Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Information Structures to Control Formation Splitting and Merging

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***Abstract*— This paper focuses on developing techniques and strategies for the analysis and design of sensor and network topologies required to achieve rigid formations of mobile autonomous agents for cooperative tasks. These strategies ensure minimum number of changes in the set of sensing and communication links between agents during splitting and merging operations. That is, in splitting, all the links between agents in the same post-split sub-formation are preserved and a minimum number of links are inserted into each post-split sub-formation to regain minimal rigidity. In merging, all the links in each pre-merged rigid sub-formation are preserved and a minimum number of links are inserted between sub-formations to create one single post-merged minimally rigid formation.**

I. Introduction

A *formation* is defined as a group of mobile agents moving in real 2- or 3-dimensional space. This paper addresses "rigid formations." A formation is *rigid* if the distance between each pair of agents does not change over time, at least under ideal conditions. In the context of this paper, "agents" are considered to be autonomous vehicles such as autonomous underwater vehicles (AUVs), microsatellites, uninhabited air vehicles (UAVs), mobile ground-based robots.

A key element in all future multi-agent systems will be the role of sensor and communications networks as an integral part of coordination. In a rigid formation, distances between agents are held fixed by measurements and information gathered through "sensing and communication links" between agents. One of the challenges in building sensor and communications networks between agents is the "topology" of the network. By *topology*, we mean the interconnection structure of sensing and communication links among agents. Two networks have the same topology if the interconnection structure is the same, although the networks may differ in physical interconnections, distances between agents, transmission rates, and signal types. Rigid formations with the minimum number of sensing and communication links required to achieve rigidity are called *minimally rigid formations*. We refer the reader to the companion paper by Eren et al. [5] for an introduction to rigid formations.

Formations of autonomous agents usually operate under time-varying conditions where sensor and network topologies need to be restructured. Such conditions can be changes in the environment, obstacles along the trajectories of agents or departures of agents from formation. In this paper, we focus on such topological changes during "operations" on formations. By an *operation*, we mean missions and maneuvers that include agent departures, splitting, and merging, which result in changes in agent set and/or interconnection structure of sensing and communication links.

First, we consider the problem of splitting rigid formations. By *splitting*, we mean creating two or more rigid post-split sub-formations from a rigid pre-split formation. When a formation encounters an obstacle, splitting may be useful to maneuver around the obstacle. Instead of all agents moving to the same side of the obstacle, it might be more efficient in terms of trajectory lengths of agents, if some agents move to one side of the obstacle and the others move to the other side.

One strategy to solve the splitting problem would be determining entirely new sets of links for the post-split sub-formations. Olfati-Saber and Murray present such a strategy [7]. When splitting a rigid formation, it is necessary to break the links between agents belonging to different post-split sub-formations. However we can preserve a link between two agents belonging to the same sub-formation. Generating entirely new sets of links for post-split sub-formations is impractical when we can preserve the links that can be maintained. Such a strategy achieves splitting with a minimum number of changes in the topology of sensing and communication links. Our goal is to find such a strategy by inserting a minimum number of links in post-split non-rigid sub-formations to make each of them rigid while preserving the links between agents belonging to the same sub-formations. To motivate our discussion of splitting a rigid formation, we have the following example:

Example: Consider a rigid formation as shown in Figure 1 in 3-dimensional space. We would like to split the formation in such a way that agents with labels $\{1,2,3,4,5\}$ belong to one post-split sub-formation and agents with labels $\{6,7,8,9,10\}$ belong to the other post-split sub-formation.

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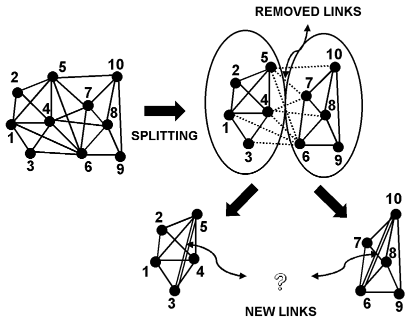

Fig. 1. A rigid formation is split into two sub-formations by removing the links shown with dashed edges, i.e., (1,6), (3,6), (4,6), (4,7), (4,8), (5,6), (5,7), (5,10), which results in two non-rigid sub-formations. The splitting problem is to find the new links that need to be inserted into each non-rigid post-split sub-formation so that each post-split sub-formation regains rigidity. In this example, the new links are (3,5), (6,10) shown with double lines.

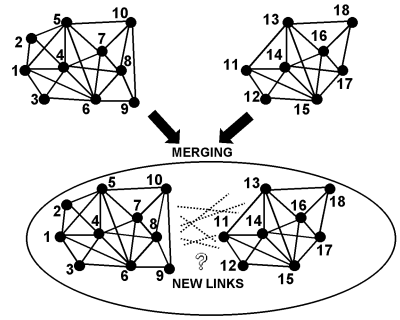

Fig. 2. Two rigid sub-formations are merged to form one single rigid formation. Finding the new links to be inserted between these two formations, which will make the whole formation rigid, is the merging problem.

Splitting can be achieved by removing the links between agents belonging to different post-split sub-formations while preserving the links between agents belonging to the same post-split sub-formations. In this example, the links (1,6), (3,6), (4,6), (4,7), (4,8), (5,6), (5,7), (5,10) (shown with dashed lines) are removed. This results in two non-rigid post-split sub-formations. The splitting problem is to find new sets of links to insert into each non-rigid post-split sub-formation resulting in rigid post-split sub-formations. In this example, those new links are $(3,5)$ and $(6,10)$ shown with double lines.

Second, we address merging rigid sub-formations. By *merging*, we mean inserting links between these rigid sub-formations which results in a single post-merged rigid formation. During a merging operation, it is a natural starting point to preserve the links in each pre-merged rigid sub-formation. Hence a reasonable goal is to create a new post-merged rigid formation by inserting a minimum number of links between sub-formations. A merging operation, for example, can be used to create one single rigid formation after split sub-formations pass around an obstacle. To motivate our discussion of merging a rigid formation, we have the following example:

Example: Consider two rigid sub-formations in 3-dimensional space as shown in Figure 2. We would like to merge these two formations resulting in a single rigid formation in such a way that all pairs of links in each formation are preserved and a minimum number of links is inserted between these two sub-formations.

Merging rigid bodies has been studied in rigidity theory. We refer the reader to Whiteley [8] for a detailed explanation. Here, we use a different approach to find the new links for merging formations. This approach can be used for both splitting and merging formations.

The approach in this paper is based on the strategies developed in Eren et al. [4]. Olfati-Saber and Murray gave an approach to merging sub-formations in 2-dimensional space so that the resulting formation is rigid [7]. The approach we develop here allows us to merge two rigid sub-formations with different types of combinations of inserting links including their strategy. Furthermore the approach in this paper also solves the merging problem in 3-dimensional space.

The splitting and merging problems can be considered as special cases of the "minimal cover problem." The *minimal cover problem* is basically to find new links to insert into a non-rigid formation so that it becomes rigid. To solve the minimal cover problem, we develop a novel procedure. This procedure can be used for creating minimally rigid post-split sub-formations from non-rigid post-split sub-formations and also for creating a minimally rigid post-merged formation from rigid pre-merged sub-formations.

The paper is organized as follows: We address the minimal cover problem in §II. Splitting rigid formations and merging rigid sub-formations are addressed in §III and §IV, respectively.

II. The Minimal Cover Problem

Before giving the definition of the minimal cover problem, we state our assumption:

Assumption: Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ represent the underlying graph of each post-split formation in the splitting problem in d-space (d=2,3), and the union of the underlying graphs of pre-merged sub-formations in the merging problem in d-space (d=2,3). In splitting, we assume that $\mathbb{G} = (\mathcal{V}, \mathcal{L}) \subseteq \bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{L}})$, where $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{L}})$ is a graph created by a Henneberg sequence in d-space (d=2,3) as explained in the companion paper [5]. In splitting, $\bar{\mathbb{G}}$ refers to the pre-split rigid formation created by a Henneberg sequence. In merging, we assume that each pre-merged rigid subgraph is created by a Henneberg sequence.

We note that the assumption is not a limitation on pre-split and pre-merged graphs in 2-dimensional space since there is a Henneberg sequence for all minimally rigid graphs in 2-dimensional space. However, the assumption is a limitation on pre-split and pre-merged graphs in 3-dimensional space since it is currently unknown whether there is a Henneberg sequence for all minimally rigid graphs in 3-dimensional space as explained in the companion paper [5]. The *minimal cover problem* is to find a set of new edges to be inserted into graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ so that the resulting graph $\mathbb{G}^* = (\mathcal{V}, \mathcal{L}^*)$ after insertions, is minimally rigid. Note that $\mathbb{G}$ and $\mathbb{G}^*$ have the same vertex set. We have the following lemma.

Lemma 1. *The edge set of a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ that satisfies the assumption of the minimal cover problem is a set of independent edges.*

We refer the reader to Eren [2] and Eren et al. [3] for the proofs of the lemmas and theorems in this paper.

The minimal cover problem can be posed using concepts from lattice theory. Let $\mathcal{G}$ denote the set of all simple graphs $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ with vertex set $\mathcal{V}$ including the graph with empty edge set which is called the *edgeless graph*. *Containment*, denoted by $\subset$, is a relation on $\mathcal{G}$ such that $\mathbb{G}_1 = (\mathcal{V}, \mathcal{L}_1)$ is contained in $\mathbb{G}_2 = (\mathcal{V}, \mathcal{L}_2)$ if $\mathcal{L}_1 \subset \mathcal{L}_2$. Containment is a partial ordering on $\mathcal{G}$, and the complete graph and edgeless graph are $\mathcal{G}$'s largest and smallest elements with respect to this ordering (see for example MacLane and Birkhoff [6]). Every graph $\mathbb{G}$ in $\mathcal{G}$ is contained in at least one rigid graph. We are interested in minimally rigid graphs in $\mathcal{G}$ that contain $\mathbb{G}$, and we call such a minimally rigid graph a *minimal cover*. The set of edges we want to preserve is represented by $\mathcal{L}$.

We can define the minimal cover problem in terms of partial ordering and graph rigidity. Suppose that a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L}) \in \mathcal{G}$, which satisfies the assumption of the minimal cover problem, is given. The minimal cover problem is to find some $\mathbb{G}^* = (\mathcal{V}, \mathcal{L}^*) \in \mathcal{G}$ such that $\mathbb{G} \subset \mathbb{G}^*$ and $\mathbb{G}^*$ is minimally rigid. In other words, we want to find a set of new edges, namely $\mathcal{L}_{new}$, between the vertices of $\mathcal{V}$ to add to the set $\mathcal{L}$ such that the resulting graph $\mathbb{G}^* = (\mathcal{V}, \mathcal{L}^*)$ is minimally rigid, where $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}_{new}$. Note that $\mathbb{G}^*$ is not necessarily unique.

Generic rigidity is directly related to the rank of a matrix [5]. As such, it has all the "exchange properties" associated with the independence of rows of a matrix [8]. For example, we note that minimally rigid graphs are also maximally independent graphs, corresponding to bases in vector spaces as minimal spanning sets and maximal independent sets. Given any independent set of edges $\mathcal{I}$ which is a subset of a basis (maximally independent set of edges) $\mathcal{B}$ for a vertex set $\mathcal{V}$, a set of edges $\mathcal{J}$ is a minimal cover of $\mathcal{I}$, if the union of $\mathcal{I}$ and $\mathcal{J}$ is a (new) basis $\mathcal{B}'$ for $\mathcal{V}$.

One crude approach to solve the minimal cover problem is based on the "generate and test" method. It is as follows: Given a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L}) \in \mathcal{G}$, we test whether with the addition of a new edge e, the graph $\mathbb{G} = (\mathcal{V}, \mathcal{L} \cup \{e\})$ is independent or not. If it is then we add e to $\mathcal{L}$. For testing independence, one could contemplate picking coordinate positions for the vertices at random, and forming a numerical rigidity matrix and testing additional rows of it. We repeat adding such new edges until we have a set of $2n-3$ independent edges in 2-dimensional space, and $3n-6$ independent edges in 3-dimensional space, where n is the number of vertices. The resulting graph with the vertex set $\mathcal{V}$ and those independent edges is the minimal cover of $\mathbb{G}$. This approach works based on random trials. If a new randomly generated edge gives us a set of independent edges then it turns out to be a success, otherwise it is a failure.

A. Planar Case

We present a systematic strategy to solve the planar minimal cover problem. We note that the knowledge of the original Henneberg sequence (i.e. the sequence used to create $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{L}})$) is needed to solve the minimal cover problem in 3-dimensional space. The resulting algorithm has linear time complexity. On the other hand, this information is not needed in 2-dimensional space. If we use this information of the original Henneberg sequence in 2-dimensional space, we have a linear time algorithm. If we do not use it, we have an exponential time algorithm. In this section, which is on planar case, we present the strategy that does not use the information of the original Henneberg sequence. In the next section, which is on spatial case, we present the strategy that uses the information of the original Henneberg sequence. First, we state the following lemma, then we introduce two types of reduction steps that will be used in the sequel.

Lemma 2. *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ denote a graph satisfying the assumption of the minimal cover problem in 2-dimensional space. Then there exists a vertex of degree 3 or less in $\mathbb{G}$.*

Reduction Step - Type I: Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph satisfying the assumption of the minimal cover problem in 2-dimensional space. Let i be a vertex of degree $\rho(i)$, where $\rho(i) \in \{0, 1, 2\}$. Suppose that we create a set of $2 - \rho(i)$ new edges incident to i and denote this set of new edges by $\mathcal{L}_{i_{new}}$. Let $\mathbb{G}' = (\mathcal{V}, \mathcal{L}')$ denote this new graph where $\mathcal{L}' = \mathcal{L} \cup \mathcal{L}_{i_{new}}$. We register $\mathcal{L}_{i_{new}}$ to use in subsequent steps. It can be easily verified that $\mathcal{L}'$ is independent. Now let us remove i and all the edges incident to i in $\mathbb{G}' = (\mathcal{V}, \mathcal{L}')$. Let $\mathbb{G}'' = (\mathcal{V}', \mathcal{L}'')$ denote this *reduced graph* where $\mathcal{V}' = \mathcal{V} \setminus \{i\}$ and $\mathcal{L}'' = \mathcal{L}' \setminus \{\text{all edges incident to } i\}$. It can again be verified that the edge set of $\mathbb{G}'' = (\mathcal{V}', \mathcal{L}'')$ is also independent. Therefore there exists a vertex of degree 3 or less in $\mathbb{G}''$ by Lemma 2.

Reduction Step - Type II: Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph satisfying the assumption of the minimal cover problem in 2-dimensional space. Let i be a vertex of degree 3 and adjacent to a set of vertices denoted by $\mathcal{N}_i$. We remove i and its three edges and create a new edge (precisely one edge) between arbitrary vertices in $\mathcal{N}_i$ forming a reduced graph $\mathbb{G}' = (\mathcal{V}', \mathcal{L}')$ where $\mathcal{V}' = \mathcal{V} \setminus \{i\}$ and $\mathcal{L}' = \mathcal{L} \setminus \{$ all edges incident to $i\} \cup \{$ a new edge between the vertices in $\mathcal{N}_i\}$ such that $\mathcal{L}'$ is a set of independent edges. [1] Therefore there exists a vertex of degree 3 or less in $\mathbb{G}'$ by Lemma 2.

[1]The existence of such a new edge can be seen from the proof of the edge splitting operation in Whiteley [8].

Now we present a reduction sequence in which the two types of reduction steps previously described are used as main steps. This sequence is used to reduce a set of independent edges down to a set of two vertices connected by an edge.

Reduction Sequence: Suppose that a graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ satisfying the assumption of the minimal cover problem in 2-dimensional space is given. From Lemma 2 it follows that there exists a vertex of degree 3 or less in $\mathbb{G}$. Hence, at least one of the reduction steps (type I or type II) can be applied to $\mathbb{G}$. Note that the reduced graphs we obtain after applying any one of these reduction steps are also independent. We apply these two types of steps repeatedly until we are left with only two vertices. While we apply the reduction steps to the vertices, we number those vertices in a descending order as n, $n-1$, $n-2$, $\dots$, 4, 3. For example, we number the vertex removed in the first reduction step with n, we number the vertex removed at the second reduction step with $n-1$ and so on until we are left with two vertices. The last vertex on which a reduction step is applied is numbered with 3. Then the very last two remaining vertices are numbered with indices 2 and 1. Each time we apply the reduction step type I on a vertex i, we keep registering its new set of edges $\mathcal{L}_{i_{new}}$ as described in reduction step type I. Depending on the initial set of independent edges, there may or may not be left an edge between the last two vertices after the execution of the reduction sequence. If there is no edge between them at the end of the reduction sequence, we create such an edge and register it as $\mathcal{L}_{2_{new}}$. If there is already an edge between vertices indexed by 1 and 2, then we register $\mathcal{L}_{2_{new}} = \emptyset$. The union of the registered sets of the new edges is $\mathcal{L}_{new} = \bigcup_i \mathcal{L}_{i_{new}}$ where i denotes the label of the vertices removed with a type I reduction step and the vertex with index 2.

Theorem 3. *(Planar Minimal Cover Theorem) Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph satisfying the assumption of the minimal cover problem in 2-dimensional space. Suppose that we apply the reduction sequence described above on $\mathbb{G}$ and find $\mathcal{L}_{new}$. Then $\mathbb{G}^* = (\mathcal{V}, \mathcal{L} \cup \mathcal{L}_{new})$ is a minimal cover of $\mathbb{G}$.*

Note that $\mathcal{L}_{new}$ obtained in the reduction sequence is not unique because the edges in $\mathcal{L}_{new}$ depend on the choice of the order of vertices in the reduction sequence. We will not consider algorithmic complexities in this paper. But one can argue that each time the reduction step type II is applied, there are up to three possible insertions for a new edge. This results in an exponential time algorithm for the reduction sequence. To overcome this problem, as we explained at the beginning of this section, the approach for the spatial case presented in the sequel can be easily applied to the planar case. This approach gives a linear time algorithm. Alternatively one can use a polynomial time algorithm called "the pebble game" for the planar case. There is a recent paper by Berg and Jordan [1] which addresses this kind of algorithms.

B. Spatial Case

The approach in the Planar Minimal Cover Theorem can be translated to 3-dimensional space with an additional condition in the order of the reduction sequence. The reason behind this condition is that if a reduction sequence similar to the planar case is applied then there is a possibility of reaching a graph with a set of vertices all of which are of degree 5. If the conjectures for adding a 5-valent vertex to minimally rigid graphs that we explain in the companion paper [5] are proven, then the reduction sequence presented for the planar case can be easily extended to 3-dimensional space. Since this is a long-standing unsolved problem, to get around those difficulties, we present an alternative reduction sequence with an additional condition in the ordering of reduction sequence. We have the following lemma in 3-dimensional space.

Lemma 4. *Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ denote a graph satisfying the assumption of the minimal cover problem in 3-dimensional space. Then there exists a vertex of $\mathbb{G}$ of degree 4 or less.*

The two reduction steps presented for the planar case can be directly extended for 3-dimensional space.

Reduction Step - Type I: Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph satisfying the assumption of the minimal cover problem in 3-dimensional space. Let i be a vertex of degree $\rho(i)$, where $\rho(i) \in \{0, 1, 2, 3\}$. Suppose that we create a set of $3 - \rho(i)$ new edges incident to i and denote this set of new edges by $\mathcal{L}_{i_{new}}$. Let $\mathbb{G}' = (\mathcal{V}, \mathcal{L}')$ denote this new graph where $\mathcal{L}' = \mathcal{L} \cup \mathcal{L}_{i_{new}}$. We register $\mathcal{L}_{i_{new}}$ to use in subsequent steps. It can be easily verified that $\mathcal{L}'$ is independent. Now let us remove i and all the edges incident to i in $\mathbb{G}' = (\mathcal{V}, \mathcal{L}')$. Let $\mathbb{G}'' = (\mathcal{V}', \mathcal{L}'')$ denote this *reduced graph* where $\mathcal{V}' = \mathcal{V} \setminus \{i\}$ and $\mathcal{L}'' = \mathcal{L}' \setminus \{$all edges incident to $i\}$. It can be easily verified that $\mathbb{G}'' = (\mathcal{V}', \mathcal{L}'')$ also satisfies the assumption of the minimal cover problem. Therefore there exists a vertex of degree 4 or less in $\mathbb{G}''$ by Lemma 4.

Reduction Step - Type II: Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph satisfying the assumption of the minimal cover problem in 3-dimensional space. Let i be a vertex of degree 4 and adjacent to a set of vertices denoted by $\mathcal{N}_i$. We remove i and its four edges and create a new edge (precisely one edge) between arbitrary vertices in $\mathcal{N}_i$ forming a reduced graph $\mathbb{G}' = (\mathcal{V}', \mathcal{L}')$ where $\mathcal{V}' = \mathcal{V} \setminus \{i\}$ and $\mathcal{L}' = \mathcal{L} \setminus \{$ all edges incident to $i\} \cup \{$ a new edge between the vertices in $\mathcal{N}_i\}$ such that $\mathbb{G}'$ satisfies the assumption of the minimal cover problem. [2] Therefore there exists a vertex of degree 4 or less in $\mathbb{G}'$ by Lemma 4.

Now we present the modified reduction sequence, in which the two types of reduction steps previously described are used as main steps. This sequence is used to reduce a graph satisfying the assumption of the minimal cover problem in 3-dimensional space down to a set of three vertices. It makes use of the order of vertices in the

[2] The existence of such an edge will become clear in the special reduction sequence presented in the sequel.

Henneberg sequence that was used to create the original graph $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{L}})$.

Reduction Sequence: Let $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{L}})$ be a minimally rigid graph created by the vertex addition and edge splitting operations in 3-dimensional space, where vertices are indexed $\{1, 2, \ldots, n\}$ with respect to their order of addition in the Henneberg sequence. If i denotes a vertex added by the edge splitting operation, then let e_i denote the edge removed in this operation. Let the graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be created by removing some of the edges and vertices of the graph $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{L}})$. To complete $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ to a minimally rigid graph, we do the following. Starting from the vertex with the highest index, we apply reductions steps type I and II repeatedly on the vertices with descending order of indices. For example, let i denote the vertex with the highest index in the remaining graph at some step in the reduction sequence. If i is of degree $\rho(i)$, where $\rho(i) \leq 3$, we apply reduction step type I. Each time we apply the reduction step type I on a vertex i, we keep registering its new set of edges $\mathcal{L}_{i_{new}}$ as described in reduction step type I. If the vertex i is of degree 4, we then apply reduction step type II by inserting the edge e_i. We continue this until three vertices are left. Depending on the initial set of independent edges, there may or may not be three edges left between the last three vertices after the execution of the reduction sequence. If there are not three edges between them at the end of the reduction sequence, we complete the number of edges to three and register them as $\mathcal{L}_{3_{new}}$. If there are already three edges between vertices labelled $1, 2$ and 3, then we register $\mathcal{L}_{3_{new}} = \emptyset$. The union of the registered sets of the new edges is $\mathcal{L}_{new} = \bigcup_i \mathcal{L}_{i_{new}}$ where i denotes the label of the vertices removed with a type I reduction step and the vertex with index 3.

Lemma 5. *At each reduction step, the vertex with the highest index is of degree 4 or less.*

Theorem 6. *(Spatial Minimal Cover Theorem) Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a graph satisfying the assumption of the minimal cover problem in 3-dimensional space. Suppose that we apply the reduction sequence described above on $\mathbb{G}$ and find $\mathcal{L}_{new}$. Then $\mathbb{G}^* = (\mathcal{V}, \mathcal{L} \cup \mathcal{L}_{new})$ is a minimal cover of $\mathbb{G}$.*

As in the planar case, $\mathcal{L}_{new}$ obtained in the reduction sequence is not unique because the edges in $\mathcal{L}_{new}$ depend on the choice in the reduction sequence.

III. Splitting Formations

To find a strategy for splitting a rigid formation into two rigid post-split sub-formations, it is convenient to introduce a suitable definition of the splitting problem in terms of graph rigidity. Let $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ be a minimally rigid graph. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ represent the two subsets of $\mathcal{V}$ such that $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$. Let $\mathcal{L}_1 \subset \mathcal{L}$ be the set of all edges whose both end-vertices are in $\mathcal{V}_1$ and $\mathcal{L}_2 \subset \mathcal{L}$ be the set of all edges whose both end-vertices are in $\mathcal{V}_2$. Let $\mathcal{L}_r = \mathcal{L} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)$ be the set of all edges whose one end-vertex is in $\mathcal{L}_1$ and the other end-vertex is in $\mathcal{L}_2$. Let $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$. When the graph $\mathbb{G} = (\mathcal{V}, \mathcal{L})$ is split into $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$, all edges in $\mathcal{L}_r$ are removed. The *splitting problem* is to find new sets of edges $\mathcal{L}_{1_{new}}$ to insert into $\mathbb{G}_1$ and $\mathcal{L}_{2_{new}}$ to insert into $\mathbb{G}_2$ such that the resulting graphs $\mathbb{G}_1^* = (\mathcal{V}_1, \mathcal{L}_1 \cup \mathcal{L}_{1_{new}})$ and $\mathbb{G}_2^* = (\mathcal{V}_2, \mathcal{L}_2 \cup \mathcal{L}_{2_{new}})$ are minimally rigid.

With the minimal cover problem in mind as defined in the previous section, the splitting problem reduces to finding the minimal covers of $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$. As detailed in the analysis in the previous section, the underlying graphs of the resulting post-split sub-formations determined by the reduction sequences are minimally rigid by Theorem 3 and Theorem 6.

IV. Merging Formations

As we did in the case of the splitting problem, we introduce a suitable definition of the merging problem in terms of graph rigidity. Let $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ be two minimally rigid graphs representing the underlying graphs of two minimally rigid point formations. The *merging problem* is to find a new set of edges $\mathcal{L}_{new}$ to insert between $\mathbb{G}_1$ and $\mathbb{G}_2$ by choosing one end-vertex in $\mathcal{V}_1$ and the other end-vertex in $\mathcal{V}_2$ such that the resulting graph $\mathbb{G}^* = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_{new})$ is minimally rigid. As in the case of splitting, the merging problem reduces to finding the minimal cover of $\mathbb{G}' = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{L}_1 \cup \mathcal{L}_2)$. We exemplify this in the sequel.

Example: Let $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ be two minimally rigid graphs in 2-dimensional space. Suppose that we apply the reduction sequence first on $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ as described in §II-A. Since $\mathbb{G}_2$ is minimally rigid, we obtain two vertices connected by an edge at the end of the reduction sequence on $\mathbb{G}_2$ without inserting any new edges. We denote this remaining edge by (i, j). At this intermediate step, we are left with $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and the edge (i, j) on which we continue applying the reduction sequence. Suppose that we apply the reduction step type I on i, by inserting an edge (i, k) where $k \in \mathcal{V}_1$. Then we apply the reduction step type I on j by inserting the edges $(j, k), (j, l)$ or by inserting the edges $(j, l), (j, r)$ where $k, l, r \in \mathcal{V}_1$. After applying those reduction steps on i, j, we are left with the graph $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ only, and we apply the reduction sequence on $\mathbb{G}_1$ without inserting any new edges. Therefore the two possible combinations of merging $\mathbb{G}_1$ and $\mathbb{G}_2$ create the set of edges $(i, k), (j, k), (j, l)$ or $(i, k), (j, l), (j, r)$. We depict these two different strategies of merging two rigid sub-formations in Figures 3a and b. $\mathbb{G}_1$ and $\mathbb{G}_2$ denote the underlying graphs of these two rigid point formations.

We can pursue a different strategy. Let $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ be two minimally rigid graphs in 2-dimensional space. Suppose that we apply the reduction sequence first on $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ as described in §II-A until we are left with three vertices i, j, k connected by three edges $(i,j),(i,k),(j,k)$. Since $\mathbb{G}_2$ is minimally rigid, we obtain these three vertices connected by three edges at the end of the reduction sequence on $\mathbb{G}_2$ without inserting any new edges. First, let us insert a new edge (i,r) where $r \in \mathcal{V}_1$. Now, i is of degree 3. Then, we can apply the reduction step type II on i by inserting (j,r). Note that while (i,r) is a new inserted edge which needs to be registered in the reduction sequence process, (j,r) is not. (j,r) is simply a result of reduction step type II. Now let us insert a new edge (j,s) to make j of degree 3 where $s \in \mathcal{V}_1$. Then we can apply the reduction step type II on j by inserting an edge (k,s). Note again that (j,s) is a new edge which needs to be registered in the reduction sequence process but (k,s) is not. Then we apply the reduction step type I on k by inserting a new edge (k,t) where $t \in \mathcal{V}_1$. At this stage, we are only left with $\mathbb{G}_1$. Then we continue the reduction sequence on $\mathbb{G}_1$. Since $\mathbb{G}_1$ is minimally rigid, the reduction sequence can be applied without inserting any other extra edges. Therefore another strategy for merging $\mathbb{G}_1$ and $\mathbb{G}_2$ creates the set of edges $(i,r),(j,s),(k,t)$. We depict this strategy in Figure 3c.

It can be verified that six links are needed in 3-dimensional space to merge two minimally rigid sub-formations to form a minimally rigid post-merged formation provided that we use at least three points in each sub-formation as an end-point of these six new links. Here, we use the solution of the minimal cover problem to determine these new links. We depict three possible strategies to determine these six links (Figures 4a, b, c) and explain one of them in detail (Figure 4a). One can find different strategies by using a modified version of the idea presented here by selecting a different combination of vertices and reduction steps type I and II. Here, we do not go into each such combinations because the idea is essentially similar to the planar version.

Let $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ be two minimally rigid graphs in 3-dimensional space. Suppose that we apply the reduction sequence first on $\mathbb{G}_2 = (\mathcal{V}_2, \mathcal{L}_2)$ as described in §II-A. Since $\mathbb{G}_2$ is minimally rigid, we obtain three vertices connected by three edges at the end of the reduction sequence on $\mathbb{G}_2$ without inserting any new edges. Let us denote this remaining vertices by i,j,k and edges by $(i,j),(i,k),(j,k)$. At this intermediate step, we are left with

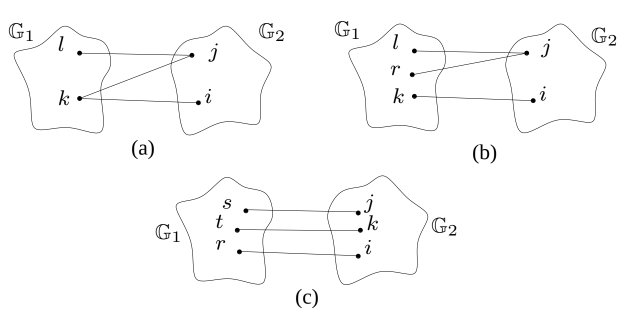

Fig. 3. Merging rigid sub-formations in 2-dimensional space.

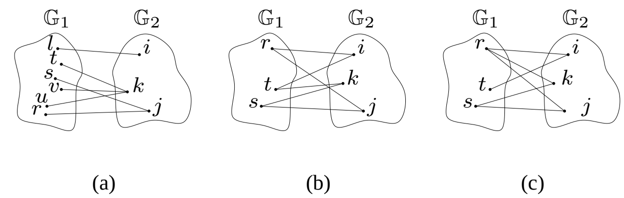

Fig. 4. Three different strategies of merging rigid sub-formations in 3-dimensional space.

$\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ and the edge $(i,j),(i,k),(j,k)$ on which we continue applying the reduction sequence. Suppose that we apply the reduction step type I on the vertex i, by inserting an edge (i,l) where $l \in \mathcal{V}_1$. Then we apply the reduction step type I on the vertex j by inserting the edges $(j,r),(j,s)$ where $r,s \in \mathcal{V}_1$. Then we apply the reduction step type I on the vertex k by inserting the edges $(k,t),(k,u),(k,v)$ where $t,u,v \in \mathcal{V}_1$. After applying those reduction steps on vertices i,j,k, we are left with the graph $\mathbb{G}_1 = (\mathcal{V}_1, \mathcal{L}_1)$ only, and we apply the reduction sequence on $\mathbb{G}_1$ without inserting any new edges. Therefore the two possible combinations of merging $\mathbb{G}_1$ and $\mathbb{G}_2$ create the set of edges $(i,l),(j,r),(j,s),(k,t),(k,u),(k,v)$. We depict this strategy of merging two rigid sub-formations in Figure 4(a). $\mathbb{G}_1$ and $\mathbb{G}_2$ denote the underlying graphs of these two rigid point formations.

V. Concluding Remarks

First, we note that the reduction strategies developed in this paper can be extended to include other types of operations such as vertex splitting. Second, solving the minimal cover problem for rigid formations, which are not necessarily minimally rigid, is an open problem.

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