mid r_\beta \rangle = \ast_\alpha \mathbb{Z} / \langle r_\beta \rangle$

$\therefore X \cong \bigvee_\alpha S^1 \cup_{\phi_\beta} D^2_\beta$, $\phi_\beta : S^1 \longrightarrow \bigvee_\alpha S^1$ is

$r_\beta \in \ast_\alpha \mathbb{Z} \cong \pi_1(\bigvee_\alpha S^1)$

<u>Def.</u> X is topological n-manifold if it is locally homeomorphic to $\mathbb{R}^n$ (and Hausdorff)

<u>Prop</u> if $n \geq 4$ and G is finitely presented (i.e. set of α, β finite), then exists a compact topological n-manifold X_G^n with $\pi_1(X_G) \cong G$

Prop. only abelian groups G so that $\pi_1(X^3) \cong G$ are $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}^3$

(and many X^3 are determined by $\pi_1(X^3)$)

Covering spaces

Def. covering space of X is space $\tilde{X}$ with a map $p : \tilde{X} \to X$ such that

- for any $x \in X$, exists open nbd $U \ni x$ s.t. $p^{-1}(U)$ is disjoint union $\bigsqcup_\alpha U_\alpha \subset \tilde{X}$ so that $p|_{U_\alpha} : U_\alpha \xrightarrow{\sim} U$

Homotopy lifting property:

- given $p : \tilde{X} \to X$ covering space and homotopy $f_t : Y \to X$ and lift $\tilde{f}_0$ of f

(i.e.
$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y \arrow[ur, "\tilde{f}_0"] \arrow[r, "f_0"'] & X \end{tikzcd}$$
), then exists unique lift $\tilde{f}_t$ of f_t

Ex. $Y =$ point p, $f_t : Y \to X$ is a path and $\tilde{f}_t$ is lift of path (i.e. path-lifting property)

Cor. $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective

Pf. • $\tilde{f}_0, \tilde{f}_1$ in $\pi_1(\tilde{X})$ so that $p \circ \tilde{f}_0, p \circ \tilde{f}_1$ homotopic in X via f_t

• can lift f_t to $\tilde{f}_t$

• if $p : \tilde{X} \to X$ covering space, $|p^{-1}(x_0)|$ ← fiber cardinality is locally constant (constant if X connected)

$\Longrightarrow$ $|p^{-1}(x)|$ is number of sheets

Prop. number of sheets of $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$

i.e. $|\pi_1(X, x_0)| \,/\, |p_* \pi_1(\tilde{X}, \tilde{x}_0)|$

Pf.

$$\text{cosets of } p_*(\pi_1(\tilde{X}, x_0)) \longleftrightarrow p^{-1}(x_0)$$

$$g\gamma \in g\, p_*(\pi_1(\tilde{X}, x_0)) \longrightarrow \tilde{g}\tilde{\gamma}(1)$$

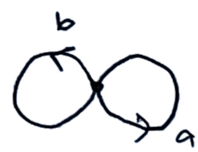
$$p_*(\text{path bet } \tilde{x}_0, \tilde{x}_0') \longleftarrow \tilde{x}_0'$$

Ex. • $\mathbb{R} \xrightarrow{} S^1$

; $\pi_1(\mathbb{R}) \longrightarrow \pi_1(S^1)$, $\pi_1(\mathbb{R}) = \{0\}$

$$\Longrightarrow |\pi_1(S^1)| = |\pi_1(S^1)| / |\pi_1(\mathbb{R})| = |p^{-1}(0)| = \mathbb{Z}$$

Ex. covering spaces of $S^1 \vee S^1$



$\pi_1(\tilde{X}) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

$\downarrow$ $a,\ b^2,\ bab^{-1}$

$\pi_1(X) = \mathbb{Z} * \mathbb{Z}$

Image $= \langle a, b^2, bab^{-1} \rangle \subset \langle a, b \rangle$