Lecture 8 02/13/20

Last time:

correspondence between subgroups of $\pi_1(X, x_0)$ and covering spaces of (X, x_0)

Today:

symmetries of covering spaces
(already discussed isomorphisms of covering spaces)

- $p: \tilde{X} \to X$ a covering space

Def. deck transformations ("symmetries") are isomorphisms of $(\tilde{X}, p)$
ie. $f: \tilde{X} \to \tilde{X}$ so that
$$\begin{tikzcd} \tilde{X} \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[dl] \\ & X & \end{tikzcd}$$
commutes

Rmk 1) here $\tilde{X}$ is fixed, not $\tilde{X} \xrightarrow{\sim} \tilde{X}'$ for some other $\tilde{X}'$
2) no basepoint in $\tilde{X}$ fixed

let $G(\tilde{X})$ = set of deck transformations of $(\tilde{X}, p)$

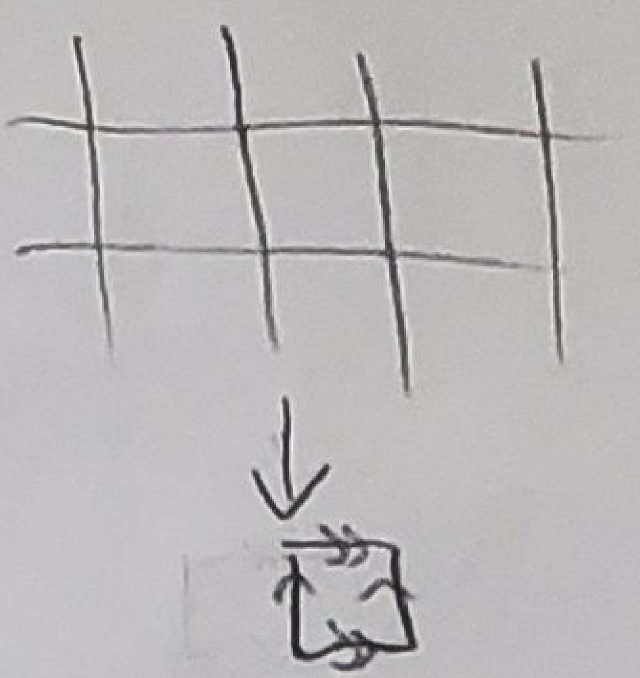
Prop $G(\tilde{X})$ is a group

Pf. if
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f", "\sim"'] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$
and
$$\begin{tikzcd} \tilde{X} \arrow[rr, "h", "\sim"'] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$
, then
$$\begin{tikzcd} \tilde{X} \arrow[rr, "h \circ f", "\sim"'] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

- inverses exist since deck transformations isomorphism
- identity map is identity element

General principle: if you have a space X with some structure (e.g. covering space structure, geometric, smooth str etc) then isomorphisms of X preserving structure form a group

Ex. $\mathbb{R}^2 \xrightarrow{p} T^2$



and $G(\mathbb{R}^2, p) \cong \mathbb{Z}^2$ generated by
$\psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\psi(x,y) = (x, y+1)$
$\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $\phi(x,y) = (x+1, y)$

Recall lifting property: $(Y, y_0) \xrightarrow{f} (X, x_0)$ has a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$, with $p \circ \tilde{f} = f : (Y, y_0) \to (X, x_0)$

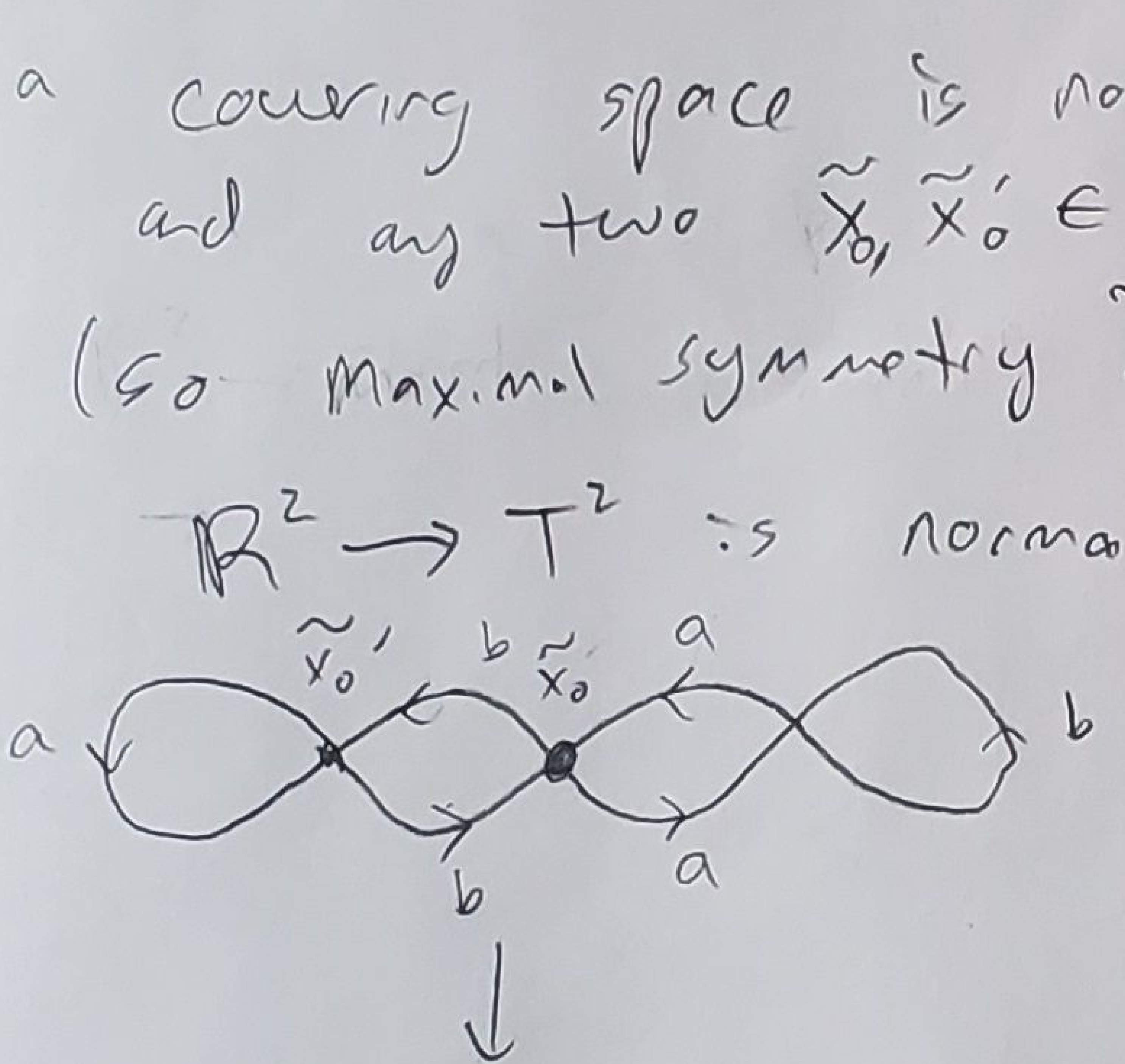
iff $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$. if lift exists, it is unique,

$\Rightarrow$ $f \in G(\tilde{X})$ determined by $f(\tilde{x}_0) \in p^{-1}(x_0)$

Def. a covering space is normal if for any $x_0 \in X$ and any two $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$, exists $g \in G(\tilde{X})$ so $g(\tilde{x}_0) = \tilde{x}_0'$
(so maximal symmetry)

Ex. $\mathbb{R}^2 \to T^2$ is normal

Ex.



$\tilde{X}$ is not normal

$\downarrow$

$S^1 \vee S^1$

- no deck transformation taking $\tilde{x}_0$ to $\tilde{x}_0'$ since $\tilde{x}_0$ has 2 circles on each side, $\tilde{x}_0'$ has 1, 3 circles
- formally $\pi_1(\tilde{X} \setminus \tilde{x}_0) \neq \pi_1(\tilde{X} \setminus \tilde{x}_0')$

(in fact, no homeomorphism ϕ with $\phi(\tilde{x}_0) = \tilde{x}_0'$)
(but there is a map ϕ <u>homotopic</u> to Id with $\phi(\tilde{x}_0) = \tilde{x}_0'$)

- Let $H \subset \pi_1(X, x_0)$ subgroup and $(X_H, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ corresponding covering space, ie. $p_* \pi_1(X_H) = H \subset \pi_1(X, x_0)$

Prop. 1) $X_H \to X$ is normal covering space iff $H \subset \pi_1(X, x_0)$ is a normal subgroup, ie. $gHg^{-1} = H$ for all $g \in \pi_1(X, x_0)$

2) if X_H is normal, $G(X_H) \cong \pi_1(X, x_0)/H$

Pf. (1) only if

- let $g \in \pi_1(X, x_0)$, ie. $g : [0,1] \to (X, x_0)$, and let $\tilde{g} : [0,1] \to \tilde{X}$ be a lift with $\tilde{g}(0) = \tilde{x}_0$ (exists by path-lifting property) and $\tilde{g}(1) := \tilde{x}_0'$
- then $p_* \pi_1(X_H, \tilde{x}_0') = gHg^{-1}$ (changing basepoint changes subgroup by conjugation)
- since $X_H \to X$ normal, exists

$$\begin{tikzcd} (X_H, \tilde{x}_0') \arrow[rr, "\phi"] \arrow[dr, "p"'] & & (X_H, \tilde{x}_0) \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

so that $\phi(\tilde{x}_0) = \tilde{x}_0'$

$\Rightarrow gHg^{-1} = p_* \pi_1(X_H, \tilde{x}_0') \subset p_* \pi_1(X_H) = H$

$\Rightarrow gHg^{-1} = H$

if $gHg^{-1} = H \Rightarrow p_* \pi_1(X_H, \tilde{x}_0') \subset p_* \pi_1(X_H, \tilde{x}_0)$

so by lifting criterion, exists map

$$\begin{tikzcd} (X_H, \tilde{x}_0') \arrow[rr, "\phi_g"] \arrow[dr] & & (X_H, \tilde{x}_0) \arrow[dl] \\ & X & \end{tikzcd}$$

and isomorphism since inverse exists (use $p_* \pi_1(X_H, \tilde{x}_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0')$)

(2) there is a group homomorphism

$$\pi_1(X, x_0) \to G(X_H)$$
$$g \to \phi_g^{-1}$$

$\boxed{\begin{array}{l} \phi_g \text{ determined by } g \\ \text{since determined by} \\ \phi_g^{-1}(\tilde{x}_0) = \tilde{x}_0' = \tilde{g}(1) \\ \text{and } \tilde{g} \text{ determined by } g \end{array}}$

group homomorphism since

$$(\Phi_g^{-1} \circ \Phi_h)^{-1}(\tilde{x}_0) = \Phi_h^{-1} \circ \Phi_g^{-1}(\tilde{x}_0) = \Phi_h^{-1}(\tilde{g}(1))$$

$$\Phi_{g*h}^{-1}(\tilde{x}_0) = \widetilde{g*h}(1)$$

^ Φ_h is obtaind by lifting path corresponding to h

$\Rightarrow$ $(\Phi_g \circ \Phi_h)^{-1} = \Phi_{g*h}^{-1}$ by uniqueness of lifts

($\gamma * \gamma'$ in $\pi_1(X, x_0)$ is $\begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \gamma'(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$

$g * h$ in $G(X)$ is $h \circ g$)

<u>Surjectivity:</u>

- for any $\tilde{x}_0' \in p^{-1}(x_0)$, exists $g \in \pi_1(X, x_0)$ so that $\tilde{g}(1) = \tilde{x}_0'$ by path-lifting property

$\Rightarrow$ $\Phi_g^{-1}(\tilde{x}_0) = \tilde{g}(1) = \tilde{x}_0'$

- since arbitrary $\Phi \in G(\tilde{X})$ determined by $\Phi(\tilde{x}_0) \in p^{-1}(x_0)$ map is surjective

<u>Kernel:</u> • $\Phi_g^{-1} = \Phi_h^{-1} \in G(\tilde{X}) \Rightarrow \tilde{g}(1) = \Phi_g^{-1}(\tilde{x}_0) = \Phi_h^{-1}(\tilde{x}_0) = \tilde{h}(1)$

$\Rightarrow$ $\tilde{g} * \tilde{h}^{-1}$ form a loop in X_H i.e. $\tilde{g} * \tilde{h}^{-1} \in \pi_1(X_H, \tilde{x}_0)$

$\Rightarrow$ $g * h^{-1} = p_*(\tilde{g} * \tilde{h}^{-1}) \in p_* \pi_1(X_H, \tilde{x}_0) = H$

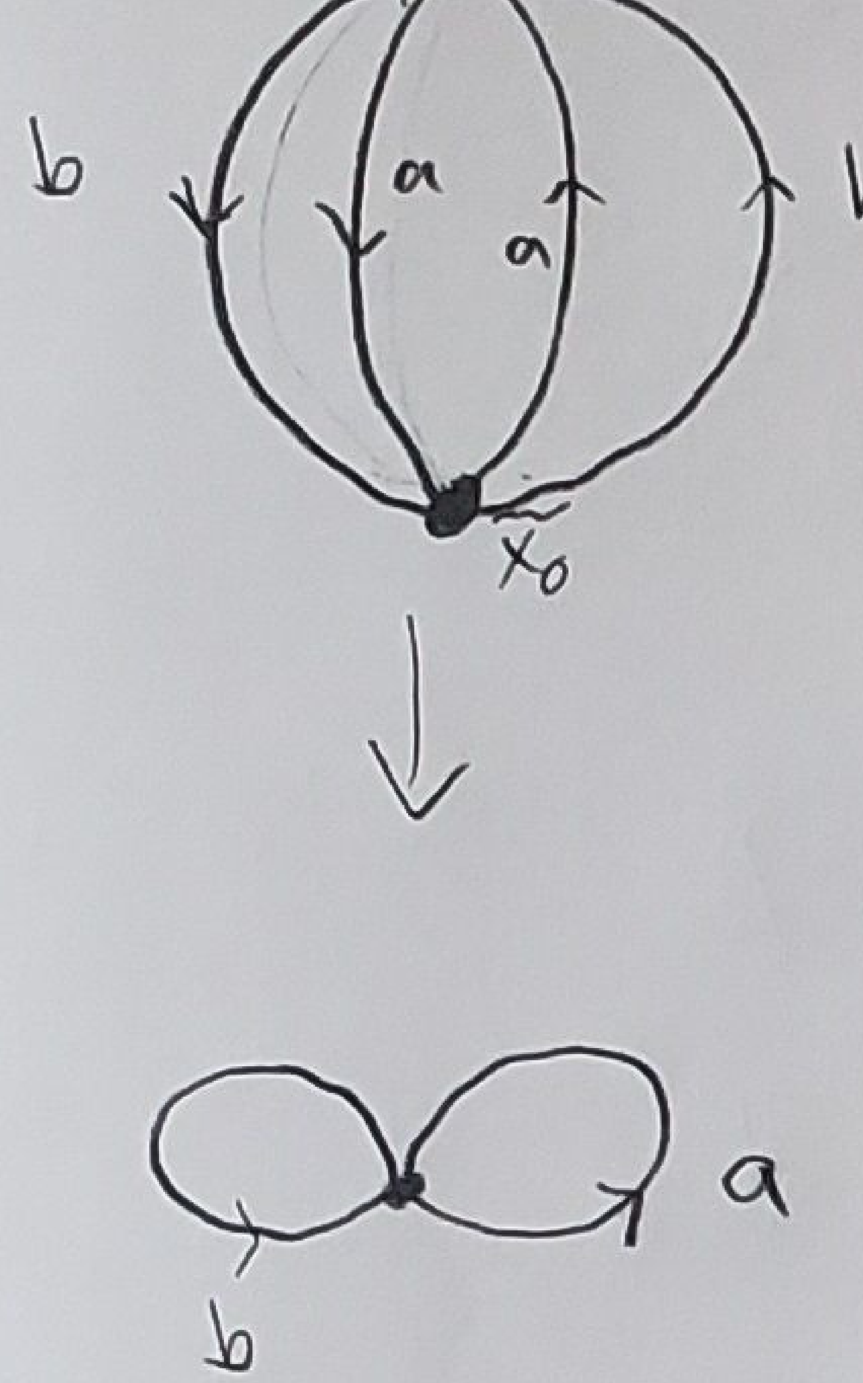
• Kernel $\subset H$

- also $H = p_* \pi_1(X_H, \tilde{x}_0) \subset$ kernel since $\tilde{h}(1) = \tilde{x}_0$ (5)

so $\Phi_h^{-1}(\tilde{x}_0) = \tilde{x}_0 \Rightarrow \Phi_h^{-1} = Id$ on X_H

- by first isomorphism theorem, $\pi_1(X, x_0)/H \cong G(X_H)$

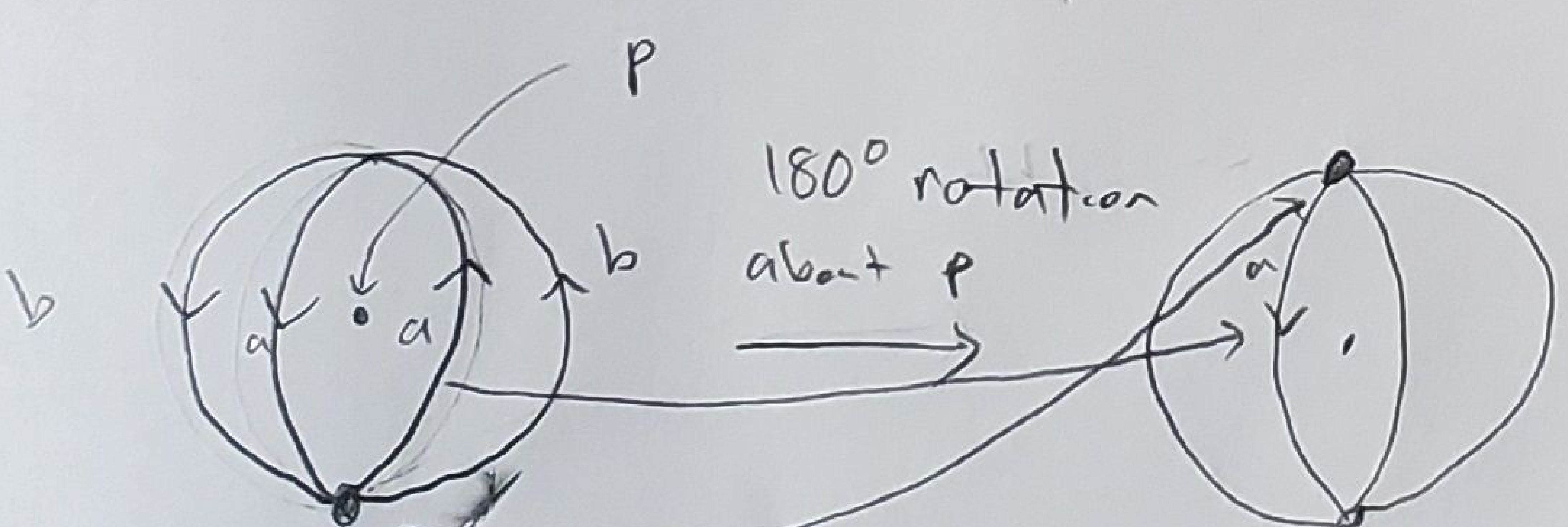
Ex.



is normal cover $X \xrightarrow{p} S^1 \vee S^1$

- $p_* \pi_1(X, \tilde{x}_0) = \langle a^2, b^2, ab \rangle \subset \pi_1(S^1 \vee S^1) = \langle a, b \rangle$
- $\langle a, b \rangle / \langle a^2, b^2, ab \rangle \cong \mathbb{Z}/2$ generated by $[a] = -[b]$

so $G(X) \cong \mathbb{Z}/2$ obtained by lifting a



$G(X) \cong \langle 180°$ rotation $\rangle$