Lecture 9

Last time:

- deck transformations $G(\tilde{X})$: isomorphisms
$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$
- normal covering space: for any $\tilde{x}_0, \tilde{y}_0' \in p^{-1}(x_0)$ exists $g \in G(\tilde{X})$ with $g(\tilde{x}_0) = \tilde{x}_0'$
- $H \subset \pi_1(X)$ normal $\leftrightarrow$ $X_H \to X$ normal covering space

Prop. if $X_H \to X$ is normal, $G(\tilde{X}) \cong \pi_1(X) / \underbrace{p_* \pi_1(X_H)}_{= H}$

Today: what spaces can Y cover? group actions

Def.
- a group G action on a set S (write $G \circlearrowright S$) is a map
$$G \times S \to S$$
$$(g, s) \to g(s)$$
and $g_1(g_2(s)) = (g_1 g_2)(s)$, $Id(s) = s$

so get group homomorphism $G \to Aut(S)$
- if Y is a space, $g : Y \to Y$ homeomorphism

and so $G \to Homeo(Y)$

Ex. group action of $G(\tilde{X})$ on $\tilde{X}$, i.e. group homomorphism $G(\tilde{X}) \to Homeo(\tilde{X})$

• if $g \in G(\tilde{X})$, then $g : p^{-1}(x) \to p^{-1}(x)$, action on $p^{-1}(x)$, so $G(\tilde{X}) \to \mathrm{Aut}(p^{-1}(x)) = S_n$ if $|p^{-1}(x)| = n$

permutations of set $\{1, \dots, n\}$

HW: relate this description with previous description?

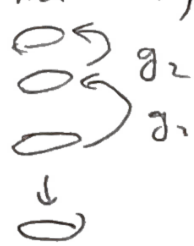
Ex.  $\mathbb{Z} = \pi_1(S^1) \to S_3 = \mathrm{Aut}(\{1,2,3\})$

• now study converse:

• given $G \circlearrowright Y$, can form $Y/G := Y/_{y \sim g(y) \text{ for all } g \in G}$ orbit space

Ex. if S is a group, $G \subset S$ subgroup, then $G \circlearrowright S$ by multiplication and S/G are cosets (quotient group if $G \subset S$ normal)

Ex. $G(\tilde{X}) \circlearrowright \tilde{X}$ and $\tilde{X}/G(\tilde{X}) \cong X$

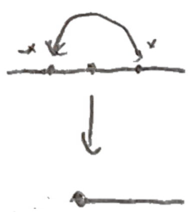


quotient map $S \xrightarrow{p} S/G$

$s \to [s]$

Q. when is this map a covering space?

Ex. $\mathbb{Z}/2 \circlearrowright \mathbb{R}$ by $\phi(x) = -x$ $(\phi^2 = Id)$



$\longleftrightarrow p^{-1}(u)$

$\to u$

$\Big\}$ not homeomorphic

Ex. $\mathbb{Z} \circlearrowright \mathbb{R}$, $\phi(x) = x+1$

then $\begin{array}{c}\mathbb{R}\\ \downarrow \\ S^1\end{array}$  covering space

<u>Def.</u> $G \circlearrowright Y$ is a <u>covering space action</u> 

if ~~every~~ $y \in Y$ has a neighborhood $U \ni y$

so that $g(U) \cap U = \emptyset$ for all $g \neq Id \in G$

$\Longrightarrow$ no fixed points (if G finite, $\Longleftarrow$)

<u>Prop.</u> if $G \circlearrowright Y$ covering space action, then

1) $\begin{array}{ccc} Y & \xrightarrow{p} & Y/G \\ y & \longrightarrow & [y] \end{array}$ is a <u>normal</u> covering space

2) deck transformations $G(\tilde{Y}) \cong G$

if Y is path-connected

($\Longrightarrow \pi_1(Y/G)/p_*\pi_1(Y) \cong G$)

<u>Pf.</u> • $Y \longrightarrow Y/G$ maps $\bigsqcup_{g \in G} g(U) \xrightarrow{p} U$

and $g(U) \xrightarrow{\sim} U$

• normal: $p^{-1}(x_0) = \bigsqcup_{g \in G} g(\tilde{x}_0)$ so all related by G-action

• $G \subseteq G(\tilde{Y})$ and $G(\tilde{Y}) \subseteq G$ since deck transformation ϕ determined by $\phi(\tilde{x}_0) \in p^{-1}(\tilde{x}_0)$ and exists g with $g(\tilde{x}_0) = \phi(\tilde{x}_0)$

- normal covering spaces $Y \xrightarrow{p} X$ for some X $\iff$ covering space actions $G \curvearrowright Y$ for some G (with $\pi_1(X)/p_*\pi_1(Y) \cong G$)

Ex. $\mathbb{Z}/2 \curvearrowright S^n$, $x \to -x$

$$S^n \xrightarrow{p} S^n/(\mathbb{Z}/2) = \mathbb{R}P^n \text{ covering space}$$

$$\Rightarrow \mathbb{Z}/2 \cong G(S^n \to \mathbb{R}P^n) \cong \pi_1(\mathbb{R}P^n)/p_*\cancel{\pi_1(S^n)}^{0} \cong \pi_1(\mathbb{R}P$$

- so way to compute $\pi_1(\mathbb{R}P^n)$

Q. for which G is there covering space action $G \curvearrowright S^n$ (i.e. if $X_{univ} \cong S^n$, what can $\pi_1(X)$ be?)

Thm. if n even, $G \cong \mathbb{Z}/2$ is only possibility

(Milnor). every abelian subgroup of G is cyclic $*$ and G has at most one element of order 2

(Madsen, Thomas, Wall) given a group G satisfying $*$, exists n so that $G \curvearrowright S^n$ is covering action

Ex. $\mathbb{Z}/m\mathbb{Z} \curvearrowright S^{2k-1} \subset \mathbb{R}^{2k} = \mathbb{C}^k$

by $\phi(v) = e^{2\pi i/m} v$, $\phi^m = Id$

- no fixed points and finite $\Rightarrow$ covering action
- $S^{2k-1}/\mathbb{Z}/m\mathbb{Z}$ called lens space, generalize $\mathbb{R}P^{n-1}$

- can get non-cyclic groups

<u>Ex.</u> $n=3$, S^3 is actually a group! (called Lie group)

<u>quaternion algebra</u> $\mathbb{H} = \mathbb{R}^4 = \mathbb{R}\langle 1, i, j, k \rangle$

$i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$

- $S^3 = \{ |a| = 1 \} \subset \mathbb{H}$, $|\;|$ Euclidean norm

and $|ab| = |a| \cdot |b|$, so $S^3 \subset \mathbb{H}$ is subgroup

- so any finite subgroup $G \subset S^3$

gives $G \curvearrowright S^3$ covering space action

quaternion <u>group</u> $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset S^3$

- $Q_{4m} = \langle \underset{a}{\underset{\parallel}{e^{2\pi i/m}}}, \underset{b}{\underset{\parallel}{j}} \rangle = \langle a, b \mid a^m = b^2 = -1, bab^{-1} = a^{-1} \rangle$