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# A moonshine path from $E_8$ to the monster

(previous title of preprint: A  $3C$ -path for Glauberman-Norton theory)

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## Abstract

One would like an explanation of the provocative McKay and Glauberman-Norton observations connecting the extended  $E_8$ -diagram with pairs of  $2A$  involutions in the Monster sporadic simple group. We propose a down-to-earth model for the  $3C$ -case which exhibits a logic to these connections.

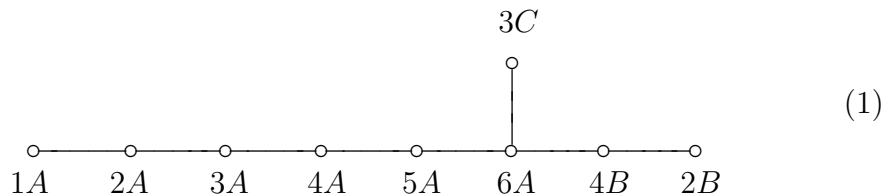
## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Compact Summary of Strategy . . . . .	4
1.2	Details on steps . . . . .	5
1.3	Notation and Terminology . . . . .	9

<b>2</b>	<b>McKay's <math>E_8</math> diagram and Leech lattice</b>	<b>12</b>
2.1	Lie algebra $sl_{n+1}(\mathbb{C})$ . . . . .	14
2.2	From $E_8$ to $A_2 \otimes E_8$ . . . . .	16
<b>3</b>	<b>Overlattices and gluing</b>	<b>20</b>
<b>4</b>	<b>The subgroup <math>2^2 \cdot O^+(8, 2)</math> of <math>2^2 \cdot O^+(8, 3)</math></b>	<b>22</b>
4.0.1	Creating double covers of $Sym_9$ in $W$ with triality . . .	22
<b>5</b>	<b>Some properties of <math>Q \perp R</math></b>	<b>25</b>
<b>6</b>	<b>Overlattices of <math>Q \perp R</math></b>	<b>28</b>
6.1	The new gluing map . . . . .	29
<b>A</b>	<b>Alternate proof that <math>2 \cdot Alt_9</math> occurs for a gluing</b>	<b>31</b>
<b>B</b>	<b>Automorphism group of <math>V_\Lambda^+</math></b>	<b>32</b>
<b>C</b>	<b>Niemeier lattices that contain <math>Q \perp R</math></b>	<b>34</b>
<b>D</b>	<b>Centralizers of pairs of <math>2A</math>-involutions for the <math>3C</math>-case</b>	<b>39</b>

# 1 Introduction

In 1979, John McKay [25] noticed a remarkable correspondence between  $\tilde{E}_8$ , the extended  $E_8$ -diagram, and pairs of  $2A$ -involutions in  $\mathbb{M}$ , the Monster (the largest sporadic finite simple group).



There are 9 conjugacy classes of such pairs  $(x, y)$ , and the orders of the 8 products  $|xy|$ , for  $x \neq y$ , are the coefficients of the highest root in the  $E_8$ -root system. Thus, the 9 nodes are labeled with 9 conjugacy classes of  $\mathbb{M}$ . There is no obvious reason why there should be such a correspondence involving high-level theories from different parts of the mathematical universe.

In 2001, George Glauberman and Simon Norton [10] enriched this theory by adding details about the centralizers in the Monster of such pairs of involutions and relations involving the associated modular forms. Let  $(x, y)$  be such a pair and let  $n(x, y)$  be its associated node. Let  $n'(x, y)$  be the subgraph of  $\tilde{E}_8$  which is supported at the set of nodes complementary to  $\{n(x, y)\}$ . If  $(x, y)$  is a pair of  $2A$  involutions and  $z$  is a  $2B$  involution which commutes with  $\langle x, y \rangle$ , Glauberman and Norton give a lot of detail about  $C(x, y, z)$ . In particular, they explained how  $C(x, y, z)$  has a “new” relation to the extended  $E_8$ -diagram, namely that  $C(x, y, z)/O_2(C(x, y, z))$  looks roughly like “half” of the Weyl group corresponding to the subdiagram  $n'(x, y)$ .

The important and provocative McKay-Glauberman-Norton observations seemed like looking across a great foggy space, from one high mountain top to another. We want to realize their connections in a manner which is more down-to-earth, like walking along a path, making natural steps with familiar mathematical objects. These objects are lattices, vertex operator algebras, Lie algebras, Lie groups and finite groups.

In this paper, we propose a specific path for the  $3C$ -case (i.e.,  $n'(x, y)$  is an  $A_8$ -diagram). The  $3C$ -case seems to be especially rich. Several Niemeier lattices are involved. They include  $E_8^3$  and the Leech lattice  $\Lambda$ . Triality for  $D_4$  plays a role. An explanation for occurrence of just “half” the Weyl group (of type  $A_8$ ) arises naturally. We hope to develop similar paths for other nodes.

## 1.1 Compact Summary of Strategy

This subsection contains a brief outline of how one may start with a node of the extended diagram  $\widetilde{E}_8$  and move to a pair of  $2A$ -involutions in the monster,  $\mathbb{M}$ .

For simplicity, we describe two paths, one beginning with a node of  $\widetilde{E}_8$  and the second one beginning with a pair of  $2A$ -involutions in  $\mathbb{M}$ . Each path ends with a subVOA generated by a pair of conformal vectors, for which theories on dihedral subVOAs give isomorphisms and enable us to splice the paths. Our Glauberman-Norton path consists of the path from  $\widetilde{E}_8$  followed by the reverse of the above path  $\mathbb{M}$ .

### Path starting in $\widetilde{E}_8$ :

node  $\rightarrow$

sublattice  $K$  of finite index in  $E_8 \rightarrow$

element  $r \in E_8(\mathbb{C})$  of order  $|E_8 : K|$ , defined by exponentiation  $\rightarrow$

$\text{cvcc}\frac{1}{2} e, f$  in  $V_{EE_8} \leq V_{E_8 \oplus E_8} \rightarrow$

conjugacy of  $r$  to an element  $h$  in torus normalizer  $N(\mathbb{T})$

so  $h$  acts on the root lattice without eigenvalue 1  $\rightarrow$

a pair of  $EE_8$  lattices  $M, M' < E_8^3$  and  $\text{cvcc}\frac{1}{2} e_M, e_{M'}$  such that  $\text{subVOA}\langle e_M, e_{M'} \rangle \cong \text{subVOA}\langle e, f \rangle \rightarrow$

Niemeyer lattice  $N$  with automorphism  $h''$  so that  $N^+(h'')$  and  $N_+(h'')$  are related to  $K$ ; find overlattice of  $N^+(h'') \perp N_+(h'')$  isometric to Leech lattice.

### Path starting in $\mathbb{M}$ :

distinct  $2A$ -involutions  $x, y \in \mathbb{M} \rightarrow$

$x, y$  correspond to unique  $\text{cvcc}\frac{1}{2} e', f'$  (Miyamoto bijection) in  $V^\natural$ ; we may replace  $x, y$  by conjugates to take  $e, f$  in  $V_\Lambda^+$

### At the endpoints of these two paths

Existing results on dihedral subalgebras of VOAs prove that  $\text{subVOA}\langle e', f' \rangle \cong \text{subVOA}\langle e, f \rangle$  if and only if  $n(x, y)$  is the node in the  $\widetilde{E}_8$  procedure [21, 24].

### Observation:

We use triality for  $D_4$  to find a Leech lattice as exceptional overlattice of  $N^+(h') \perp N_+(h')$ , resulting in visible loss of half the Weyl group, going from  $n(x, y)$  to  $x, y$  (and then on to  $C(x, y, z)$ ).

## 1.2 Details on steps

Our Glauberman-Norton path starting in  $\widetilde{E}_8$  involves several steps, which we preview here.

**Step I.** We show that the subdiagram  $n'(x, y)$  defines an automorphism  $r = r(x, y)$  of exponential type in  $Aut(V_{E_8})$ . Then we construct a pair of conformal vectors of central charge  $1/2$  (abbreviated as  $cvcc_{\frac{1}{2}}$ )  $e$  and  $f$  in  $V_{E_8}$ .

Let  $e \in V$  be a  $cvcc_{\frac{1}{2}}$ , i.e., the subVOA  $Vir(e)$  generated by  $e$  is isomorphic to  $L(1/2, 0)$ . It is well known that  $L(1/2, 0)$  is rational,  $C_2$ -cofinite and has three irreducible  $L(1/2, 0)$ ,  $L(1/2, 1/2)$  and  $L(1/2, 1/16)$  (cf. [8]).

Let  $V_e(h)$  be the sum of all irreducible  $Vir(e)$ -submodules of  $V$  isomorphic to  $L(1/2, h)$  for  $h = 0, 1/2, 1/16$ . Then one has an isotypical decomposition:

$$V = V_e(0) \oplus V_e(1/2) \oplus V_e(1/16).$$

Define a linear automorphism  $\tau_e$  on  $V$  by

$$\tau_e = \begin{cases} 1 & \text{on } V_e(0) \oplus V_e(1/2), \\ -1 & \text{on } V_e(1/16). \end{cases}$$

Miyamoto [23] showed that  $\tau_e$  defines an automorphism of the VOA  $V$ . This automorphism is often called *the Miyamoto involution* associated to  $e$ . It is also known that 2A-involutions of  $\mathbb{M}$  are in one-to-one correspondence with conformal vectors of central charge  $1/2$  in  $V^{\natural}$  through the construction of Miyamoto involutions [2, 23]. Thus, given a pair of 2A-involutions  $x, y$ , one can associate a pair of conformal vectors  $e', f' \in V^{\natural}$  of central charge  $1/2$  so that  $x$  is the Miyamoto involution for  $e'$  and  $y$  is the Miyamoto involution for  $f'$ . By using the above correspondence, one can show that the dihedral group  $\langle x, y \rangle$  is uniquely determined by the subVOA generated by  $e'$  and  $f'$  [2, 24, 21].

The diagram  $n'(x, y)$  defines an automorphism  $r(x, y)$  of  $V_{E_8}$  induced by a character  $\lambda$  of  $E_8$  with  $K = Ker(\lambda)$ , i.e.,

$$r(x, y)(u \otimes e^\alpha) = \xi^{\lambda(\alpha+K)} u \otimes e^\alpha \quad \text{for } u \in M(1), \alpha \in E_8,$$

where  $\xi$  is a primitive  $n$ -th root of unity,  $n = |E_8/K|$ ,  $K$  is the root lattice associated to the diagram  $n'(x, y)$  and  $M(1)$  is an irreducible  $\hat{\mathfrak{h}}$ -module. Here  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} E_8$  and  $\hat{\mathfrak{h}}$  is the affine Lie algebra of  $\mathfrak{h}$  (see Section 2 for details).

Let  $L = E_8 \oplus E_8$  and  $M = \{(x, x) \mid x \in E_8\}$ . Then  $M \cong EE_8$ . Let  $e = e_M$  be the cvcc $\frac{1}{2}$  defined in Notation 2.1 and  $f = (r(x, y) \otimes 1)(e)$ . Then both  $e$  and  $f$  are cvcc $\frac{1}{2}$ .

The key observation for this step is the following proposition.

**Proposition 1.1** (cf. [21, 24]). *The subVOA  $\langle e, f \rangle$  generated by  $e$  and  $f$  in  $V_{EE_8}$  is isomorphic to the subVOA in  $V^{\natural}$  generated by the conformal vectors associated to the 2A involutions  $x$  and  $y$ . Moreover, the centralizer of the dihedral group  $\langle \tau_e, \tau_f \rangle$  in  $\text{Aut}(V_{EE_8})$  is isomorphic to  $2^8 \cdot \text{Sym}_9$ , where  $\text{Sym}_9$  is the Weyl group of  $A_8$ .*

**Step II.** We explain that  $r(x, y)$  is conjugate in  $\text{Aut}(V_{E_8})$  to an automorphism  $\hat{h}(x, y)$  in a torus normalizer in  $\text{Aut}(V_{E_8}) \cong E_8(\mathbb{C})$  such that  $\hat{h}(x, y)$  induces a fixed point free isometry  $h$  on  $E_8$  by the natural action of the torus normalizer on the root lattice. We then derive a pair of  $EE_8$ -sublattices  $M$  and  $M'$  in  $E_8^3$  as follows.

Set  $\rho := r(x, y) \otimes 1 \otimes 1$  and  $\eta := \eta(x, y) := \hat{h}(x, y) \otimes 1 \otimes 1$ . We identify  $V_{E_8^3}$  with  $V_{E_8}^{\otimes 3}$ , so that  $\rho$  and  $\eta$  may be considered automorphisms of  $V_{E_8^3}$ . We also take the two  $EE_8$ -sublattices of  $E_8^3$ :

$$M = \{(a, a, 0) \mid a \in E_8\} \quad \text{and} \quad M' = \{(ha, a, 0) \mid a \in E_8\}.$$

The following are the main results of this step.

**Theorem 1.2.**  *$\rho$  is conjugate in  $\text{Aut}(V_{E_8^3})$  to  $\eta$  and  $\eta$  is in a torus normalizer.*

**Theorem 1.3.** *Let  $e_M$  and  $e_{M'}$  be cvcc $\frac{1}{2}$  supported at  $M$  and  $M'$ , respectively (cf. Notation 2.1). Then, the subVOA  $\langle e_M, e_{M'} \rangle$  generated by  $e_M$  and  $e_{M'}$  is isomorphic to  $\langle e, f \rangle$ .*

Therefore, we may transfer the study of the dihedral group  $\langle x, y \rangle < \mathbb{M}$  to the study of cvcc $\frac{1}{2}$   $e_M$  and  $e_{M'}$  in  $V_{\Lambda}^+ \subset V^{\natural}$ .

We trade  $\rho$  for  $\eta$  since  $\eta$  looks like a “permutation of roots” and gives a map on a lattice, so can be interpreted as a map on the VOA  $V_{\Lambda}^+$  associated with the Leech lattice  $\Lambda$ , whereas  $\rho$  is “exponential”, so cannot have a direct interpretation as an exponential on  $V_{\Lambda}^+$  (since this VOA has a finite automorphism group).

**Step III.** In this step, we shall take the pair  $x, y$  to a pair of Miyamoto involutions associated to conformal vectors  $e_M, e_{M'}$  of central charge  $\frac{1}{2}$  which lie in  $V_{\Lambda}^+$ .

We first determine the isometry type of  $Q := M + M'$  and show that  $Q$  can be embedded into the Leech lattice  $\Lambda$ . The main theorem is as follows.

**Theorem 1.4.** *The Leech lattice  $\Lambda$  contains a sublattice isometric to  $Q \cong A_2 \otimes E_8$  and hence  $U = \langle e, f \rangle$ , the subVOA generated by  $e$  and  $f$ , can be embedded into  $V_\Lambda^+$ . Moreover, the annihilator  $R := \text{ann}_\Lambda(Q)$  of  $Q$  in  $\Lambda$  is isometric to  $\sqrt{3}E_8$ .*

As a consequence, the subVOA generated by  $e_M$  and  $e_{M'}$  can be embedded into  $V_\Lambda^+$ . Recall that the moonshine VOA  $V^\natural$  is constructed by [9] as a  $\mathbb{Z}_2$ -orbifold of the Leech lattice VOA  $V_\Lambda$ , that means,

$$V^\natural = (V_\Lambda)^+ \oplus (V_\Lambda^T)^+, \quad (2)$$

where  $V_\Lambda^T$  is the unique irreducible  $\theta$ -twisted module for  $V_\Lambda$  and  $(V_\Lambda^T)^+$  is the fixed point subspace of  $\theta$  in  $V_\Lambda^T$ . Thus  $U = \langle e, f \rangle$  can also be embedded into the Moonshine VOA  $V^\natural$ . We shall note that  $\eta$  leaves the subVOA  $V_Q$  invariant. Thus, it induces an automorphism  $\eta_Q$  on  $V_Q$  by restriction. We also show that  $\eta_Q$  can be extended to an automorphism  $\eta_\Lambda$  in  $\text{Aut}(V_\Lambda)$ . Thus,  $\eta$  has a life on  $V_\Lambda$  and  $V_\Lambda^+$ . Since  $V_\Lambda^+ \cong (V^\natural)^z$  for a  $2B$  involution  $z \in \mathbb{M}$  and  $\text{Aut}(V_\Lambda^+) \cong C_{\mathbb{M}}(z)/\langle z \rangle \cong 2^{24} \cdot C_{O_1}$ , we can study the centralizer of  $\langle x, y, z \rangle$  in  $\mathbb{M}$  by using the configuration of  $M, M'$  and their sum  $Q$  in  $\Lambda$ . This leads us to study the overlattices of  $Q \perp R$  and the corresponding gluing maps. It turns out that the stabilizer of a gluing map is exactly the normalizer of  $\eta$  in the isometry group of the overlattice (5.4).

**Step IV.** Our analysis at the stage where we enlarge  $Q \perp R$  to  $\Lambda$  leads to an analysis of gluing maps. There exists one whose stabilizer is a subgroup  $\text{Sym}_3 \times 2 \cdot \text{Alt}_9$ . Our proof makes use of triality for groups of type  $D_4$ . Since the half-spin representations play a role, it is clear that we lose the ‘outer’ part of our subgroup of type  $\text{Sym}_9$ .

In this step, we first start with a gluing map  $\alpha : \mathcal{D}(Q) \rightarrow \mathcal{D}(R)$  such that the associated overlattice  $L_\alpha$  is isometric to  $E_8^3$ . We also construct a subgroup  $K_0$  of  $\text{Spin}^+(8, 3)$  so that  $K_0$  is a covering group of  $\text{Sym}_9$ ,  $K := K'_0 \cong 2 \cdot \text{Alt}_9$  and  $K_0/K \cong 2$ . The main idea is to choose such a  $K_0$  so that the action of  $K$  comes from a subgroup of  $O(Q) \times O(R)$ , but not so for  $K_0$ . We then twist  $\alpha$  by an element  $u \in K_0 \setminus K$  to get a new gluing map  $\beta =: u\alpha u^{-1}$ . The result is:

**Theorem 1.5.** *The associated overlattice  $L_\beta$  is even unimodular and rootless, so is isometric to the Leech lattice. Its stabilizer is a subgroup  $\text{Sym}_3 \times 2 \cdot \text{Alt}_9$  of the group  $O(Q \perp R) \cong \text{Sym}_3 \times O(E_8) \times O(E_8)$ .*

Thus we can, in a sense, witness loss of half the Weyl group of type  $A_8$  for the node  $n(x, y)$ . This is an explanation for one of the Glauberman-Norton observations.

Note that  $u$  gives a map from  $\mathcal{D}(Q \perp R)$  to itself. Hence, it induces a permutation on the set of all irreducible modules for  $V_{Q \perp R}$  since the irreducible modules for  $V_{Q \perp R}$  are parametrized by  $\mathcal{D}(Q \perp R)$  [4]. Therefore, the construction of  $L_\beta (\cong \Lambda)$  from  $L_\alpha$  can also be interpreted as an orbifold construction of  $V_{L_\beta}$  from  $V_{L_\alpha}$  using a subgroup  $A \cong 3^8$  of  $Aut(V_{L_\alpha})$  such that the fixed point subVOA  $(V_{L_\alpha})^A$  is isomorphic to  $V_{Q \perp R}$  (cf. [5, 6, 22]).

This ends the preview of our  $3C$ -path construction. It begins with one set of data (the extended  $E_8$  diagram) and ends with  $V_\Lambda$ . In the latter VOA, we find concrete realizations of the second set of data (dihedral groups generated by pairs of  $2A$ -involutions), namely pairs of conformal vectors of central charge  $\frac{1}{2}$  which represent all 9 types of these dihedral groups. The monster group does not act as automorphisms of this VOA, but rather does so on an orbifold of it, called  $V^\natural$ . Both  $V_\Lambda$  and  $V^\natural$  contain a subVOA  $V_\Lambda^+$ , where suitable pairs of conformal vectors may be found (so we felt no need to add details about  $V^\natural$  in this article). In [20], all  $cvcc_{\frac{1}{2}}$  in the VOA  $V_\Lambda^+$  were classified. There are two types of  $cvcc_{\frac{1}{2}}$ . The first type ( $AA_1$ -type) is associated to a norm 4 vector  $\alpha$  in  $\Lambda$  and denoted by  $\omega^\pm(\alpha)$  (cf. Notation B.6). The corresponding Miyamoto involution is defined by

$$\tau_{\omega^\pm(\alpha)}(u \otimes e^\beta) = (-1)^{\langle \alpha, \beta \rangle} u \otimes e^\beta \quad \text{for } u \in M(1), \beta \in \Lambda.$$

The second type ( $EE_8$ -type) is associated to an  $EE_8$ -sublattice  $M$  of  $\Lambda$ . The corresponding Miyamoto involution induces an isometry of  $\Lambda$ , which acts as  $-1$  on  $M$  and  $1$  on  $ann_\Lambda(M)$  (see Notation 2.1 and Appendix B). Our recent classification [17, 18] of configurations of  $EE_8$ -lattices is used to analyze relevant pairs of conformal vectors.

Building materials for our path come from several highly developed mathematical theories (Lie theory, lattices, vertex operator algebras, finite groups). More aspects of these theories could play roles in the future. We hope for a wide moonshine road, making the study of moonshine more concrete and enabling the transporting of ideas. In particular, this ought to illuminate connections between the extended  $E_8$ -diagram and the monster.

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### 1.3 Notation and Terminology

In this article, all group actions are assumed to be on the left. Our notation for the lattice vertex operator algebra

$$V_L = M(1) \otimes \mathbb{C}[L] \quad (3)$$

associated with a positive definite even lattice  $L$  is also standard [9]. In particular,  $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$  is an abelian Lie algebra and we extend the bilinear form to  $\mathfrak{h}$  by  $\mathbb{C}$ -linearity. Also,  $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k$  is the corresponding affine algebra and  $\mathbb{C}k$  is the 1-dimensional center of  $\hat{\mathfrak{h}}$ . The subspace  $M(1) = \mathbb{C}[\alpha(n) | \alpha \in \mathfrak{h}, n < 0]$ , where  $\alpha(n) = \alpha \otimes t^n$ , is the unique irreducible  $\hat{\mathfrak{h}}$ -module such that  $\alpha(n) \cdot 1 = 0$  for all  $\alpha \in \mathfrak{h}$  and  $n$  positive, and  $k = 1$ . Also,  $\mathbb{C}[L] = \{e^\beta | \beta \in L\}$  is the twisted group algebra of the additive group  $L$  such that  $e^\beta e^\alpha = (-1)^{\langle \alpha, \beta \rangle} e^\alpha e^\beta$  for any  $\alpha, \beta \in L$ . The vacuum vector  $\mathbf{1}$  of  $V_L$  is  $1 \otimes e^0$  and the Virasoro element  $\omega$  is  $\frac{1}{2} \sum_{i=1}^d \beta_i (-1)^2 \cdot \mathbf{1}$  where  $\{\beta_1, \dots, \beta_d\}$  is an orthonormal basis of  $\mathfrak{h}$ . For the explicit definition of the corresponding vertex operators, we shall refer to [9] for details.

#### Notation and Terminology

Notation	Explanation	Examples in text
$2A, 2B, 3A, \dots$	conjugacy classes of the Monster, the first number denotes the order of the elements and the second letter is arranged in descending order of the size of the centralizers	Equation (4)
$A_1, \dots, E_8$	root lattice for root system $\Phi_{A_1, \dots, E_8}$	Sec. 2
$AA_1, \dots, EE_8$	lattice isometric to $\sqrt{2}$ times the lattice $A_1, \dots, E_8$	Sec. 2
$AAA_1, \dots, EEE_8$	lattice isometric to $\sqrt{3}$ times the lattice $A_1, \dots, E_8$	Remark 2.20

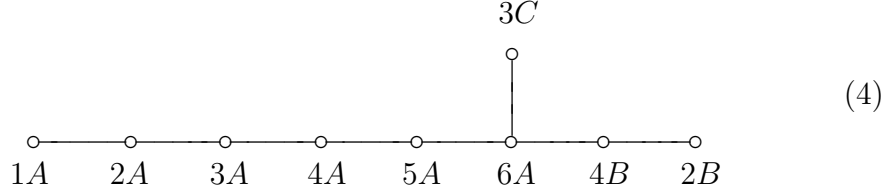
Notation	Explanation	Examples in text
$E_{i,j}$	a square matrix whose $(i, j)$ -th entry is 1 and all other entries are 0	Sec. 2.1, Equation (10)
$e_M$	a principal conformal vector of $V_M$ , i.e, $e_M = \frac{1}{16}\omega_M + \frac{1}{32}\sum_{\alpha \in M(4)} e^\alpha,$ where $M \cong EE_8$	Notation 2.1
$\eta$ or $\eta_{E_8^3}$	the automorphism $\tilde{h} \otimes 1 \otimes 1$ of $V_{E_8^3} \cong V_{E_8} \otimes V_{E_8} \otimes V_{E_8}$	Notation 2.11
$\eta_Q$	the restriction of $\eta$ to $V_Q \subset V_{E_8^3}$	Step III of Introduction
$h$	a fixed point free automorphism of $E_8$ of order 3	Notation 2.5
$\tilde{h}$	a lift of $h$ in $V_{E_8}$ , i.e, $\tilde{c}(M(1)) \subset M(1)$ , $\tilde{h}(e^x) = \epsilon_x e^{h(x)}$ , $\epsilon_x = \pm 1$	Equation (16)
$h_{A_n}$	a Coxeter element in $Weyl(A_n)$	Sec. 2.1, Equation (14)
$\tilde{h}_{A_n}$	a lift of $h_{A_n}$ in $Aut(sl_{n+1}(\mathbb{C}))$ (See Equation (10) for the precise definition)	Equation (10)
$K$ or $K_{nX}$	the lattice associated with the Dynkin subdiagram of $\hat{E}_8$ with the $nX$ -node removed	Equation (5)
$L^+(\theta), L_+(\theta)$	the fixed point sublattice of theta, its annihilator, resp.	Proposition 5.3
$L(k)$	the set of all norm $k$ vectors in $L$ , i.e., $L(k) = \{a \in L \mid \langle a, a \rangle = k\}$	Notation 2.1
$\mathbb{M}$	the Monster simple group	Compact Summary, Appendix D
$M(\phi)$	overlattice defined by gluing map $\phi$	Notation 6.6
Niemeyer lattice	a rank 24 even unimodular lattice	Introduction, Appendix C
$N(X)$	Niemeyer lattice whose root system has type $X$	Appendix C
$O(X)$	the isometry group of the quadratic space $X$	Remark 2.20, Lemma 5.3
$O(X, Y, \dots)$	$O(X) \cap O(Y) \dots$	

Notation	Explanation	Examples in text
$\varphi_x$	an automorphism of $V_{EE_8}$ defined by $\varphi_x(u \otimes e^\alpha) = (-1)^{\langle x, \alpha \rangle} u \otimes e^\alpha$ for $u \in M(1)$ and $\alpha \in EE_8$	Equation (8)
$Q$	$A_2 \otimes_{\mathbb{Z}} E_8$ , lattice isometric to the tensor product of $A_2$ and $E_8$	Notation 2.8
$R$	$EE_8$ , lattice isometric to $\sqrt{3}$ times the root lattice $E_8$	Notation 2.8, Remark 2.20
$r$ or $r(nX)$	an automorphism of $V_{E_8}$ induced by a character of $E_8/K_{nX}$	Notation 2.9
$r_M$ or $r_M(nX)$	an automorphism of $V_M$ , $M \cong EE_8$ induced by a character of $E_8/K_{nX}$	Equation (6)
$r_{A_n}$	an automorphism of $sl_{n+1}(\mathbb{C})$ defined by $r_{A_n}(E_{i,j}) = \omega^{i-j}(E_{i,j})$ , $\omega = e^{2\pi i/(n+1)}$	Equation (11), Equation (15)
$\rho$	the automorphism $r \otimes 1 \otimes 1$ of $V_{E_8^3} \cong V_{E_8} \otimes V_{E_8} \otimes V_{E_8}$	Notation 2.11
$s_{A_n}$	an automorphism of $sl_{n+1}(\mathbb{C})$ such that $r_{A_n} = s_{A_n} \tilde{h}_{A_n} s_{A_n}^{-1}$	Def. 2.4
$s$	$s = s_{A_2} \otimes s_{A_2} \otimes s_{A_2} \otimes s_{A_2}$ is an automorphism of $V_{E_8}$	Equation (17)
$\sigma$	$\sigma = s \otimes s \otimes s$ , an automorphism of $V_{E_8^3}$	Notation 2.13
$\tau_e$	the Miyamoto involution associated to a conformal vector $e$ , i.e., $\tau_e$ acts as $-1$ on $W_{1/16}$ and $1$ on $W_0 \oplus W_{1/2}$ , where $W_h$ is the sum of all irreducible $Vir(e)$ -submodules isomorphic to $L(1/2, h)$ , $h = 0, 1/2, 1/16$	Prop 1.1
$\theta$ or $\theta_L$	an involution of $V_L$ defined by $\theta(x_1(-n_1) \dots x_k(-n_k) \otimes e^x) =$ $(-1)^{k+\langle x, x \rangle/2} x_1(-n_1) \dots x_k(-n_k) \otimes e^{-x}$	Equation (18)
$V_L$	the lattice VOA associated with an even lattice $L$	Equation (3)
$Weyl(A_n), \dots,$ $Weyl(E_8)$	the Weyl group of the corresponding root system	Equation (14), Equation (17)

## 2 McKay's $E_8$ diagram and Leech lattice

We now set up notation for the  $3C$  case and establish our path.

Consider the McKay diagram.



By removing the node labeled  $3C$ , the remaining subdiagram is a Dynkin diagram of type  $A_8$ .

Let  $M \cong EE_8$  and  $\hat{M} = \{\pm e^\alpha \mid \alpha \in M\}$  a central extension of  $M$  by  $\pm 1$  such that  $e^\alpha e^\beta = \pm e^{\alpha+\beta}$  and  $e^\alpha e^\beta = (-1)^{\langle \alpha, \beta \rangle} e^\beta e^\alpha$ . Since  $M \cong EE_8$  is doubly even, we may arrange that  $\{e^\alpha \mid \alpha \in M\}$  forms a subgroup of  $\hat{M}$  [9].

Let  $K$  be a sublattice of  $M$  which is isometric to  $AA_8$ . Then  $|M/K| = 3$ , say

$$M = K \cup (\beta + K) \cup (-\beta + K), \text{ for some } \beta \in M \setminus K. \tag{5}$$

Then the lattice VOA  $V_M$  decomposes as

$$V_M = V_K \oplus V_{\beta+K} \oplus V_{-\beta+K}$$

and we can define an automorphism  $r_M$  of  $V_M$  by

$$r_M := \begin{cases} 1 & \text{on } V_K, \\ \xi & \text{on } V_{\beta+K}, \\ \xi^2 & \text{on } V_{-\beta+K}, \end{cases}$$

where  $\xi := e^{2\pi i/3}$ . Note that

$$r_M = \exp(2\pi i \gamma_0) \tag{6}$$

for some  $\gamma \in K^*$  (the subscript 0 refers the  $0^{th}$  operator associated to  $\gamma$  by the vertex operator). For example, if we identify

$$K = \{\sqrt{2}(a_0, a_1, \dots, a_8) \mid a_i \in \mathbb{Z}, \sum a_i = 0\}, \text{ and}$$

$$\beta = \frac{\sqrt{2}}{3}(1, 1, 1, 1, 1, 1, -2, -2, -2),$$

we may take  $\gamma = \frac{\sqrt{2}}{9}(1, 1, 1, 1, 1, 1, 1, 1, -8)$ .

**Notation 2.1.** *Let*

$$e_M = \frac{1}{16}\omega_M + \frac{1}{32} \sum_{\alpha \in M(4)} e^\alpha, \quad (7)$$

where  $\omega_M$  is the Virasoro element of  $V_M$  and  $M(4) = \{\alpha \in M \mid \langle \alpha, \alpha \rangle = 4\}$ .

It is shown in [7] that  $e_M$  is a simple conformal vector of central charge  $1/2$ .

Recall that

$$M^* = \{\alpha \in \mathbb{Q} \otimes_{\mathbb{Z}} M \mid \langle \alpha, \beta \rangle \in \mathbb{Z} \text{ for all } \beta \in M\} = \frac{1}{2}M.$$

For  $x \in M^*$ , define a  $\mathbb{Z}$ -linear map

$$\begin{aligned} \langle x, \cdot \rangle : M &\rightarrow \mathbb{Z}_2 \\ y &\mapsto \langle x, y \rangle \pmod{2}. \end{aligned}$$

Clearly the map

$$\begin{aligned} \varphi : M^* &\longrightarrow \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}_2) \\ x &\longmapsto \langle x, \cdot \rangle \end{aligned}$$

is a group homomorphism and  $\text{Ker} \varphi = 2M^* = M$ . Hence, we have

$$\text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}_2) \cong M^*/2M^* \cong \frac{1}{2}M/M.$$

For any  $x \in M^* = \frac{1}{2}M$ ,  $\langle x, \cdot \rangle$  induces an automorphism  $\varphi_x$  of  $V_M$  given by

$$\varphi_x(u \otimes e^\alpha) = (-1)^{\langle x, \alpha \rangle} u \otimes e^\alpha \quad \text{for } u \in M(1) \text{ and } \alpha \in M. \quad (8)$$

Note that

$$\varphi_x(e_M) = \frac{1}{16}\omega_M + \frac{1}{32} \sum_{\alpha \in M(4)} (-1)^{\langle x, \alpha \rangle} (e^\alpha + \theta(e^\alpha))$$

is also a simple conformal vectors of central charge  $1/2$ . Since  $\varphi_x$  commutes  $\theta$ ,  $\varphi_x(e_M)$  is also contained in  $V_M^+$ .

We call  $\varphi_x(e_M)$  a conformal vector of central charge  $1/2$  supported at  $M$ .

**Notation 2.2.** *Let*

$$e := e_M \quad \text{and} \quad f := r_M e_M \tag{9}$$

and let  $U := \langle e, f \rangle$  be the subVOA of  $V_M$  generated by  $e$  and  $f$ .

**Remark 2.3.** It was shown in [21] and [24] that the subVOA  $\langle e, f \rangle$  generated by  $e$  and  $f$  in  $V_{EE_8}$  is isomorphic to the subVOA in  $V^\natural$  generated by the  $\text{cvcc}\frac{1}{2}$  associated to the  $2A$  involutions  $x$  and  $y$ . Therefore, we can transfer the study of the dihedral group  $\langle x, y \rangle$  to the study of some subVOA of  $V^\natural$  isomorphic to  $\langle e, f \rangle$ .

Next we shall explain how to derive from  $e$  and  $f$  a pair of  $EE_8$ -sublattices in a suitable Niemeier lattice,  $N$ , such that their sum is isometric to  $Q = A_2 \otimes E_8$ . We shall also embed  $U$  into  $V_\Lambda^+$  and study the corresponding Miyamoto involutions in  $V_\Lambda^+$ ,  $V_\Lambda$ ,  $V^\natural$ , etc. We carry out this program for  $N = E_8^3$ , though it should be possible to do in any Niemeier lattice which contains a sublattice isometric to  $A_2 \otimes E_8$ . Such Niemeier lattices are classified in an appendix to this paper.

## 2.1 Lie algebra $sl_{n+1}(\mathbb{C})$ .

Let  $\mathcal{G} = sl_{n+1}(\mathbb{C})$  be the simple Lie algebra of type  $A_n$ . Let  $\epsilon_1, \dots, \epsilon_{n+1}$  be an orthonormal basis of  $\mathbb{R}^{n+1}$ . Then the root lattice system for  $\mathcal{G}$  can be identified with

$$\{\epsilon_i - \epsilon_j \mid 0 \leq i \neq j \leq n+1\}.$$

Let  $\mathcal{T}$  be the set of all diagonal matrices in  $sl_{n+1}(\mathbb{C})$  and denote by  $E_{i,j}$  the matrix whose  $(i, j)$ -th entry is 1 and all other entries are zero. Then  $\mathcal{T}$  is a Cartan subalgebra and the root space for the root  $(\epsilon_i - \epsilon_j), i \neq j$  is  $\text{span}\{E_{i,j}\}$ .

Next we shall define several automorphisms of  $sl_{n+1}(\mathbb{C})$ .

Let  $\omega = e^{2\pi i/(n+1)}$  and denote

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

and

$$B = \frac{1}{\sqrt{n+1}} [\omega^{ij}]_{1 \leq i, j \leq n+1} = \frac{1}{\sqrt{n+1}} \begin{pmatrix} \omega & \omega^2 & \cdots & \omega^{n-1} & \omega^n & 1 \\ \omega^2 & \omega^4 & \cdots & & \omega^{2n} & 1 \\ \vdots & & & \ddots & & \vdots \\ \omega^n & \omega^{n-1} & \cdots & \omega^2 & \omega & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{pmatrix}.$$

**Definition 2.4.** Define  $\tilde{h}_{A_n} : sl_{n+1}(\mathbb{C}) \rightarrow sl_{n+1}(\mathbb{C})$  and  $s_{A_n} : sl_{n+1}(\mathbb{C}) \rightarrow sl_{n+1}(\mathbb{C})$  by

$$\tilde{h}_{A_n}(A) = P^{-1}AP \quad \text{and} \quad s_{A_n}(A) = B^{-1}AB$$

for  $A \in sl_{n+1}(\mathbb{C})$ .

Then

$$\tilde{h}_{A_n}(E_{i,j}) = E_{i+1,j+1}, \quad (10)$$

where  $i, j$  are viewed as integers  $\text{mod}(n+1)$ .

Let  $\mathcal{C} = \mathcal{G}^{\tilde{h}_{A_n}}$ . Then  $\mathcal{C}$  is also a Cartan subalgebra of  $\mathcal{G}$ . Note that  $\dim(\mathcal{C}) = n$  and  $\mathcal{C} = \mathcal{G}^{\tilde{h}_{A_n}}$  is spanned by  $P, P^2, \dots, P^n$  and

$$B^{-1}PB = \text{diag}(\omega, \omega^2, \dots, \omega^n, 1).$$

Moreover, we have  $s_{A_n}(\mathcal{C}) = \mathcal{T}$  and

$$s_{A_n} \tilde{h}_{A_n} s_{A_n}^{-1}(E_{i,j}) = \omega^{j-i} E_{i,j}.$$

Let  $r_{A_n} := s_{A_n} \tilde{h}_{A_n} s_{A_n}^{-1}$ . Then

$$r_{A_n} = \exp\left(\frac{2\pi i}{n+1} \left(\frac{n}{2}, \frac{n}{2} - 1, \dots, -\frac{n}{2} + 1, -\frac{n}{2}\right)_0\right). \quad (11)$$

Define  $\theta : \mathcal{G} \rightarrow \mathcal{G}$  by

$$\theta(A) = -A^t, \quad \text{for } A \in \mathcal{G}. \quad (12)$$

By direct computation, we have

$$\theta s_{A_n} \theta s_{A_n}^{-1}(A) = (BB^t)A(BB^t)^{-1} \quad (13)$$

and

$$BB^t = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

Note that  $BB^t$  is symmetric and it is a permutation matrix of order 2.

Let  $h_{A_n} : A_n \rightarrow A_n$  be the  $\mathbb{Z}$ -linear map defined by

$$h_{A_n}(\epsilon_i - \epsilon_{i+1}) = \epsilon_{i+1} - \epsilon_{i+2}, \quad (14)$$

where  $i$  is again viewed as an integer  $mod(n+1)$ . Then  $h_{A_n}$  is an isometry of  $A_n$  and it also defines a Coxeter element in  $Weyl(A_n)$ .

Now identify  $(V_{A_n})_1$  with  $\mathcal{G}$  by identifying  $(\epsilon_i - \epsilon_j)(-1) \cdot \mathbf{1}$  with  $E_{i,i} - E_{j,j}$  and  $e^{\epsilon_i - \epsilon_j}$  with  $E_{i,j}$ . Then we have

$$\tilde{h}_{A_n}(e^\alpha) = e^{h_{A_n}\alpha}$$

by (10) and

$$\theta(e^\alpha) = -e^{-\alpha}$$

by (12), for any root  $\alpha \in A_n$ .

## 2.2 From $E_8$ to $A_2 \otimes E_8$

In this section, we shall describe how to derive a pair  $EE_8$ -sublattices  $M, M'$  in  $E_8^3$  such that the subVOA  $\langle e_M, e_{M'} \rangle$  generated by  $e_M$  and  $e_{M'}$  is isomorphic to  $U = \langle e, f \rangle$  (9).

Let  $L := E_8 \perp E_8$ . We first show that  $L$  contains a sublattice isometric to  $A_2 \otimes E_8$ .

**Notation 2.5.** Let  $h$  be a fixed point free automorphism of  $E_8$  of order 3.

Set  $M = \{(x, x) \in E_8 \perp E_8 \mid x \in E_8\}$  and  $M' = \{(hx, x) \mid x \in E_8\}$ . Then both  $M$  and  $M'$  are isometric to  $EE_8$ .

**Lemma 2.6.**  $M + M'$  is rootless.



*Proof.* Let  $(x + hy, x + y)$  be an element of  $M + M'$ .

If  $x + y = 0$ , then  $x + hy = (h - 1)y$  has norm  $\geq 6$ .

If  $x + hy = 0$ , then  $x = -hy$  and  $x + y = (1 - h)y$  has norm  $\geq 6$ .

If  $x + y \neq 0$  and  $x + hy \neq 0$ , then  $(x + hy, x + y) \geq 2 + 2 = 4$ .  $\square$

**Lemma 2.7.**  $M + M' \cong A_2 \otimes E_8$ .

*Proof.* Clearly  $M' = (h \oplus 1)(M)$  and  $(h \oplus 1)$  has order 3. Since  $h$  is fixed point free,  $M \cap M' = 0$ .

Since  $M + M'$  is rootless, by the  $EE_8$ -theory established in [18],  $M + M' \cong DIH_6(16) \cong A_2 \otimes E_8$ .  $\square$

**Notation 2.8.** Set  $Q := M + M' \cong A_2 \otimes E_8$  and  $R := \sqrt{3}E_8 = EEE_8$ .

Now let  $A_2 \perp A_2 \perp A_2 \perp A_2$  be a sublattice of  $E_8$ .

Set  $\gamma = (1, 0, -1) \in A_2$  and define

$$r_{A_2} = \exp\left(\frac{2\pi i}{3}\gamma_0\right) \quad (15)$$

**Notation 2.9.** Define  $r = r_{A_2} \otimes r_{A_2} \otimes r_{A_2} \otimes r_{A_2} = \exp\left(\frac{2\pi i}{3}\tilde{\gamma}_0\right)$  as an automorphism of  $V_{E_8}$ , where  $\tilde{\gamma} = (\gamma, \gamma, \gamma, \gamma) \in A_2^{\perp 4}$ .

**Lemma 2.10.**  $V_{E_8}^r = V_{A_8}$ .

*Proof.* Note that the sublattice

$$\{\alpha \in E_8 \mid (\alpha, \tilde{\gamma}) \equiv 0 \pmod{3}\}$$

is isometric  $A_8$ .  $\square$

Now by (11), we have

$$r_{A_2} = s_{A_2} \tilde{h}_{A_2} s_{A_2}^{-1},$$

where  $s_{A_2}$  and  $\tilde{h}_{A_2}$  are defined as before. Thus,  $r := r_{A_2} \otimes r_{A_2} \otimes r_{A_2} \otimes r_{A_2}$  is conjugate to

$$\tilde{h} := \tilde{h}_{A_2} \otimes \tilde{h}_{A_2} \otimes \tilde{h}_{A_2} \otimes \tilde{h}_{A_2} \quad (16)$$

in  $Aut(V_{E_8})$ . In fact,

$$r = s \tilde{h} s^{-1}, \quad (17)$$

where  $s := s_{A_2} \otimes s_{A_2} \otimes s_{A_2} \otimes s_{A_2}$ .

Recall that  $\tilde{h}_{A_2}$  induces an element  $h_{A_2} \in \text{Weyl}(A_2)$  (cf. (14)). Thus  $h := (h_{A_2}, h_{A_2}, h_{A_2}, h_{A_2})$  defines an isometry on  $E_8$  and it acts fixed point freely on  $E_8$ .

Fix  $h$  as above and embed

$$\begin{aligned} E_8 \perp E_8 &\longrightarrow E_8 \perp E_8 \perp E_8 \\ (\alpha, \beta) &\longmapsto (\alpha, \beta, 0) \end{aligned}$$

We shall choose a section of  $E_8^3$  in  $\hat{E}_8^3$  such that  $e^{(0,0,0)}$  is the identity element of  $\hat{E}_8^3$  and  $e^{(\alpha,\beta,\gamma)} = e^{(\alpha,0,0)} \cdot e^{(0,\beta,0)} \cdot e^{(0,0,\gamma)}$ , where  $\alpha, \beta, \gamma \in E_8$  [9, Chapter 5].

**Notation 2.11.** Define  $\rho := r \otimes 1 \otimes 1$  and  $\eta = \tilde{h} \otimes 1 \otimes 1$  as automorphisms of  $V_{E_8^3} \cong V_{E_8}^{\otimes 3}$ .

**Lemma 2.12.**  $\rho$  keeps  $V_M$  invariant and  $V_M^p \cong V_{AA_8}$ .

For any even lattice  $L$ , we define  $\theta : V_L \rightarrow V_L$  by

$$\begin{aligned} &\theta(\alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^\alpha) \\ &= (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes ((-1)^{\langle \alpha, \alpha \rangle / 2} e^\alpha) \end{aligned} \tag{18}$$

(cf. [9, 23]). Note that if  $L = A_n$  is a root lattice of type  $A_n$ , by identifying  $(V_L)_1$  with  $sl_{n+1}(\mathbb{C})$ ,  $(\epsilon_i - \epsilon_j)(-1)$  with  $E_{i,i} - E_{j,j}$  and  $e^{\epsilon_i - \epsilon_j}$  with  $E_{i,j}$ , we have

$$\theta|_{sl_{n+1}(\mathbb{C})}(A) = -A^t, \quad A \in sl_{n+1}(\mathbb{C})$$

Now let  $e := e_M$  be a conformal vector in  $V_M$  as defined in (7) and define  $f := \rho e$ .

By the definition of  $\theta$ , it is clear that

$$\theta(e^{(\alpha,\alpha,0)}) = e^{-(\alpha,\alpha,0)} \quad \text{for all } \alpha \in E_8$$

and hence  $e$  is fixed by  $\theta$ .

By (13), we have

$$\theta s_{A_2} \theta s_{A_2}^{-1} = (BB^t)A(BB^t)^{-1},$$

where  $BB^t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  is a permutation matrix of order 2. Thus,  $\theta s \theta s^{-1}$

induces an isometry  $\mu := \overline{\theta s \theta s^{-1}}$  of  $E_8$ . This implies

$$\theta s \theta s^{-1}(M(1)) \subset M(1)$$

and

$$\theta s \theta s^{-1}(e^\alpha) = \epsilon(\alpha) e^{\mu\alpha}, \quad \text{for } \alpha \in E_8, \quad (19)$$

where  $\epsilon(\alpha) = \pm 1$ .

**Notation 2.13.** Define  $\sigma := s \otimes s \otimes s \in \text{Aut}(V_{E_8}^{\otimes 3})$ , considered as an automorphism of  $V_{E_8^3}$ .

**Lemma 2.14.**  $\theta \sigma \theta \sigma^{-1}(e^{(\alpha, \alpha, 0)}) = e^{(\mu\alpha, \mu\alpha, 0)}$  for any  $\alpha \in E_8$ .

*Proof.* By (19),

$$\theta \sigma \theta \sigma^{-1}(e^{(\alpha, \alpha, 0)}) = (\epsilon(\alpha) e^{(\mu\alpha, 0, 0)}) (\epsilon(\alpha) e^{(0, \mu\alpha, 0)}) = e^{(\mu\alpha, \mu\alpha, 0)}.$$

for any  $\alpha \in E_8$ .  $\square$

Hence, we have the following corollaries.

**Corollary 2.15.**  $\theta \sigma \theta \sigma^{-1}$  fixes  $e$ .

**Corollary 2.16.**  $\sigma \theta \sigma^{-1}$  fixes  $e$ .

*Proof.* First we note that  $\sigma \theta \sigma^{-1} = \theta(\theta \sigma \theta \sigma^{-1})$ . Since  $\theta$  and  $\theta \sigma \theta \sigma^{-1}$  both fix  $e$ , so does  $\sigma \theta \sigma^{-1}$ .  $\square$

**Lemma 2.17.**  $\sigma^{-1}e$  and  $\sigma^{-1}f = \sigma^{-1}\rho e$  are fixed by  $\theta$ .

*Proof.* Since  $e$  is fixed by  $\sigma \theta \sigma^{-1}$ , we have

$$\theta \sigma^{-1}e = \sigma^{-1}(\sigma \theta \sigma^{-1}(e)) = \sigma^{-1}e.$$

Moreover,

$$\begin{aligned} \theta \sigma^{-1}\rho e &= \theta \eta \sigma^{-1}e && (\text{since } \rho = \sigma \eta \sigma^{-1}) \\ &= \eta \theta \sigma^{-1}e && (\text{since } \theta \eta = \eta \theta) \\ &= \eta \sigma^{-1}e \\ &= \sigma^{-1}\rho e. \end{aligned}$$

Thus,  $\sigma^{-1}f$  is fixed by  $\theta$ .  $\square$

**Lemma 2.18.** Set  $e' = \sigma^{-1}e$  and  $f' = \sigma^{-1}f$ . Then  $e' \in V_M^+$  and  $f' \in V_{M'}^+$ , and hence  $e = \varphi_x e_M$  and  $f' = \varphi_y e_{M'}$  for some  $x \in \frac{1}{2}M$  and  $y \in \frac{1}{2}M'$ , where  $\varphi_x$  and  $\varphi_y$  are defined as in Notation 2.1.

*Proof.* Since  $\sigma^{-1}$  keeps  $V_M$  invariant, we have  $\sigma^{-1}e \in V_M$  and thus  $e' \in V_M^+$  as  $\sigma^{-1}e$  is fixed by  $\theta$ .

On the other hand,  $\eta$  maps  $V_M$  to  $V_{M'}$ . Therefore,

$$f' = \sigma^{-1}\rho e = \eta\sigma^{-1}e \in V_{M'}$$

and thus  $f' \in V_{M'}^+$ .

Next we note that  $\tau_e$  acts on  $(V_M)_1 = (M(1))_1$  as  $-1$ . Thus,  $\tau_{\sigma^{-1}e} = \sigma\tau_e\sigma^{-1}$  also acts as  $-1$  on  $(V_M)_1$ . Now by the classification of conformal vectors of central charge  $1/2$  in  $V_{E_8}^+$  (cf. [16, 20]), we have  $\sigma^{-1}e = \varphi_x e_M$  for some  $x \in E_8$ . Similarly, we have  $f' = \varphi_y e_{M'}$  for some  $y \in E_8$ .  $\square$

**Theorem 2.19.** *The Leech lattice  $\Lambda$  contains a sublattice isometric to  $A_2 \otimes E_8$  and hence  $U := \langle e, f \rangle$ , the subVOA generated by  $e$  and  $f$ , can be embedded into  $V_\Lambda^+$ .*

*Proof.* An explicit embedding of  $A_2 \otimes E_8$  into  $\Lambda$  can be found in Appendix of [18]. Thus,

$$U \cong \sigma^{-1}U \subset V_{A_2 \otimes E_8}^+ \subset V_\Lambda^+$$

as desired.  $\square$

**Remark 2.20.** One can also obtain an embedding of  $Q \cong A_2 \otimes E_8$  into  $\Lambda$  as follows: Let  $h \in O(\Lambda)$  such that  $h$  has order 3 and trace 0. The fixed point sublattice of  $h$  in  $\Lambda$  is isometric to  $R \cong \sqrt{3}E_8$  and the annihilator of  $R$  in  $\Lambda$  is

$$\text{ann}_\Lambda(R) \cong Q = A_2 \otimes E_8.$$

Recall that  $N_{O(\Lambda)}(h) \cong \text{Sym}_3 \times 2 \cdot \text{Alt}_9$  in this case [1].

**Remark 2.21.** Since  $\rho$  is conjugate to  $\eta$  in  $\text{Aut}(V_{E_8^3})$ , it is clear that the subVOA  $\langle e_M, \rho e_M \rangle \cong \langle e_M, \eta e_M \rangle$ . Note also that  $\eta e_M \in V_{M'}$  is a cvcc $_{\frac{1}{2}}$  supported at  $M'$ . Thus, we may study the properties of the dihedral group  $\langle \tau_e, \tau_f \rangle$  in  $\text{Aut}(V_\Lambda^+)$  or  $\text{Aut}(V^\natural)$  by examining the configuration  $(M, M')$  in  $\Lambda$ .

### 3 Overlattices and gluing

The goal is to discuss overlattices for  $Q \perp R$  which are isometric to  $\Lambda$ , the Leech lattice. We explain how  $Q \perp R$  is contained in a copy of  $E_8^3$  and  $\Lambda$

in such a way that the common stabilizer is a group  $2 \cdot Alt_9$  and triality of groups of type  $D_4$  is involved.

Our argument uses triality to prove existence of a Leech lattice and explain the occurrence of the group  $2 \cdot Alt_9$  as the stabilizer of a relevant gluing map. We shall give an easy proof that  $2 \cdot Alt_9$  occurs in a gluing based on existence of a Leech lattice in the appendix.

We discuss the following situation.

**Notation 3.1.** We fix an orthogonal direct sum of integral lattices,  $Q \perp R$ . Suppose that an index  $m$  is given and that we are to study the set  $\mathfrak{X} := \{L \mid Q \perp R \leq L \leq Q^* \perp R^*, |L : Q \perp R| = m, L \cap \mathbb{Q} \otimes Q = Q, L \cap \mathbb{Q} \otimes R = R\}$ . We wish to understand the orbits of  $O(Q) \times O(R)$  on  $\mathfrak{X}$ . Let  $\mathfrak{Y} := \{L \in \mathfrak{X} \mid L \text{ is integral}\}$ .

**Notation 3.2.** We define

$$\begin{aligned} \mathfrak{T} := \{ & (A, B, \psi) \mid A \text{ is a subgroup of order } m \text{ in } \mathcal{D}(Q), \\ & B \text{ is a subgroup of order } m \text{ in } \mathcal{D}(R), \\ & \psi \text{ is an isomorphism of } A \text{ to } B \}. \end{aligned}$$

**Proposition 3.3.** (i)  $\mathfrak{X}$  is in bijection with the set of triples  $\mathfrak{T}$ .

(ii)  $L$  is integral if and only if  $\{(a, \psi a) \mid a \in A\}$  is a totally singular subspace of the quadratic space  $\mathcal{D}(Q) \perp \mathcal{D}(R)$  with natural  $\mathbb{Q}/\mathbb{Z}$ -valued bilinear form.

(iii) The totally singular condition holds if and only if for all  $a \in A$ ,  $(a, a) + (\psi a, \psi a) = 0 \in \mathbb{Q}/\mathbb{Z}$ . In particular, there exists a scalar so that  $\psi$  is a scaled isometry.

Special case: the spaces  $\mathcal{D}(Q)$  and  $\mathcal{D}(R)$  have a scaled isometry, e.g.  $Q = A_2 \otimes E_8$  and  $R = EEE_8$ .

**Definition 3.4.** A group action is assumed to be on the left. Suppose that the group  $G$  acts on the set  $A$  and the group  $H$  acts on the set  $B$ . We have an action of  $G \times H$  on  $Maps(A, B)$  as follows. If  $f$  is a map, then  $(g, h) \cdot f$  is the map which takes  $a$  to  $h(f(g^{-1}a))$ .

**Definition 3.5.** A similitude is a linear map between quadratic spaces which is a scaled isometry. The set of self-similitudes of a quadratic space is a group which contains the orthogonal group as a normal subgroup.

Now let  $G_Q$  be the group of similitudes on  $\mathcal{D}(Q)$  and  $G_R$  the group of similitudes on  $\mathcal{D}(R)$ . Let  $Z = Q$  or  $R$ . For  $g$  in one of these groups  $G_Z$ , define  $\lambda(g)$  to be the scaling factor, i.e., the nonzero scalar such that  $\lambda(g) \cdot (x, y) = (gx, gy)$  for all  $x, y \in \mathcal{D}(Z)$ .

The above definition gives an action of  $G_Q \times G_R$  on  $\mathfrak{X}$ . The subgroup  $G_{Q,R} := \{(g, g') \in G_Q \times G_R \mid \lambda(g) = \lambda(g')\}$  is the stabilizer in  $G_Q \times G_R$  of the condition  $(a, a) + (\psi a, \psi a) = 0 \in \mathbb{Q}/\mathbb{Z}$  in (3.3)(ii) and of the set  $\mathfrak{Y}$ .

## 4 The subgroup $2^2 \cdot O^+(8, 2)$ of $2^2 \cdot O^+(8, 3)$

The structure of  $2 \cdot O^+(8, 2) \cong Weyl(E_8)$  is well known. It embeds in  $O^+(8, 3)$  as a subgroup generated by reflections. One gets such an embedding by taking the  $E_8$  lattice modulo 3 with the associated quadratic form.

The group  $O^+(8, 3)$  has the property that its second derived group has index 8, is a perfect central extension of  $\Omega^+(8, 3)$  and gives the quotient  $Dih_8$ .

Its order is therefore  $2^{15}3^{12}5 \cdot 7 \cdot 13$ . It contains  $Weyl(E_8)$  with index  $2 \cdot 3^7 13$ .

We need a few standard facts. For all  $q$ , the group  $\Omega^+(8, q)$  has a group of graph automorphisms isomorphic to  $Sym_3$ . This group acts faithfully on the Schur multiplier when this is isomorphic to  $2 \times 2$ , i.e., for  $q = 2$  and  $q$  odd.

**Lemma 4.1.** *Let  $F$  be a field of characteristic not 2 and  $n \geq 2$ . An involution in  $SO(n, F)$  lifts to an element of order 2 or 4 in  $Spin(n, F)$ . It lifts to an element of order 4 if and only if the multiplicity of  $-1$  in its spectrum on the natural  $n$ -dimensional module is  $2 \pmod{4}$ .*

**Proof.** This is a standard fact. A proof may be found in [13].  $\square$

We have  $X := Weyl(E_8)' / \{\pm 1\} \cong \Omega^+(8, 2)$ . There are three conjugacy classes of maximal parabolic subgroups with Levi factors of type  $A_3$ . Let  $P_i$  be representatives,  $i = 1, 2, 3$ . For each  $i$ ,  $P_i$  lifts in the covering group  $\widehat{X}$  to a group  $Q_i$  of the shape  $(2 \times 2_+^{1+6})GL(4, 2)$ . In a quotient of  $\widehat{X}$  by a group of order 2, two of these  $Q_i/Z$  are isomorphic to  $2_+^{1+6}GL(4, 2)$  and the other is isomorphic to  $2^7:GL(4, 2)$ .

### 4.0.1 Creating double covers of $Sym_9$ in $W$ with triality

**Proposition 4.2.** *Let  $X_1 < X_2$  be a containment of perfect groups isomorphic to  $2^2 \cdot \Omega^+(8, 2)$  and  $Spin^+(8, 3)$  respectively.*

There exists a subgroup  $\Sigma \cong \text{Sym}_3$  of  $\text{Aut}(X_2)$  which complements  $\text{Inn}(X_2)$  and such that  $\Sigma$  stabilizes  $X_1$ .

**Proof.** Let  $r$  be an element in  $\text{Aut}(X_2)$  corresponding to a reflection in a representation  $\rho$  of  $X_2$  on its natural quadratic space  $V := \mathbb{F}_3^8$ . We assume that  $r$  normalizes  $\rho(X_1)$  and so  $\langle \rho(X_1), r \rangle \cong \text{Weyl}(E_8)$ . We extend  $\rho$  to a representation of the semidirect product  $X_2 \langle r \rangle$ .

Let  $h$  be an automorphism of order a power of 3 which is outer and is inverted by  $r$  under conjugation.

We consider an arbitrary representation  $\sigma$  of  $X_1$  on the quadratic space  $V$  such that the kernel of  $\sigma$  has order 2.

It has the property that exactly one of the three conjugacy classes of maximal parabolic subgroups of  $X_1$  with Levi factor of type  $A_3$  acts by  $\rho$  as a monomial group  $2^7:\text{Alt}_8$  (we use the term *parabolic* for a subgroup of  $X_1$  if it contains  $Z(X_1)$  and maps modulo  $Z(X_1)$  to a parabolic of the group of Lie type  $X_1/Z(X_1)$ ). Let  $P$  be such a maximal parabolic.

Then,  $\sigma(P)$  can be conjugated by an element of  $\rho(X_2)$  to  $\rho(Q)$ , where  $Q$  is a parabolic subgroup of  $X_1$  such that  $\rho(Q)$  acts monomially with respect to some basis, say  $\mathcal{A}$  of  $V$ . We may assume that  $r$  is chosen to normalize  $Q$ . Our hypotheses imply that  $\rho(\langle Q, r \rangle)$  is a uniquely determined index 2 subgroup of the full orthogonal monomial group on  $\mathcal{A}$ .

The group  $\rho(\langle X_1, r \rangle)$  is generated by  $\rho(\langle Q, r \rangle)$  together with a product  $r_1 r_2$  of commuting reflections, one of which, say  $r_1$ , is a reflection at  $\pm b \pm b'$ , for some  $b, b' \in \mathcal{A}$ . The other reflection,  $r_2$ , may be taken as reflection at some element  $s$  of the quadratic space which has the property that for all  $b \in \mathcal{A}$ ,  $(s, b) \in \{-1, 1\}(\text{mod } 3)$ . It is clear that any two such  $s$  are in the same orbit under the monomial group on  $\mathcal{A}$ .

We apply above remarks to the composition  $\sigma = \rho h$ . It follows that there exists  $g \in X_2 \langle r \rangle$  so that  $\rho(g)\rho(h(X_1))\rho(g)^{-1} = \rho(X_1)$ . Let  $i_g \in \text{Aut}(X_2)$  be conjugation by  $g$ . It follows that  $i_g h \in \text{Aut}(X_2)$  takes  $X_1$  to itself and induces a group of order 3 on  $Z(X_1) = Z(X_2)$ . This proves the result since  $\langle \{i_k \mid k \in X_1\}, r, i_g h \rangle \cong \text{Aut}(X_2)$ .  $\square$

**Proposition 4.3.** *We use the notation of (4.2) and its proof. Let  $X$  be a subgroup of  $X_1$  so that  $\rho(X) \cong \text{Alt}_9$ . Let  $\alpha \in \Sigma$  so that  $\alpha$  does lie in the group  $\text{Inn}(X_2) \langle r \rangle$ . Then  $\rho(\alpha(X)) \cong 2 \cdot \text{Alt}_9$ , the covering group of  $\text{Alt}_9$ .*

**Proof.** The hypotheses on  $\alpha$  imply that  $\alpha$  does not stabilize the subgroup  $\text{Ker}(\rho) \cap X \cong 2$ . Therefore, the image of  $\alpha(X)$  in  $\rho(X_2)$  is isomorphic to  $2 \cdot \text{Alt}_9$  (4.1).  $\square$

**Notation 4.4.** Let  $L = E_8$  and  $\Phi$  the root system. Let  $\Phi_0$  be a sub root system of type  $A_8$ ,  $\Phi_0 \subset \Phi$ . Let  $W$  be the Weyl group of  $\Phi$  and let  $W_0$  be the Weyl group of  $\Phi_0$ . Then  $W_0 \cong Sym_9$  and its action on  $L/3L$  has constituents of dimensions 1 and 7. There are submodules of these dimensions and each is nonsingular.

It is straightforward to check the last two statements above with a standard model of the relevant root lattices.

**Notation 4.5.** Let  $q$  be the reflection at the nonsingular 1-dimensional module described in (4.4). We therefore have the subgroup  $\pi(W_0) \times \langle q \rangle \cong 2 \times Sym_9$  of  $O(\mathcal{D}(Q)) \times O(\mathcal{D}(R))$ . Its commutator subgroup is isomorphic to  $Alt_9$  and the commutator quotient is  $2 \times 2$ . The procedure of (4.2) and (4.3) gives a subgroup  $K_0$  of  $G_0 \cong Spin^+(8, 3)$  so that  $K := K'_0$  satisfies  $K \cong 2 \cdot Alt_9$  and  $K_0/K \cong 2$ . Thus,  $K_0$  is a covering group of  $Sym_9$  (there are two such covering groups, depending on whether a transposition is represented by an element of order 2 or 4).

**Lemma 4.6.** (i) *The group  $Z(W) \times W_0$  is maximal in  $W$ .*

(ii) *The group  $Z(W) \times W'_0$  is maximal in  $W'$ .*

**Proof.** (i) This follows from the classification of root systems.

(ii) Since  $Z(W) \times W'_0$  does not contain reflections, this is more difficult. By use of  $Aut(W/Z(W))$ , we see that the proof is equivalent to proving that  $K$  is maximal in  $W'$ , where  $K \cong 2 \cdot Alt_9$  is the group created in Proposition 4.3.

We let  $\alpha$  be a root and  $X := Stab_W(\alpha) \cong 2 \times Sp(6, 2)$ , a group of order  $2^{10}3^45 \cdot 7$ .

Suppose that there is a subgroup  $S$  so that  $K < S < W'$ . Define  $T := Stab_S(\alpha)$ ,  $T_0 := Stab_K(\alpha) \cong SL(2, 8):3$ . We have  $|S : K| = |T : T_0|$ . By (6.4),  $TZ(X) = T_0Z(X)$  or  $X$ . Since  $T_0$  and  $X$  are generated by their odd order elements and  $Z(X)$  is a 2-group,  $T = (T \cap Z(X)) \times (T \cap X')$ . The left factor has order 1 or 2 and the right factor is  $T_0$  or  $X'$ .

If  $T \cap X' = T_0$ , either  $K = S$ , which is impossible, or  $|S : K| = 2$ , which would mean that  $K$  is normal in  $S$ . But this would mean that  $W'_0$  is contained in  $N_W(W'_0)$  with index divisible by 4. This is clearly impossible since  $W_0$  is self-normalizing in  $W$ .

We conclude that  $T \cap X' = X'$ . This means that  $S$  has index 1 or 2 in  $W'$ , which is a perfect group. Therefore  $S = W'$ , a contradiction.  $\square$



**Notation 4.7.** We define the group  $H$  to be a natural  $2 \cdot \text{Alt}_8$  subgroup of  $K$  where  $K \cong 2 \cdot \text{Alt}_9$  is the group defined in Proposition 4.3.

**Lemma 4.8.** *The group  $H$  acts transitively on roots. A stabilizer has the form  $2^3:7:3$ . For the action of  $K$  on roots, a stabilizer has the form  $SL(2, 8):3$ .*

**Proof.** We start with the Barnes-Wall viewpoint for  $E_8 \cong BW_3$ . Consider a standard frame  $F$  of minimal vectors. In the BRW group,  $G$ ,  $\text{Stab}_G(F) \cong 2^{1+6}2^3 \cdot GL(3, 2)$  and for  $\alpha \in F$ ,  $J := \text{Stab}_G(\alpha)$  has the form  $2^3 \cdot 2^3 \cdot GL(3, 2)$ .

We may replace  $H$  by a conjugate to assume that its intersection with  $J$  contains a group of the form  $7:3$ . The intersection has order bounded below by  $8!/240 = 2^3 \cdot 3 \cdot 7$ . If the intersection were larger, it would have order of the form  $2^a 3 \cdot 7$ , for some  $a \geq 3$ . By Sylow 7-theory,  $a$  is divisible by 3, whence  $a = 6$ . Thus, the intersection would contain a maximal subgroup of a Sylow 2-group  $P$  of  $H$  which meets  $Z(H)$  trivially. This is impossible by group transfer theory (since  $Z(H) \leq H'$  implies  $Z(H) \leq P'$ ). It follows that the stabilizer order is exactly  $8!/240 = 2^3 \cdot 3 \cdot 7$ . Transitivity follows. Finally, we argue that a stabilizer,  $S$ , has the form  $2^3:7:3$ . Since  $S$  is contained in a group of the form  $2^3 \cdot 2^3 \cdot GL(3, 2)$ , if the statement is false,  $S \cong GL(3, 2)$ .

In the stabilizer of a root, there is up to conjugacy just one Sylow 7-subgroup and up to conjugacy just two subgroups of the form  $2^3:7$ , because the action of a group of order 7 on  $O_2(J)$  is completely reducible with two non-isomorphic (and dual) constituents. Each constituent has order  $2^3$ . One constituent is represented by  $O_2(G) \cap J$ . The second constituent is represented by a subgroup of the group  $A$  of permutation matrices in  $G$ , which is isomorphic to  $AGL(3, 2)$ . It is clear that  $O_2(A)$  fixes the root  $(\frac{1}{2}^8)$ , in the standard  $E_8$  notation. It follows that  $S$  has the form  $2^3:7:3$ , rather than  $GL(3, 2)$ .

Now let  $T$  be the stabilizer of  $\alpha$  in  $K$ . Then  $|Y : S| = 9$ . Thus,  $T$  is a triply transitive group of degree 9. By a classification [27],  $T \cong SL(2, 8):3$ .  $\square$

## 5 Some properties of $Q \perp R$

**Lemma 5.1.** *The minimal vectors in  $Y := A_2 \otimes E_8$  have norm 4 and are expressed as the union of the three sets  $\alpha \otimes \Psi$ , where  $\alpha$  runs over three pairwise nonproportional vectors of the  $A_2$ -factor and  $\Psi$  is the set of roots for the second factor.*

(i) These three sets are maximal sets of pairwise doubly even sets (i.e.  $(x, y) \in 2\mathbb{Z}$  for all  $x, y$  in the set) of minimal vectors;

(ii) A doubly even set of minimal vectors of cardinality at least 240 equals one of these sets. In particular, a doubly even set of minimal vectors which meets every coset of  $3Y$  in  $Y$  which contains a minimal vector is one of the above sets.

(iii) These sets are permuted by the isometry group of the lattice. We have  $O(Y) = U \times T$ , where  $T$  acts on each  $\text{span}_{\mathbb{Z}}(\alpha \otimes \Psi)$  as its full isometry group, isomorphic to  $O(E_8)$ , and where  $U \cong \text{Sym}_3$  permutes the three sets  $\alpha \otimes \Psi$ . We may take three nonproportional vectors  $\alpha_1, \alpha_2, \alpha_3$  whose sum is 0 and choose identifications  $U \cong \text{Sym}_3$  and  $Y = A_2 \otimes E_8$  so that the permutation  $p$  corresponds to the isometry  $p(\alpha_i \otimes x) = \alpha_{p(i)} \otimes x$ , for all  $x \in E_8$ .

**Proof.** (i) Let  $\Delta$  be the set of roots of the first factor.

Choose a single minimal vector, say  $\alpha \otimes \gamma$ . The set of norm 4 vectors which have even inner product with it is  $E := (\alpha \otimes \Psi) \cup (\Delta \otimes \gamma)$ . The set of elements of  $E$  which have even inner product with every element of  $E$  is just  $\alpha \otimes \Psi$  and any  $\beta \otimes \gamma$  has odd inner product with at least one member of  $\alpha \otimes \Psi$ . It follows that  $\alpha \otimes \Psi$  is a doubly even set, maximal under containment.

(ii) The second statement follows from the first, which we now prove.

Suppose  $S$  is a doubly even set of minimal vectors with  $|S| \geq 240$ . Let  $S$  be the union of sets  $\alpha \otimes P, \beta \otimes Q, \gamma \otimes R$ , where  $\alpha, \beta, \gamma$  are pairwise nonproportional vectors in  $\Delta$ . We want to prove that  $S$  is one of these. Suppose that this is not so. Then none of  $P, Q, R$  equals  $\Psi$  and at least one of them, say  $P$ , has cardinality at least  $240/3 = 80$ , which means that  $P$  represents at least 40 nonsingular cosets of  $E_8 \bmod 2$ . Therefore, the span of  $P + 2E_8$  has dimension  $d \geq 6$ . Since  $\beta \otimes Q$  has even inner product with  $\alpha \otimes P$ ,  $(P, Q) \leq 2\mathbb{Z}$ . Therefore  $Q$  represents nonsingular cosets in the annihilator space of the above span of  $P + 2E_8$ . This annihilator space has dimension  $8 - d$ , so  $|Q| \leq 6$ . Similarly,  $|R| \leq 6$ . If  $Q \neq \emptyset$ , then  $|P| \leq 126$  and so  $|P \cup Q \cup R| \leq 126 + 6 + 6 < 240$ , a contradiction to  $|S| \geq 240$ . We conclude that  $Q = R = \emptyset$ .

(iii) This follows from the characterization of (ii). The obvious map  $O(Y) \rightarrow \text{Sym}_3 \times O(E_8)$  is an isomorphism of groups.  $\square$

**Lemma 5.2.** *Suppose that  $A$  is a free abelian group and that  $n > 1$  so that the finite order automorphism  $g \neq 1$  acts trivially on  $A/nA$ . Then  $n = 2$ ,  $g$  has order 2 and  $A$  is the direct sum of  $A^+ := \{a \in A \mid ga = a\}$  and  $A^- := \{a \in A \mid ga = -a\}$ .*

**Proof.** Suppose that  $g$  has order  $p^a > 2$  for a prime number  $p$  and integer  $a \geq 1$ . There exists a direct summand  $B$  of  $A$  so that on  $B$ , the minimum polynomial of  $g$  is the cyclotomic polynomial  $\Phi_{p^a}$  of degree  $p^a - p^{a-1}$ . In the ring of integers,  $\mathbb{Z}[\sqrt[p^a]{1}]$ , if  $\pi$  is a primitive  $p^a$ -th root of 1, then  $\pi^{p^a - p^{a-1}}$  generates the ideal  $p\mathbb{Z}[\sqrt[p^a]{1}]$  [28].

It follows that if  $g$ , of arbitrary finite order greater than 1, acts trivially on  $B/nB$ , then  $n$  is a power of  $p$  and  $p^a - p^{a-1} = 1$ , whence  $p^a = 2$ .

We therefore may assume that  $g$  has order 2 and  $n = 2^f$ , for some  $f \geq 1$ . In this case  $g$  acts trivially on  $A/2A$ . If we prove that the decomposition  $A = A^+ \oplus A^-$  holds, then  $f \leq 1$  follows (since  $A^- \neq 0$ ). We may therefore assume that  $n = 2$ .

There exists an endomorphism  $E$  of  $A$  so that  $g = 1 + 2E$ . Then  $1 = g^2 = 1 + 4(E + E^2)$ , whence  $E(E + 1) = 0$  in  $\text{End}(A)$ . For  $a \in A$ , we have  $g(Ea) = (1 + 2E)Ea = (E + 2E^2)a = -Ea$  and  $g(E + 1)a = (1 + 2E)(E + 1)a = (2E^2 + 3E + 1)a = (E + 1)a$ , so  $a = (E + 1)a - Ea \in A^+ + A^-$ .  $\square$

**Lemma 5.3.** *Suppose that  $L$  is an overlattice of  $Q \perp R$  such that  $L \cap \mathbb{Q}Q = Q$ ,  $L \cap \mathbb{Q}R = R$  and  $L$  is stable under  $O_3(O(Q))$ . Write  $O(Q) = X \times Y$ , where  $X \cong \text{Sym}_3$  and  $Y \cong O(E_8)$  (5.1). Then  $\text{Stab}_{X \times Z(Y)}(L) \cong \text{Sym}_3$  and  $C_{X \times Y}(\text{Stab}_{X \times Z(Y)}(L)) = Y$ , the subgroup of  $X \times Y$  which fixes each of the sets  $\alpha \otimes \Phi$ .*

**Proof.** Let  $h$  generate  $O_3(Q)$ . Since  $h$  acts trivially on  $R$ ,  $h$  acts trivially on  $\text{Proj}_Q(L)/Q$ , which means  $(h - 1)\text{Proj}_Q(L) \leq Q$ . Since  $O(Q) = X \times Y$ , where  $X \cong \text{Sym}_3$  and  $Y \cong O(E_8)$  (5.1), the fact that  $\mathcal{D}(Q)$  is an absolutely irreducible module for  $Y$  means that elements of  $X$  act as scalars on  $\mathcal{D}(R)$ . If an involution  $t$  of  $X \setminus Z(X)$ , acts as the scalar  $c \in \{\pm 1\}$  on  $\mathcal{D}(R)$ , there exists  $z \in Z(Y)$  which acts on  $\mathcal{D}(Q)$  as  $c$ . We have  $\text{Stab}_{X \times Z(Y)}(L) = O_3(X \times Z(Y))\langle tz \rangle \cong \text{Sym}_3$ . The last statement is clear.  $\square$

**Proposition 5.4.** *Suppose that  $L$  is a Niemeier lattice and that  $\theta \in O(L)$  has order 3 and satisfies  $L^+(\theta) \cong R$  and  $L_+(\theta) \cong Q$ . Then the stabilizer in  $O(L)$  of the gluing map for  $L$  over  $Q \perp R$  is  $N_{O(L)}(\langle \theta \rangle)$ .*

**Proof.** Note that both  $L^+(\theta)$  and  $L_+(\theta)$  are direct summands of  $L$ . Let  $\alpha$  be the gluing map and  $S$  its stabilizer in  $O(L)$ , i.e.,  $\{g \in O(L) \mid g(Q) = Q, g(L) = L, g \circ \alpha = \alpha\}$ . If  $g \in N_{O(L)}(\langle \theta \rangle)$ , clearly  $g$  fixes both  $L^+(\theta)$  and its annihilator  $L_+(\theta)$ . Since it fixes  $L$  and commutes with projections, it fixes the gluing map.

Now, we prove that  $S \leq N_{O(L)}(\langle \theta \rangle)$ . Since  $S$  acts on  $Q$ , it permutes the set  $F$  of norm 4 vectors. There is a partition of  $F$  into three sets  $F_i$ ,  $i = 1, 2, 3$  so that  $Q_i$ , the  $\mathbb{Z}$ -span of  $F_i$ , is an  $EE_8$  lattice in which  $F_i$  is the set of minimal vectors. It follows from (5.1) that  $S$  permutes these three sets.

We now refer to the notation of (5.3). Since  $Stab_{X \times Z(Y)}(L)$  acts on  $\{F_1, F_2, F_3\}$  as  $Sym_3$ ,  $S = Stab_{X \times Z(Y)}(L)T$  where  $T$  is the subgroup of  $S$  which normalizes each of  $Q_1, Q_2, Q_3$ . By (5.3),  $T \leq C(Stab_{X \times Z(Y)}(L))$ . It follows that  $S \leq N_{O(L)}(\langle \theta \rangle)$ .  $\square$

## 6 Overlattices of $Q \perp R$

We continue to use the notations  $Q, R$  (Notation 2.8). This section will explain which  $L$  may arise in (5.4).

**Lemma 6.1.** *There exist embeddings of  $Q \perp R$  in  $E_8^3$ . In fact, there are at least two kinds of embeddings.*

(i) (3-cycle type) *there exist embeddings such that  $R$  is the fixed point sublattice of an automorphism of order 3 which permutes the three direct summands cyclically; and*

(ii) (1+2 type) *there exist embeddings such that  $\mathbb{Q}R \cap E_8^3$  is an orthogonal direct summand of  $E_8^3$ .*

**Proof.** (i) is trivial. Compare Appendix C. (ii) Let  $A, B, C$  be the three indecomposable summands of a lattice  $L$  isometric to  $E_8^3$ . Fix an isometry  $\phi : A \rightarrow B$ . We take faithful actions of  $O(L)$  on  $A, B, C$  for which  $\phi$  is an invariant map.

There is an automorphism  $h$  of order 3 of  $E_8$  which does not have eigenvalue 1. Then, the endomorphism  $h - 1$  triples norms.

So,  $(h - 1)C \cong R$ . Now, define a lattice  $J \leq A \perp B$  by  $J := (h - 1)A + (h - 1)B + K$ , where  $K := \{a + \phi(a) \mid a \in A\} \cong EE_8$ . Then  $J$  is the sum of  $K$  and  $K' := \{a + \phi(ha) \mid a \in A\} \cong EE_8$ . It is easy to prove that  $J$  is rootless (since any element of  $A + B$  of the form  $a + b$ , where  $a \neq 0, b \neq 0$  has norm at least 4). Since  $K \cap K' = 0$ , the classification [18] identifies  $J$  as isometric to  $A_2 \otimes E_8$ .  $\square$

**Notation 6.2.** We fix an overlattice of  $Q \perp R$  which is isometric to  $E_8^3$  and is of type (i) in (6.1). Let  $\alpha$  be the associated gluing map, of  $\mathcal{D}(Q)$  with  $\mathcal{D}(R)$ . If  $\gamma$  is any gluing map, let  $L_\gamma$  be the overlattice associated to it. So,  $L_\alpha = L$  is our initial choice of  $E_8^3$ -overlattice.

We seek a new gluing map which gives a rootless Niemeier lattice. Such a lattice would be isometric to the Leech lattice, by a well-known classification.

**Notation 6.3.** Let  $\pi$  be the representation of  $O(Q) \times O(R)$  on  $\mathcal{D}(Q) \perp \mathcal{D}(R)$ . The image of  $\pi$  lies in  $O(\mathcal{D}(Q)) \times O(\mathcal{D}(R))$ .

## 6.1 The new gluing map

We look for a similitude on our quadratic spaces which respects a subgroup isomorphic to  $2 \cdot \text{Alt}_9$  and defines a Leech overlattice.

**Lemma 6.4.** *Suppose that  $A \cong SL(2, 8):3$  and  $A \leq B \cong Sp(6, 2)$ . Then  $A$  is a maximal subgroup.*

**Proof.** The embedding is essentially unique, by the 2-modular representation theory of  $SL(2, 8)$ . A Sylow 7-normalizer in  $A$  is a Sylow 7-normalizer in  $B$ . We have  $|B : A| = 2^6 \cdot 3 \cdot 5$ . The only divisors of this which are  $1 \pmod{7}$  are products of a subset of  $2^3, 2^3, 3 \cdot 5$ . Now suppose that  $S$  is a subgroup,  $A < S < B$ . If  $|B : S| \leq 15$ , we have a contradiction since  $Sp(6, 2)$  does not embed in  $Sym_{15}$ . Therefore,  $|S : A| \leq 15$ . If  $|S : A| = 8$ , then  $A' \cong SL(2, 8)$  is normal in  $S$ , which is impossible by above Sylow 7-theory. We conclude that  $|S : A| = 15$  and  $|B : A| = 2^6$ . Therefore, in the action of  $S$  on the left cosets of  $A$ ,  $A$  fixes 6 cosets and has a single orbit of length 9. We now use the fact that if  $T \in \text{Syl}_2(A) \subset \text{Syl}_2(S)$ ,  $N_S(T)$  operates transitively on the fixed points of  $T$ . Here,  $T$  fixes 7 of the 15 points. A group of order 7 in the normalizer acts by a 7-cycle on the  $A$ -orbit of length 9 and trivially on the other seven points. This is a contradiction. Therefore,  $S$  does not exist.  $\square$

**Notation 6.5.** We take the groups  $K_0$  and  $K$  constructed in (4.5). Let  $u \in K_0 \setminus K$ .

We define a new glue map by conjugating with  $u$ :  $\beta : x+Q \rightarrow u(\alpha(u^{-1}(x+Q)))$ , for  $x \in Q^*$ .

The stabilizer of  $\beta$  in  $W$  is  $W \cap uWu^{-1}$ , which contains  $K$ . (Actually,  $K$  is a maximal subgroup of  $W'$  (4.6) and  $W'$  is the only maximal subgroup of  $W$  which contains  $K$ ).

**Notation 6.6.** If  $Q$  has minimal vectors  $\alpha \otimes \gamma$ , for roots  $\alpha \in A_2$  and  $\gamma \in E_8$ , then the minimum norm vectors in  $Q^*$  are the norm  $\frac{4}{3}$  vectors of the form  $\frac{1}{3}(\alpha' - \alpha'') \otimes \gamma$ , where  $\alpha, \alpha', \alpha''$  are roots in  $A_2$  such that an isometry of

order 3 takes  $\alpha \mapsto \alpha' \mapsto \alpha''$ , and where  $\gamma$  is a root of  $E_8$ . The minimum norm vectors in  $R^*$  are the vectors of the form  $\frac{1}{3}\delta$ , where  $\delta$  is one of the 240 minimal vectors in  $R \cong \sqrt{3}E_8$ . Take any isomorphism  $\phi : E_8 \rightarrow R$  which triples norms. Note that the vectors  $(\alpha' - \alpha'') \otimes \gamma$  and  $(\alpha' - \alpha'') \otimes \gamma'$  have inner product  $-(\gamma, \gamma')$ . Thus the three sets  $\{\frac{1}{3}(\alpha - \alpha') \otimes \gamma + \phi(\frac{1}{3}\gamma) \mid \gamma \in E_8\}$ ,  $\{\frac{1}{3}(\alpha' - \alpha'') \otimes \gamma + \phi(\frac{1}{3}\gamma) \mid \gamma \in E_8\}$  and  $\{\frac{1}{3}(\alpha'' - \alpha) \otimes \gamma + \phi(\frac{1}{3}\gamma) \mid \gamma \in E_8\}$ , are pairwise orthogonal root systems of type  $E_8$ . Define  $M(\phi)$  to be the overlattice of  $Q \perp R$  which is the  $\mathbb{Z}$ -span of these three sets. Then  $M(\phi) \cong E_8^3$ . Any overlattice of  $Q \perp R$  which is isometric to  $E_8^3$  equals one of these  $M(\phi)$ , of which there are  $|O(E_8)| = 2^{14}3^55^27$ .

**Lemma 6.7.** (i) *The action of  $\pi(O(Q) \times O(R)) \cong W \times W$  is transitive on the set of  $M(\phi)$ . In fact, the action of either direct factor,  $\pi(O(Q))$  or  $\pi(O(R))$ , is regular.*

(ii) *The stabilizer of a given  $M(\phi)$  in  $O(\mathcal{D}(Q)) \times O(\mathcal{D}(R))$  is a diagonal subgroup of  $\pi(O(Q) \times O(R))$ .*

(iii) *Any element of  $O(\mathcal{D}(Q)) \times O(\mathcal{D}(R))$  which moves one  $M(\phi)$  to another is in  $\pi(O(Q) \times O(R))$ .*

**Proof.** (i) Straightforward.

(ii) Let  $S$  be the stabilizer of  $M(\phi)$ . Then  $S$  acts faithfully on both  $\mathcal{D}(Q)$  and  $\mathcal{D}(R)$ . The conclusion follows.

(iii) Suppose that  $g \in O(\mathcal{D}(Q)) \times O(\mathcal{D}(R))$  moves one  $M(\phi)$  to another, say  $M(\phi')$ . By the transitivity result of (i), there exists  $h \in \pi(O(Q) \times O(R))$  which takes  $M(\phi)$  to  $M(\phi')$ . Then  $h^{-1}g$  stabilizes  $M(\phi)$ , so  $h^{-1}g \in \pi(O(Q) \times O(R))$ , by (ii).  $\square$

**Lemma 6.8.** *The lattice  $L_\beta$  is rootless, so is isomorphic to the Leech lattice.*

**Proof.** Suppose that  $M$  contains a root, say  $r$ . Then  $r$  projects to minimal vectors in each of  $Q^*$  and  $R^*$ . It follows that  $M$  contains roots  $gr$ , for all  $g \in K$ . Therefore, by transitivity (6.7),  $M$  contains at least  $3 \times 240 = 720$  roots. In fact, we can show that these roots form a root system of type  $\Phi(E_8^3)$ . This follows from the discussion of (6.6).

We now quote (6.7)(iii) to conclude that  $u \in \pi(O(Q) \times O(R))$ . However, this is impossible because of (4.6)(ii).  $\square$

**Lemma 6.9.** *The common stabilizer  $O(L_\alpha) \cap O(L_\beta)$  is isomorphic to  $Sym_3 \times 2 \cdot Alt_9$ .*

**Proof.** The intersection  $O(L_\alpha) \cap O(L_\beta)$  can not be  $O(L_\alpha)$ . Now use (4.6).  
 $\square$

## Appendices

### A Alternate proof that $2 \cdot Alt_9$ occurs for a gluing

In this section, we assume existence of  $\Lambda$ , the Leech lattice, and some of its properties.

We start with  $Q \cong A_2 \otimes E_8$  and  $R := \sqrt{3}E_8$ . There is an embedding  $Q \perp R \leq \Lambda$ : if  $h \in O(\Lambda)$  has order 3 and trace 0, we take  $R$  to be its fixed point sublattice and  $Q$  to be  $ann_\Lambda(R)$ . Let  $K$  be the common stabilizer of these three lattices. The two projections of  $\Lambda$  are  $K$ -maps and so are the associated maps of  $\Lambda/3\Lambda$  to  $\mathcal{D}(Q)$  and  $\mathcal{D}(R)$ . In fact,  $\mathcal{D}(Q)$  and  $\mathcal{D}(R)$  are isometric  $K$ -modules. The group  $K$  acts on  $\mathcal{D}(Q)$  and  $\mathcal{D}(R)$  completely reducibly, with constituents of dimensions 1 and 7. To a gluing is associated an isometry  $\varphi : \mathcal{D}(Q) \rightarrow \mathcal{D}(R)$

Now suppose that the  $\varphi$  comes from the  $E_8$ -structure on  $Q$  and  $P$ . That is,  $Q$  contains three copies of  $EE_8$ , say  $A, B, C$ , and their annihilators are isometric to  $\sqrt{6}E_8$ .

Take  $A' := ann_Q(A)$ . Let  $\psi : A' \rightarrow R$  be any isomorphism of free abelian groups which is a scaled isometry (so that the scale factor is  $\sqrt{2}$ ). Then  $Q \perp R$  and  $\{v + v\psi \mid v \in \frac{1}{3}A'\}$  span an even unimodular lattice which has roots. It is isometric to  $E_8^3$ .

To get Leech from a gluing of  $Q \perp R$ , we need  $\varphi$  which does not arise this way. To get an integral overlattice without roots, we need the property that if  $\psi$  makes the cosets  $v + Q$  and  $w + R$  correspond, then the minimum norms  $a$  in  $v + Q$  and  $b$  in  $w + R$  must satisfy  $a + b \in 2\mathbb{Z}$  but  $a + b > 2$ .

Since the Leech lattice exists, it follows that there is such a  $\psi$ . Since the Leech lattice is unique and since we know the isometry group of Leech, it follows that for any  $\psi$  which defines a Leech lattice, its stabilizer in  $Weyl(E_8)$  is isomorphic to  $2 \cdot Alt_9$ .

## B Automorphism group of $V_\Lambda^+$

The full automorphism group of  $V_\Lambda^+$  associated with the Leech lattice has been determined in [26]. In this section, we recall some basic results which we used in this article.

Let  $L$  be a positive definite even lattice and

$$1 \longrightarrow \langle \kappa \rangle \longrightarrow \hat{L} \twoheadrightarrow L \longrightarrow 1$$

a central extension of  $L$  by  $\langle \kappa \rangle$  such that  $\kappa^2 = 1$  and the commutator map  $c_0(\alpha, \beta) = \langle \alpha, \beta \rangle \pmod{2}$ ,  $\alpha, \beta \in L$ . The following theorem is well-known (cf. [9]):

**Theorem B.1.** *For an even lattice  $L$ , the sequence*

$$1 \rightarrow \text{Hom}(L, \mathbb{Z}_2) \rightarrow \text{Aut}(\hat{L}) \xrightarrow{\pi} \text{Aut}(L) \rightarrow 1$$

*is exact. In particular  $\text{Aut}(\hat{\Lambda}) \cong 2^{24} \cdot Co_0$ .*

Recall that  $\theta$  is the automorphism of  $V_\Lambda$  defined by

$$\theta(\alpha_1(-n_1) \cdots \alpha_k(-n_k)e^\alpha) = (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k)\theta(e^\alpha),$$

where  $\theta(a) = a^{-1} \kappa^{\langle \bar{a}, \bar{a} \rangle / 2}$  on  $\hat{L}$ .

**Lemma B.2** ([26]). *Let  $L$  be a positive definite even lattice without roots, i.e.,  $L(1) = \emptyset$ . Then the centralizer  $C_{\text{Aut}V_L}(\theta)$  of  $\theta$  in  $\text{Aut}V_L$  is isomorphic to  $\text{Aut}(\hat{L})$ . If  $L = \Lambda$  is the Leech lattice, we have*

$$C_{\text{Aut}V_\Lambda}(\theta) \cong \text{Aut}(\hat{\Lambda}) \cong 2^{24} \cdot Co_0.$$

**Theorem B.3** ([26]). *Let  $V_\Lambda^+ = \{v \in V_\Lambda \mid \theta(v) = v\}$  be the fixed point subVOA of  $\theta$  in  $V_\Lambda$ . Then  $\text{Aut}V_\Lambda^+ \cong C_{\text{Aut}V_\Lambda}(\theta)/\langle \theta \rangle \cong 2^{24} \cdot Co_1$  and the sequence*

$$1 \rightarrow \text{Hom}(\Lambda, \mathbb{Z}_2) \rightarrow \text{Aut}V_\Lambda^+ \xrightarrow{\pi} \text{Aut}(\Lambda)/\langle \pm 1 \rangle \rightarrow 1.$$

*is exact.*

Next we shall recall the properties of the corresponding Miyamoto involutions.



**Lemma B.4.** *Let  $L$  be an even lattice without roots and  $e$  a  $\text{cvcc}_{\frac{1}{2}}$  in  $V_L^+$ . Then,  $\tau_e \in C_{\text{Aut}(V_L)}(\theta)$ . In particular, we may view  $\tau_e$  as an element in  $\text{Aut}\hat{L}(\cong C_{\text{Aut}(V_L)}(\theta))$ .*

*Proof.* We view  $\tau_e$  as an automorphism of  $V_L$ . Since  $\theta$  fixes  $e$ , we have  $\theta\tau_e\theta = \tau_{\theta(e)} = \tau_e$ , which proves this lemma.  $\square$

**Remark B.5.** *Recall the exact sequence*

$$1 \rightarrow \text{Hom}(\Lambda, \mathbb{Z}_2) \rightarrow \text{Aut}(\hat{\Lambda}) \xrightarrow{\pi} \text{Aut}(\Lambda) \rightarrow 1$$

*defined in Theorem B.1. Hence, by Lemma B.4,  $\pi(\tau_e)$  is an isometry of  $L$  for any  $\text{cvcc}_{\frac{1}{2}}$   $e \in V_L^+$ .*

**Notation B.6.** *In [20], all  $\text{cvcc}_{\frac{1}{2}}$  in the VOA  $V_{\Lambda}^+$  were classified. There are two types of  $\text{cvcc}_{\frac{1}{2}}$ .*

**AA<sub>1</sub>-formula:** *conformal vectors supported at AA<sub>1</sub>-sublattices, i.e.,*

$$\omega^{\pm}(\alpha) = \frac{1}{4}\alpha(-1)^2 \cdot \mathbf{1} \pm \frac{1}{4}(e^{\alpha} + e^{-\alpha}), \quad \text{where } \alpha \in \Lambda(4) = \{\alpha \in \Lambda \mid \langle \alpha, \alpha \rangle = 4\}.$$

**EE<sub>8</sub>-formula:** *conformal vectors supported at EE<sub>8</sub>-sublattices, i.e.,*

$$\varphi_x(e_M), \quad \text{where } \Lambda \supset M \cong EE_8, x \in M^*.$$

**Lemma B.7.** *Let  $e$  be a  $\text{cvcc}_{\frac{1}{2}}$  in  $V_{\Lambda}^+$ .*

(1) *If  $e = \omega^{\pm}(\alpha)$ , then  $\pi(\tau_e) = 1$ . In fact,  $\tau_e = \varphi_{\alpha}$  as an automorphism of  $V_L$ , i.e.,*

$$\tau_e(u \otimes e^{\beta}) = (-1)^{\langle \alpha, \beta \rangle} u \otimes e^{\beta} \quad \text{for } u \in M(1), \beta \in L.$$

(2) *If  $e = \varphi_x(e_M)$  for some  $M \cong EE_8$  in  $\Lambda$ , then  $\pi(\tau_e)$  defines an isometry of  $\Lambda$  which acts as  $-1$  on  $M$  and  $1$  on  $\text{ann}_{\Lambda}(M)$ .*

Now let  $V^{\natural} = V_{\Lambda}^+ \oplus V_{\Lambda}^{T,+}$  be the famous Moonshine. Let  $z$  be the linear map of  $V^{\natural}$  acting as  $1$  and  $-1$  on  $V_{\Lambda}^+$  and  $V_{\Lambda}^{T,-}$  respectively. Then  $z$  is an automorphism of  $V^{\natural}$ .

**Lemma B.8.** *Let  $\alpha \in \Lambda(4)$ . Then  $\tau_{\omega^+(\alpha)}\tau_{\omega^-(\alpha)} = z$  on  $V^{\natural}$ .*

Next, we shall discuss the centralizer of  $\tau_e$  and  $z$  in  $\text{Aut}(V^{\natural})$  for any  $\text{cvcc}_{\frac{1}{2}}$   $e$  in  $V_{\Lambda}^+$ . The following lemma is well known [19, 9].

**Lemma B.9.** *The centralizer of  $z$  in  $AutV^\natural$  has the structure  $2^{1+24}.Co_1$ .*

The proofs of the following two theorems can be found in [19]

**Theorem B.10.** *Let  $\alpha \in \Lambda(4)$ . Set  $e = \omega^\varepsilon(\alpha)$ , where  $\varepsilon = +$  or  $-$ . Then the centralizer  $C_{AutV^\natural}(\tau_e, z)$  has the structure  $2^{2+22}.Co_2$ .*

If  $e = \varphi_x(e_M)$ , it turns out that the centralizer of  $\tau_e$  in  $C_{AutV^\natural}(z)$  also stabilizes the VOA  $V_M^+ \subset V_\Lambda^+$ . Moreover, we have

**Theorem B.11.** *Let  $M$  be a sublattice of  $\Lambda$  isomorphic to  $EE_8$  and  $x$  a vector in  $M/2$ . Set  $e = \varphi_x(e_M)$ . Then the centralizer  $C_{AutV^\natural}(\tau_e, z)$  has the structure  $2^{2+8+16}.\Omega^+(8, 2)$ .*

**Remark B.12.** *Let  $M$  be a sublattice of  $\Lambda$  isomorphic to  $EE_8$ . Then the stabilizer of  $e_M$  in the subgroup  $Aut(\hat{M})/\langle\theta\rangle$  of  $AutV_M^+$  is isomorphic to  $Aut(M)/\langle-1\rangle \cong O^+(8, 2)$ . In Theorem B.1, the centralizer  $C_{AutV^\natural}(\tau_{e_M}, z)$  actually acts on  $V_M^+$  as  $\Omega^+(8, 2)$ , which is the quotient of the commutator subgroup of the Weyl group of  $E_8$  by its center.*

*Let  $(M, M')$  be an  $EE_8$ -pair in  $\Lambda$ . Then we have*

$$C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_{M'}}, z) = C_{Aut(V^\natural)}(\tau_{e_M}, z) \cap C_{Aut(V^\natural)}(\tau_{e_{M'}}, z).$$

*In this case,  $C_{Aut(V^\natural)}(\tau_{e_M}, \tau_{e_{M'}}, z)$  must contain a factor group which is isomorphic to the common stabilizer of  $M$  and  $M'$  in  $Aut(\Lambda)/\pm 1$ .*

## C Niemeier lattices that contain $Q \perp R$

In this section, we shall list the Niemeier lattices that contain  $Q \perp R$  such that  $R$  is the fixed point sublattice of an isometry of order 3 and  $Q$  is its annihilator.

Our setting is as follows: Let  $h$  be an order 3 element of  $Weyl(A_2) \cong Sym_3$ . Then  $h$  defines an isometry on  $Q = A_2 \otimes E_8$  by  $h(\alpha \otimes \beta) = (h\alpha) \otimes \beta$ . It also induces an isometry on  $Q \perp R$  and  $(Q \perp R)^*$  by acting trivially on  $R$ .

Now let  $N$  be a Niemeier lattice that contains  $Q \perp R$ . We assume that  $N$  is stable under  $h$  and  $Q$  and  $R$  are direct summands in  $N$ . In this case, the fixed point sublattice of  $h$  in  $N$  is exactly  $R \cong EEE_8$  and the annihilator of  $R$  is  $ann_N(R) = Q = A_2 \otimes E_8$ .

The list of all possible Niemeier lattices (including the Leech lattice) that satisfy the above is given below.

Type of Niemeier	$C_{O(N)}(h)$
$A_8^3$	$3 \times (2 \times Sym_9)$
$D_8^3$	$3 \times 2^7 \cdot Sym_8$
$E_8^3$	$3 \times Weyl(E_8)$
$A_2^{12}$	$3 \times ((Sym_3)^4 \cdot C_{Aut(\mathcal{TG})}(h))$
$A_1^{24}$	$3 \times 2^8 \cdot L_2(7)$
$D_4^6$	$3 \times (Weyl(D_4) \times Weyl(D_4)) \cdot 3$
$\Lambda$	$3 \times 2 \cdot Alt_9$

Here  $\mathcal{TG}$  is the ternary Golay code.

**Sketch of the proof.**

Let  $N = N(\Phi)$  be a Niemeier lattice associated to a root system  $\Phi$ .

We shall first search for the element of  $O(N)$  of order 3, which acts fixed point freely on roots.

Let  $h \in O(N)$  be such an element. Suppose  $h$  preserves an irreducible component of  $\Phi$ , say,  $\Phi_1$ . Then  $g$  also acts on the corresponding root sublattice  $L_1 := span_{\mathbb{Z}}\{\Phi_1\}$ . In this case,  $\Phi_1$  is isomorphic to  $A_{3n}$ ,  $D_{3n}$ ,  $D_{3n+1}$ ,  $E_6$  or  $E_8$  since  $h$  acts fixed point freely on roots. Then by case by case checking,  $ann_{L_1}(L_1^h)$  must contain roots.

Therefore,  $h$  induces a permutation on the irreducible components of  $\Phi$  and has no fixed points. Thus,  $\Phi$  must be one of the followings:

$$A_8^3, D_8^3, E_8^3, A_4^6, D_4^6, A_2^{12}, A_1^{24} \text{ or } \emptyset.$$

For  $N = N(A_4^6)$ , the glue code  $C$  is generated by (101441), (114410), (144101), (141014), and (110144) and  $[N(A_4^6) : A_4^6] = 125$ . Nevertheless, there is no element in  $Sym_6$  of cycle shape  $3^2$  which preserves  $C$ . Thus,  $N(A_4^6)$  is also out.

The explicit embedding of  $Q \perp R$  for the remaining cases are given below.

**Case:  $N(A_8^3)$**

$[N(A_8^3) : A_8^3] = 3^3$  and the glue code is generated by (114), (141) and (411).

Then  $Aut N(A_8^3) \cong W(A_8^3) \cdot (2 \times S_3)$ . Let  $\sigma$  be the cyclic permutation of the 3 copies of  $A_8$ .

Set

$$R = \text{span}\{(\alpha, \alpha, \alpha) | \alpha \in A_8\} \cup \{(\gamma, \gamma, \gamma)\} \cong \sqrt{3}E_8,$$

and

$$Q = \text{span}\{(\alpha, -\alpha, 0), (0, \alpha, -\alpha) | \alpha \in A_8\} \cup \{(\gamma, -\gamma, 0), (0, \gamma, -\gamma)\} \cong A_2 \otimes E_8,$$

where  $\gamma = \frac{1}{3}(1^6, -2^3)$ . Note that  $(3, 3, 3)$ ,  $(3, -3, 0)$  and  $(0, 3, -3)$  are in the glue code and  $R = N(A_8^3)^\sigma$ .

In this case,

$$C_{\text{Aut } N(A_8^3)}(\sigma) = 3 \times (2 \times \text{Sym}_9),$$

where  $\text{Sym}_9$  acts diagonally on  $A_8^3$ .

**Case:  $\mathbf{N}(D_8^3)$**

$[N(D_8^3 : D_8^3) = 2^3$  and the glue code is generated by  $(122)$ ,  $(212)$ ,  $(221)$ .

Then  $\text{Aut } N(D_8^3) \cong \text{Weyl}(D_8) \wr S_3$ . Let  $\sigma$  be the cyclic permutation of the 3 copies of  $D_8$ . Then

$$N(D_8^3)^\sigma = R = \text{span}\{(\alpha, \alpha, \alpha) | \alpha \in D_8\} \cup \{(\gamma, \gamma, \gamma)\} \cong \sqrt{3}E_8$$

and

$$\begin{aligned} Q &= \text{ann}_N(R) \\ &= \text{span}\{(\alpha, -\alpha, 0), (0, \alpha, -\alpha) | \alpha \in D_8\} \cup \{(\gamma', -\gamma', 0), (0, \gamma', -\gamma')\} \\ &\cong A_2 \otimes E_8, \end{aligned}$$

where  $\gamma = \frac{1}{2}(11111111)$ ,  $\gamma' = \frac{1}{2}(11111111 - 1)$ .

In this case,

$$C_{\text{Aut } N(D_8^3)}(\sigma) = 3 \times \text{Weyl}(D_8) \cong 3 \times (2^7 : \text{Sym}_8).$$

**Case:  $\mathbf{E}_8^3$**

In this case,  $\text{Aut } E_8^3 \cong \text{Weyl}(E_8) \wr S_3$ . Let  $\sigma$  be the cyclic permutation of the 3 copies of  $E_8$ . Then

$$(E_8^3)^\sigma = R = \text{span}\{(\alpha, \alpha, \alpha) | \alpha \in E_8\} \cong EEE_8$$

and

$$Q = \text{ann}_{E_8^3}(R) = \text{span}\{(\alpha, -\alpha, 0), (0, \alpha, -\alpha) | \alpha \in E_8\} \cong A_2 \otimes E_8.$$

Moreover,

$$C_{Aut E_8^3}(\sigma) \cong 3 \times Weyl(E_8).$$

**Case:  $N(A_1^{24})$**

The glue code is isomorphic to  $G_{24}$  and  $[N(A_1^{24}) : A_1^{24}] = 2^{12}$ .

In this case,  $Aut N(A_1^{24}) \cong 2^{24}.M_{24}$ .

Let  $\sigma$  be the order 3 automorphism which has the shape  $3^8 \in M_{24}$ . Let  $\mathcal{C}$  be the subcode generated by the 14 dodecads fixed by  $\sigma$ . Then  $\mathcal{C}$  is isomorphic to the tripled Hamming code. Then,

$$N(A_1^{24})^\sigma = R = span\{(\alpha, \alpha, \alpha) | \alpha \in A_1^8\} \cup \{\frac{1}{2}\alpha_C | C \in \mathcal{C}\} \cong \sqrt{3}E_8.$$

Let  $\mathcal{T}$  be the set of sextets that are fixed by  $\sigma$  and  $\{O_1, O_2, O_3\}$  the trio fixed by  $\sigma$ . Then  $|\mathcal{T}| = 7$ . Set

$$\mathcal{H} = \{T \subset O_1 | T \text{ is a tetrad of a sextet in } \mathcal{T}\} \cup \{\emptyset, O_1\}$$

Then  $\mathcal{H}$  is isomorphic to the Hamming code. Then

$$\begin{aligned} Q &= span\{(\alpha, -\alpha, 0), (0, \alpha, -\alpha) | \alpha \in A_1^8\} \cup \{\frac{1}{2}(\alpha_T, -\alpha_T, 0), \frac{1}{2}(0, \alpha_T, -\alpha_T) | T \in \mathcal{H}\} \\ &\cong A_2 \otimes E_8 \end{aligned}$$

In this case, the centralizer is

$$C_{Aut N(A_1^{24})} = 2^8.C_{M_{24}}(\sigma) = 3 \times 2^8.L_2(7)$$

**Case:  $N(D_4^6)$**

$D_4^*/D_4 \cong F_4$  and the glue code is the Hexacode.

Let  $\sigma = (135)(246)$ . Then  $\sigma$  fixes a subcode  $I$  generated by  $(1\omega \ 1\omega \ 1\omega)$  and  $|I| = 2^2$ . Set

$$R = span\{(\alpha, \beta, \alpha, \beta, \alpha, \beta) | \alpha, \beta \in D_4\} \cup \{\alpha_c | c \in I\} \cong \sqrt{3}E_8,$$

where  $\alpha_c = ([c_1], [c_2], \dots, [c_6])$  if  $c = (c_1, c_2, \dots, c_6)$  and  $[1] = (000 - 1)$ ,  $[\omega] = \frac{1}{2}(1111)$ ,  $[\bar{\omega}] = \frac{1}{2}(-1 - 1 - 11)$ .

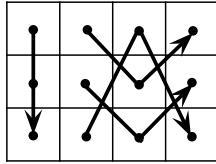
$$\begin{aligned} Q &= span\{(\alpha, \beta, -\alpha, -\beta, 0, 0), (0, 0, \alpha, \beta, -\alpha, -\beta) | \alpha, \beta \in D_4\} \\ &\quad \cup \{([c], [c], -[c], -[c], 0, 0), (0, 0, [c], [c], -[c], -[c]) | c \in \{1, \omega, \bar{\omega}\}\} \\ &\cong A_2 \otimes E_8. \end{aligned}$$

$$C_{Aut N(D_4^6)} = (Weyl(D_4) \times Weyl(D_4)) \cdot C_{Aut Hexacode}(\sigma) \cong (2^3 Sym_4 \times 2^3 Sym_4).3.2$$

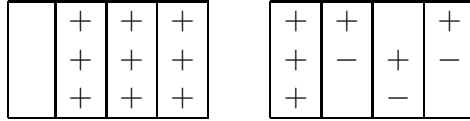
**Case:  $N(A_2^{12})$**

The glue code is the ternary Golay code  $\mathcal{TG}$  and  $[N(A_2^{12}) : A_2^{12}] = 3^6$ .  
 $Aut(\mathcal{TG}) = 2.M_{12}$ .

Let  $\sigma$  be an order 3 element given by



Then  $\sigma$  fixes a subcode  $\mathcal{C}$  generated by

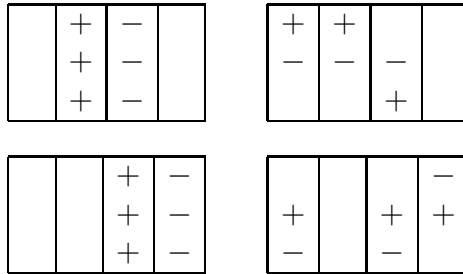


Then  $\mathcal{C}$  is isomorphic to a triple of tetra-code and

$$R = \text{span}\{(\alpha, \alpha, \alpha) | \alpha \in A_2^4\} \cup \{\alpha_c | c \in \mathcal{C}\} \cong \sqrt{3}E_8,$$

where  $\alpha_c = ([c_1], [c_2], \dots, [c_{12}])$  and  $[0] = (0, 0, 0)$ ,  $[1] = \frac{1}{3}(1, 1, -2)$ ,  
 $[2] = \frac{1}{3}(-1, -1, 2)$ .

Let  $\mathcal{H}$  be the subcode generated by



Then

$$Q = \text{ann}_{N(A_2^{12})}(R) = \text{span}A_2 \otimes (A_2^4) \cup \{\alpha_c | c \in \mathcal{H}\},$$

where  $A_2 \otimes E_8$  is generated by

$$\begin{array}{|c|c|c|c|} \hline \alpha & \beta & -\delta & \\ \hline -\alpha & \gamma & -\beta & \\ \hline & \delta & -\gamma & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline & & \delta & -\beta \\ \hline \alpha & & \beta & -\gamma \\ \hline -\alpha & & \gamma & -\delta \\ \hline \end{array}$$

In this case, the centralizer is

$$C_{\text{Aut}N(A_2^{12})}(\sigma) = (S_3 \times S_3 \times S_3 \times S_3) \cdot C_{\text{Aut}(\mathcal{TG})}(\sigma).$$

### Case: Leech lattice $\Lambda$

This case was treated in [18].

## D Centralizers of pairs of $2A$ -involutions for the $3C$ -case

For background in this section, see [11, 14]

We take a  $3C$ -pair of  $2A$  involutions,  $x, y$ , and study  $C(x, y)$  (meaning  $C_{\mathbb{M}}(\langle x, y \rangle)$ ) and  $C(x, y, z)$ , where  $z \in 2B$  and  $z \in C(x, y)$ .

Consider the  $3C$ -element  $h := xy$ . We have  $C(h) = F \times \langle h \rangle$ , where  $F \cong F_3$ , a simple group of order  $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$ . The group  $F$  has one class of involutions and they are contained in the  $2B$  class of  $\mathbb{M}$ . We take  $z \in F$  and  $\langle x, y \rangle = C(F)$ .

Let us go to  $C := C(z) \cong 2^{1+24} \cdot Co_1$ . This is a twisted holomorph in the sense of [11, 12]. The element  $h$  is in  $C$  and corresponds in  $O(\Lambda)$  to an element  $h'$  of order 3 which is a permutation in the natural  $M_{24}$  of cycle shape  $3^8$ . Its centralizer in  $O(\Lambda)$  has the form  $3 \times 2 \cdot Alt_9$ . Therefore,  $C_C(h)$  has shape  $2^{1+8} \cdot Alt_9$ .

There exist involutions  $x', y' \in O(\Lambda)$  of trace 8 so that  $h' = x'y'$ . If we choose  $x, y \in C$  to correspond to such involutions, then  $\langle x, y \rangle$  centralizes  $C_C(h)$ . Then we get  $C(x, y, z) = F$  and  $\langle x, y, C(h) \rangle = \langle x, y \rangle \times F$ .

## References

- [1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, ATLAS of finite groups. Clarendon Press, Oxford, 1985.

- [2] J.H. Conway, A simple construction for the Fisher-Griess Monster group, *Invent. Math.* 79 (1985), 513-540.
- [3] J.H. Conway and N.J.A. Sloane, Sphere packings, lattices and groups, 3rd Edition, Springer, New York, 1999.
- [4] C. Dong, Vertex algebras associated with even lattices, *J. Algebra* 161 (1993), 245-265.
- [5] C.Dong, H.S. Li, G. Mason, Simple currents and extensions of vertex operator algebras, *Commun. Math. Phys.* 180 (1996), no. 3, 671–707.
- [6] C.Dong, H.S. Li, G. Mason, Modular-Invariance of Trace Functions in Orbifold Theory and Generalized Moonshine, *Commun. Math. Phys.* 214 (2000), 1 V- 56.
- [7] C. Dong, H. Li, G. Mason and S. P. Norton, Associative subalgebras of Griess algebra and related topics, *Proc. of the Conference on the Monster and Lie algebra at the Ohio State University*, May 1996, ed. by J. Ferrar and K. Harada, Walter de Gruyter, Berlin - New York, 1998.
- [8] C. Dong, G. Mason and Y. Zhu, Discrete series of the Virasoro algebra and the moonshine module, *Proc. Symp. Pure. Math.*, American Math. Soc. **56** II (1994), 295–316.
- [9] I. B. Frenkel, J. Lepowsky, and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. 134, Academic Press, 1988.
- [10] G. Glauberman and S. P. Norton, On McKay’s connection between the affine  $E_8$  diagram and the Monster, *Proceedings on Moonshine and related topics (Montreal, QC, 1999)*, 37–42, CRM Proc. Lecture Notes, **30**, Amer. Math. Soc., Providence, RI, 2001.
- [11] Robert L. Griess, Jr., The friendly giant, *Invent. Math.*, 69, 1982, 1-102.
- [12] Robert L. Griess, Jr., The monster and its nonassociative algebra, in *Proceedings of the Montreal Conference on Finite Groups*,



Contemporary Mathematics, 45, 121-157, 1985, American Mathematical Society, Providence, RI.

- [13] Robert L. Griess, Jr., Elementary Abelian  $p$ -subgroups of algebraic groups, *Geom. Dedicata* 39, 1991, 253-305.
- [14] Robert L. Griess, Jr., Twelve Sporadic Groups, Springer Verlag, 1998.
- [15] Robert L. Griess, Jr. and Gerald Höhn, Frame stabilizers for the lattice vertex operator algebra of type  $E_8$ , *J. reine angew. Math.*, 561 (2003), 1-37.
- [16] R. L. Griess, Jr., A vertex operator algebra related to  $E_8$  with automorphism group  $O^+(10, 2)$ . *The Monster and Lie algebras* (Columbus, OH, 1996), 43–58, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.
- [17] Robert L. Griess, Jr. and Ching Hung Lam Rootless pairs of  $EE_8$  lattices, *Electronic Research Announcements in Mathematical Sciences*, 15 (2008) 52-61
- [18] Robert L. Griess, Jr. and Ching Hung Lam, Dihedral groups and  $EE_8$  lattices, preprint 87 pages. pdf ; arxiv 15 jun 08; accepted.
- [19] C.H. Lam and H. Shimakura, Ising vectors in the vertex operator algebra  $V_\Lambda^+$  associated with the Leech lattice  $\Lambda$ , *Int. Math. Res. Not.* (2007), Vol. 2007: article ID rnm132, 21 pages.
- [20] C.H. Lam, S. Sakuma and H. Yamauchi, Ising vectors and automorphism groups of commutant subalgebras related to root systems, *Math. Z.* **255** (2007) 597–626.
- [21] C. H. Lam, H. Yamada and H. Yamauchi, McKay’s observation and vertex operator algebras generated by 2 conformal vectors of central charge 1/2, *Inter. Math. Res. Paper*, **No. 3** (2005), pp. 117-181.
- [22] H.S. Li, The Physics Superselection Principle in Vertex Operator Algebra Theory, *J. Algebra* **196**, 436 –457 (1997).

- [23] M. Miyamoto, A new construction of the moonshine vertex operator algebra over the real number field, *Ann. of Math* **159** (2004), 535–596.
- [24] S. Sakuma, 6-transposition property of  $\tau$ -involutions of vertex operator algebras, *Int. Math. Res. Not.* (2007), Vol. 2007 : article ID rnm030, 19 pages, doi:10.1093/imrn/rnm030.
- [25] J. McKay, Graphs, singularities, and finite groups, Proc. Symp. Pure Mathe., Vol.37, Amer. Math. Soc., Providence, RI, 1980, 183–186
- [26] H. Shimakura, Automorphism group of the vertex operator algebra  $V_L^+$  for an even lattice  $L$  without roots, *J. Algebra* **280** (2004), 29–57.
- [27] Suzuki, Michio Finite groups in which the centralizer of any element of order 2 is 2-closed. *Ann. of Math.* (2) 82 1965 191–212.
- [28] Edwin Weiss, Algebraic Number Theory, Dover Publications, 1998.