Lecture 7: 02/11/20

Last time: classification of covering space

Thm: there is a 1-to-1 correspondence between based covering space of X path-connected, locally path-connected, semi-locally simply-connected and subgroups of $\pi_1(X, x_0)$

$$p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \longrightarrow p_* \pi_1(\tilde{X}, x_0) \subset \pi_1(X, x_0)$$

- did uniqueness of $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ last time

Pf. of existence:

- given $G \subset \pi_1(X, x_0)$ want $X_G \xrightarrow{p} X$ so $p_* \pi_1(X_G) = G$

  focus on $G = \mathrm{Id} \subset \pi_1(X)$

Ex.

- let $\tilde{X} = \{$ homotopy classes of paths γ with $\gamma(0) = x_0$ $\}$ (fixing endpoints)

Prop $p : \tilde{X} \longrightarrow X$ is a covering space

$$p(\gamma) := \gamma(1)$$

- need locally path-connected for surjectivity of p
- need semilocally simply-connected for (local) injectivity

Prop. $\pi_1(\tilde{X}) = 0$ $(\Leftrightarrow p_* \pi_1(\tilde{X}) = 0)$

Pf. • take $\gamma \in p_* \pi_1(\tilde{X})$

• let $\gamma_t = \gamma|_{[0,t]}$ so $\gamma_t \in \tilde{X}$ for each t

and get path $s(t) = \gamma_t$ in $\tilde{X}$

lifting loop γ of X

• since $\gamma \in p_* \pi_1(\tilde{X})$, $s(t)$ is a loop in $\tilde{X}$

i.e. $s(1) = x_0$; so $\gamma = \gamma_1 = s(1) = x_0$

$\Rightarrow \gamma = 0$ □

• for general $H \subset \pi_1(X, x_0)$, let

$$X_H = \left\{ \begin{array}{l} \text{homotopy classes of paths } \gamma \text{ in } X \\ \text{with } \gamma(0) = x_0 \end{array} \right\} \Big/ \sim$$

• $\gamma \sim \gamma'$ if $\gamma(1) = \gamma'(1)$

and $[\gamma(\gamma')^{-1}] \in H \subset \pi_1(X, x_0)$,

Prop. $\tilde{X} \longrightarrow X_H \longrightarrow X$ all covering spaces

and $p_* \pi_1(X_H) = H$

Pf. • if $\gamma \in p_* \pi_1(X_H)$, again construct

lift $s(t)$ and since it is a loop in X_H,

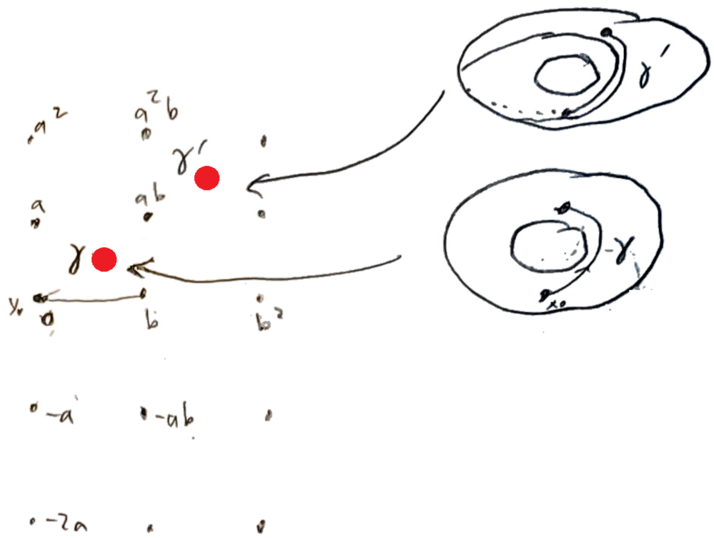
must have $\gamma_1 = x_0$, i.e. $\gamma \in H \Rightarrow p_* \pi_1(X_H) \subset H$

Rmk (next time) $H \subset \tilde{X}$ and $X_H = \tilde{X}/H$

Ex. $X =$  $S^1 \times S^1$

first find $\tilde{X} \to X$, i.e. $H = Id \subset \pi_1(X)$

- what are paths γ in X?
- if $\gamma(1) = x_0$, get $\pi_1(X) = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a,b] \rangle$



- now fill in the rest by paths not ending on x_0

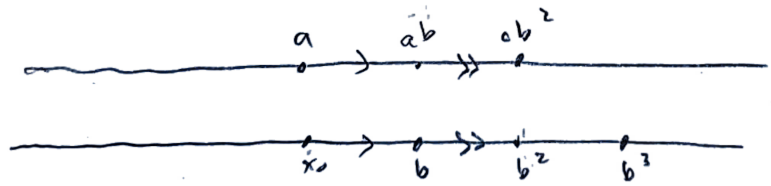
$\therefore \tilde{X} = \mathbb{R}^2$

- has projection $p: \mathbb{R}^2 \to T^2$ by identifying sides

- if $H = \langle a \rangle \subset \pi_1(X)$, then

$$\gamma \sim \gamma' \quad \text{if} \quad \gamma \cdot (\gamma')^{-1} \in \langle a \rangle$$

$$\mathbb{R}^2 / (x,y) \sim (x, y+1) \quad \left(= \mathbb{R}^2/\mathbb{Z} \right)$$

$=$  $= S^1 \times \mathbb{R}$

$\downarrow$

$S^1 \times S^1$

Cayley Complex

Recall: for any group G, exists X_G with $\pi_1(X_G) \cong G$

$$G = \langle g_\alpha \mid r_\beta \rangle \rightsquigarrow X_G = \bigvee_\alpha S^1 \sqcup D^2_\beta \Big/ \underset{x \in \partial D_\beta}{x \sim \phi_\beta(x)}$$

$$\phi_\beta : \partial D^2_\beta = S^1 \to \bigvee_\alpha S^1$$

satisfies $[\phi_\beta] = r_\beta \in \pi_1(\vee S^1) = \ast_\alpha \mathbb{Z}$

Ex. $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a,b] \rangle$

$S^1 \times S^1 =$ 

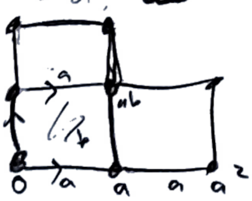
· now construct universal cover $\tilde{X}_G \to X_G$

1) one vertex g for each $g \in \pi_1(X_G) = G$

2) at each vertex g, add edge to $g_\alpha g$ for each generator $g_\alpha g$

$\Rightarrow$ result called Cayley ~~graph~~

Ex. for $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid [a,b] \rangle$



$\leftarrow$ 2 generators, so 4 edges at each vertex (one for inverse)

3) for each relation r_β, get a loop in the graph and attach 2-cell to that loop (in terms paths)

- $\widetilde{X}_G \to X_G$ by

all vertices $\to x_0$

edges $\to S^1$ in $\bigvee S^1$

disks $\to D^2_B$ in X_G

$G \circlearrowright \widetilde{X}_G$ by $g \cdot g' \to$ vertex gg'

and $X_G = \widetilde{X}_G / G$