

## DIRECTIVE WORDS OF EPISTURMIAN WORDS: EQUIVALENCES AND NORMALIZATION

AMY GLEN<sup>1,2</sup>, FLORENCE LEVÉ<sup>3</sup> AND GWÉNAËL RICHOMME<sup>3</sup>

**Abstract.** Episturmian morphisms constitute a powerful tool to study episturmian words. Indeed, any episturmian word can be infinitely decomposed over the set of pure episturmian morphisms. Thus, an episturmian word can be defined by one of its morphic decompositions or, equivalently, by a certain directive word. Here we characterize pairs of words directing the same episturmian word. We also propose a way to uniquely define any episturmian word through a normalization of its directive words. As a consequence of these results, we characterize episturmian words having a unique directive word.

**Mathematics Subject Classification.** 68R15.

### 1. INTRODUCTION

Since the seminal works of Morse and Hedlund [21], Sturmian words have been widely studied. Their beautiful properties are related to many fields like number theory, geometry, dynamical systems, and combinatorics on words (see [1,3,20,23] for recent surveys). These infinite words, which are defined on a binary alphabet, have numerous equivalent definitions and characterizations.

Many recent works have been devoted to generalizations of Sturmian words to arbitrary finite alphabets. Two very interesting generalizations are very close: the

---

*Keywords and phrases.* Episturmian word, Sturmian word, Arnoux-Rauzy sequence, episturmian morphism, directive word.

<sup>1</sup> LaCIM, Université du Québec à Montréal, C.P. 8888, succursale Centre-ville, Montréal, Québec, H3C 3P8, Canada

<sup>2</sup> The Mathematics Institute, Reykjavík University, Kringlan 1, IS-103 Reykjavík, Iceland; [amy.glen@gmail.com](mailto:amy.glen@gmail.com)

<sup>3</sup> Université de Picardie Jules Verne, Laboratoire MIS (Modélisation, Information, Systèmes), 33 rue Saint Leu, 80039 Amiens Cedex 1, France; [\[florence.leve;gwenael.richomme\]@u-picardie.fr](mailto:[florence.leve;gwenael.richomme]@u-picardie.fr)

*Arnoux-Rauzy sequences* (e.g., see [2,14,23,30]) and *episturmian words* (e.g., see [5,13,15]). The first of these two families is a particular subclass of the second one. More precisely, the family of episturmian words is composed of the Arnoux-Rauzy sequences, images of the Arnoux-Rauzy sequences by *episturmian morphisms*, and certain periodic infinite words. In the binary case, Arnoux-Rauzy sequences are exactly the Sturmian words whereas episturmian words include all recurrent *balanced* words, that is, periodic balanced words and Sturmian words (see [10,22,29] for recent results relating episturmian words to the balanced property). See also [9] for a recent survey on episturmian theory.

Episturmian morphisms play a central role in the study of these words. Introduced first as a generalization of Sturmian morphisms, Justin and Pirillo [13] showed that they are exactly the morphisms that preserve the aperiodic episturmian words. They also proved that any episturmian word is the image of another episturmian word by some so-called *pure episturmian morphism*. Even more, any episturmian word can be infinitely decomposed over the set of pure episturmian morphisms (see Sect. 2.4). This last property allows an episturmian word to be defined by one of its morphic decompositions or, equivalently, by a certain *directive word*, which is an infinite sequence of rules for decomposing the given episturmian word by morphisms. In consequence, many properties of episturmian words can be deduced from properties of episturmian morphisms. This approach is used for instance in [4,8,16,28–30] and of course in the papers by Justin *et al.* In Section 2, we recall useful results on episturmian words and explain the vision of morphic decompositions and directive words introduced by Justin and Pirillo in [13].

An episturmian word can have several directive words. The question: “When do two different words direct the same episturmian word?” was considered in [15]. Using a block-equivalence notion for directive words, Justin and Pirillo provided several results to answer this question in most cases (see Sect. 3). In Section 4, we state a complete result characterizing the form of words directing the same episturmian word, without using block-equivalence.

In [4], Berthé *et al.* showed that any Sturmian word has a unique directive word with some particular properties. In [18], the second and third authors rephrased this result and used it to characterize all quasiperiodic Sturmian words. In Section 5, we extend this result to all episturmian words by introducing a way to normalize the directive words of an episturmian word so that any episturmian word can be defined uniquely by its *normalized directive word*, defined by some factor avoidance (Thm. 5.2). This result was previously presented at the Sixth International Conference on Words [17] to characterize all quasiperiodic episturmian words (see also [11]).

As an application of the previous results, we end this paper with a characterization of episturmian words having a unique directive word.

## 2. EPISTURMIAN WORDS AND MORPHISMS

We assume the reader is familiar with combinatorics on words and morphisms (e.g., see [19,20]). In this section, we recall some basic definitions and properties relating to episturmian words which are needed later in the paper. For the most part, we follow the notation and terminology of [5,10,13,15].

## 2.1. NOTATION AND TERMINOLOGY

Let  $\mathcal{A}$  denote a finite *alphabet*. A finite *word* over  $\mathcal{A}$  is a finite sequence of letters from  $\mathcal{A}$ . The *empty word*  $\varepsilon$  is the empty sequence. Under the operation of concatenation, the set  $\mathcal{A}^*$  of all finite words over  $\mathcal{A}$  is a *free monoid* with identity element  $\varepsilon$  and set of generators  $\mathcal{A}$ . The set of *non-empty* words over  $\mathcal{A}$  is the *free semigroup*  $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$ .

Given a finite word  $w = x_1x_2 \dots x_m \in \mathcal{A}^+$  with each  $x_i \in \mathcal{A}$ , the *length* of  $w$  is  $|w| = m$ . The length of the empty word is 0. By  $|w|_a$  we denote the number of occurrences of the letter  $a$  in the word  $w$ . If  $|w|_a = 0$ , then  $w$  is said to be *a-free*. For any integer  $p \geq 1$ , the  $p$ -th power of  $w$  is the word  $w^p$  obtained by concatenating  $p$  occurrences of  $w$ .

A (right) *infinite word*  $\mathbf{x}$  is a sequence indexed by  $\mathbb{N}^+$  with values in  $\mathcal{A}$ , i.e.,  $\mathbf{x} = x_1x_2x_3 \dots$  with each  $x_i \in \mathcal{A}$ . The set of all infinite words over  $\mathcal{A}$  is denoted by  $\mathcal{A}^\omega$ . Given a non-empty finite word  $v$ , we denote by  $v^\omega$  the infinite word obtained by concatenating  $v$  with itself infinitely many times. An *ultimately periodic* infinite word is a word that can be written as  $uv^\omega = uvvv \dots$ , for some  $u, v \in \mathcal{A}^*$ ,  $v \neq \varepsilon$ . If  $u = \varepsilon$ , then such a word is *periodic*. An infinite word that is not ultimately periodic is said to be *aperiodic*. For easier reading, infinite words are hereafter typed in boldface to distinguish them from finite words.

Given a set  $X$  of finite words,  $X^*$  (resp.  $X^\omega$ ) is the set of all finite (resp. infinite) words that can be obtained by concatenating words of  $X$ . The empty word  $\varepsilon$  belongs to  $X^*$ .

A finite word  $w$  is a *factor* of a finite or infinite word  $z$  if  $z = uwv$  for some words  $u, v$  (where  $v$  is infinite iff  $z$  is infinite). Further,  $w$  is called a *prefix* (resp. *suffix*) of  $z$  if  $u = \varepsilon$  (resp.  $v = \varepsilon$ ). We use the notation  $p^{-1}w$  (resp.  $ws^{-1}$ ) to indicate the removal of a prefix  $p$  (resp. suffix  $s$ ) of the word  $w$ .

The *alphabet* of a word  $w$ , denoted by  $\text{Alph}(w)$  is the set of letters occurring in  $w$ , and if  $w$  is infinite, we denote by  $\text{Ult}(w)$  the set of all letters occurring infinitely often in  $w$ .

## 2.2. EPISTURMIAN WORDS

In this paper, our vision of episturmian words will be the characteristic property stated in Theorem 2.1. Nevertheless, to give an idea of what an episturmian word is, let us give one of the equivalent definitions of an episturmian word provided in [5]. Before doing so, we recall that a factor  $u$  of an infinite word  $\mathbf{w} \in \mathcal{A}^\omega$  is *right* (resp. *left*) *special* if  $ua, ub$  (resp.  $au, bu$ ) are factors of  $\mathbf{w}$  for some letters

$a, b \in \mathcal{A}$ ,  $a \neq b$ . We recall also that the *reversal*  $\tilde{w}$  of a finite word  $w$  is its mirror image: if  $w = x_1 \dots x_{m-1}x_m$ , then  $\tilde{w} = x_mx_{m-1} \dots x_1$ .

An infinite word  $\mathbf{t} \in \mathcal{A}^\omega$  is *episturmian* if its set of factors is closed under reversal and  $\mathbf{t}$  has at most one right (or equivalently left) special factor of each length. Moreover, an episturmian word is *standard* if all of its left special factors are prefixes of it.

In the initiating paper [5], episturmian words were defined in two steps. Standard episturmian words were first introduced and studied as a generalization of standard Sturmian words. (Note that in the rest of this paper, we refer to a standard episturmian word as an *epistandard word*, for simplicity.) Then an episturmian word was defined as an infinite word having exactly the same set of factors as some epistandard word.

Moreover, it was proved in [5] that any episturmian word is *recurrent*, that is, all of its factors occur infinitely often (actually episturmian words are uniformly recurrent but this will not be needed here). Thus ultimately periodic episturmian words are (purely) periodic.

### 2.3. EPISTURMIAN MORPHISMS

To study episturmian words, Justin and Pirillo [13] introduced *episturmian morphisms*. In particular they proved that these morphisms (defined below) are precisely the morphisms that preserve the set of aperiodic episturmian words.

Let us recall that given an alphabet  $\mathcal{A}$ , a *morphism*  $f$  on  $\mathcal{A}$  is a map from  $\mathcal{A}^*$  to  $\mathcal{A}^*$  such that  $f(uv) = f(u)f(v)$  for any words  $u, v$  over  $\mathcal{A}$ . A morphism on  $\mathcal{A}$  is entirely defined by the images of letters in  $\mathcal{A}$ . All morphisms considered in this paper will be non-erasing: the image of any non-empty word is never empty. Hence the action of a morphism  $f$  on  $\mathcal{A}^*$  can be naturally extended to infinite words; that is, if  $\mathbf{x} = x_1x_2x_3 \dots \in \mathcal{A}^\omega$ , then  $f(\mathbf{x}) = f(x_1)f(x_2)f(x_3) \dots$ .

In what follows, we will denote the composition of morphisms by juxtaposition as for concatenation of words.

Episturmian morphisms are the compositions of the permutation morphisms (the morphisms  $f$  such that  $f(\mathcal{A}) = \mathcal{A}$ ) and the morphisms  $L_a$  and  $R_a$  where, for all  $a \in \mathcal{A}$ :

$$L_a : \begin{cases} a & \mapsto a \\ b & \mapsto ab \end{cases}, \quad R_a : \begin{cases} a & \mapsto a \\ b & \mapsto ba \end{cases} \quad \text{for all } b \neq a \text{ in } \mathcal{A}.$$

Here we will work only on *pure* episturmian morphisms, i.e., morphisms obtained by composition of elements of the sets:

$$\mathcal{L}_{\mathcal{A}} = \{L_a \mid a \in \mathcal{A}\} \quad \text{and} \quad \mathcal{R}_{\mathcal{A}} = \{R_a \mid a \in \mathcal{A}\}.$$

*Note.* In [13], the morphism  $L_a$  (resp.  $R_a$ ) is denoted by  $\psi_a$  (resp.  $\bar{\psi}_a$ ). We adopt the current notation to emphasize the action of  $L_a$  (resp.  $R_a$ ) when applied to a

word, which consists in placing an occurrence of the letter  $a$  on the left (resp. right) of each occurrence of any letter different from  $a$ .

*Epistandard morphisms* are the morphisms obtained by concatenation of morphisms in  $\mathcal{L}_{\mathcal{A}}$  and permutations on  $\mathcal{A}$ . Likewise, the *pure episturmian morphisms* (resp. *pure epistandard morphisms*) are the morphisms obtained by concatenation of morphisms in  $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$  (resp. in  $\mathcal{L}_{\mathcal{A}}$ ). Note that the episturmian morphisms are exactly the *Sturmian morphisms* when  $\mathcal{A}$  is a 2-letter alphabet.

All episturmian morphisms are injective on both finite and infinite words. The monoid of episturmian morphisms is *left cancellative* (see [26], Lem. 7.2) which means that for any episturmian morphisms  $f, g, h$ , if  $fg = fh$  then  $g = h$ . Note that this fact, which is a by-product of the injectivity, can also be seen as a consequence of the invertibility of these morphisms (see [7,12,26,32]).

#### 2.4. MORPHIC DECOMPOSITION OF EPISTURMIAN WORDS

Justin and Pirillo [13] proved the following insightful characterizations of epistandard and episturmian words (see Thm. 2.1 below), which show that any episturmian word can be *infinitely decomposed* over the set of pure episturmian morphisms.

The statement of Theorem 2.1 needs some extra definitions and notation.

First we define the following new alphabet,  $\bar{\mathcal{A}} = \{\bar{x} \mid x \in \mathcal{A}\}$ . A letter  $\bar{x}$  is considered to be  $x$  with *spin R*, whilst  $x$  itself has *spin L*. A finite or infinite word over  $\mathcal{A} \cup \bar{\mathcal{A}}$  is called a *spinned* word. To ease the reading, we sometimes call a letter with *spin L* (resp. *spin R*) an *L-spinned* (resp. *R-spinned*) letter. By extension, an *L-spinned* (resp. *R-spinned*) word is a word having only letters with *spin L* (resp. *spin R*).

The *opposite*  $\bar{w}$  of a finite or infinite spinned word  $w$  is obtained from  $w$  by exchanging all spins in  $w$ . For instance, if  $v = ab\bar{a}$ , then  $\bar{v} = \bar{a}\bar{b}a$ . When  $v \in \mathcal{A}^+$ , then its opposite  $\bar{v} \in \bar{\mathcal{A}}^+$  is an *R-spinned* word and we set  $\bar{\varepsilon} = \varepsilon$ . Note that, given a finite or infinite word  $w = w_1w_2\dots$  over  $\mathcal{A}$ , we sometimes denote  $\check{w} = \check{w}_1\check{w}_2\dots$  any spinned word such that  $\check{w}_i = w_i$  if  $w_i$  has *spin L* and  $\check{w}_i = \bar{w}_i$  if  $w_i$  has *spin R*. Such a word  $\check{w}$  is called a *spinned version* of  $w$ . Sometimes we denote by  $\hat{w}$  a spinned version of a word  $w$  that is different from  $\check{w}$ .

*Note.* In Justin and Pirillo’s original papers, spins are 0 and 1 instead of *L* and *R*. It is convenient here to change this vision of the spins because of the relationship with episturmian morphisms, which we now recall.

For  $a \in \mathcal{A}$ , let  $\mu_a = L_a$  and  $\mu_{\bar{a}} = R_a$ . This operator  $\mu$  can be naturally extended (as done in [13]) to a morphism from the free monoid  $(\mathcal{A} \cup \bar{\mathcal{A}})^*$  to a pure episturmian morphism: for a spinned finite word  $\check{w} = \check{w}_1\dots\check{w}_n$  over  $\mathcal{A} \cup \bar{\mathcal{A}}$ ,  $\mu_{\check{w}} = \mu_{\check{w}_1}\dots\mu_{\check{w}_n}$  ( $\mu_{\varepsilon}$  is the identity morphism). We say that the word  $w$  *directs* or is a *directive word* of the morphism  $\mu_w$ . The following result extends the notion of directive words to infinite episturmian words.

**Theorem 2.1** [13].

- (i) An infinite word  $\mathbf{s} \in \mathcal{A}^\omega$  is epistandard if and only if there exist an infinite word  $\Delta = x_1x_2x_3\dots$  over  $\mathcal{A}$  and an infinite sequence  $(\mathbf{s}^{(n)})_{n \geq 0}$  of infinite words such that  $\mathbf{s}^{(0)} = \mathbf{s}$  and for all  $n \geq 1$ ,  $\mathbf{s}^{(n-1)} = \mu_{x_n}(\mathbf{s}^{(n)})$  ( $= L_{x_n}(\mathbf{s}^{(n)})$ ).
- (ii) An infinite word  $\mathbf{t} \in \mathcal{A}^\omega$  is episturmian if and only if there exist a spinned infinite word  $\check{\Delta} = \check{x}_1\check{x}_2\check{x}_3\dots$  over  $\mathcal{A} \cup \bar{\mathcal{A}}$  and an infinite sequence  $(\mathbf{t}^{(n)})_{n \geq 0}$  of recurrent infinite words such that  $\mathbf{t}^{(0)} = \mathbf{t}$  and for all  $n \geq 1$ ,  $\mathbf{t}^{(n-1)} = \mu_{\check{x}_n}(\mathbf{t}^{(n)})$ .

For any epistandard word (resp. episturmian word)  $\mathbf{t}$  and  $L$ -spinned infinite word  $\Delta$  (resp. spinned infinite word  $\check{\Delta}$ ) satisfying the conditions of the above theorem, we say that  $\Delta$  (resp.  $\check{\Delta}$ ) is a (*spinned*) *directive word* for  $\mathbf{t}$  or that  $\mathbf{t}$  is *directed by*  $\Delta$  (resp.  $\check{\Delta}$ ). Notice that this directive word is exactly the one that arises from the equivalent definition of epistandard words that uses *palindromic closure* [5,9,13] and, in the binary case, it is related to the continued fraction of the slope of the straight line represented by a standard word (see [20]). It follows immediately from Theorem 2.1 that, with the notation of case (ii), each  $\mathbf{t}^{(n)}$  is an episturmian word directed by  $\check{x}_{n+1}\check{x}_{n+2}\dots$ .

The natural question: “Does any spinned infinite word direct a unique episturmian word?” is answered in [13]:

**Proposition 2.2** ([13], Prop. 3.11).

- (1) Any spinned infinite word  $\check{\Delta}$  having infinitely many  $L$ -spinned letters directs a unique episturmian word beginning with the left-most letter having spin  $L$  in  $\check{\Delta}$ .
- (2) Any  $R$ -spinned infinite word  $\check{\Delta}$  directs exactly  $|\text{Ult}(\Delta)|$  episturmian words.
- (3) Let  $\check{\Delta}$  be an  $R$ -spinned infinite word, and let  $a$  be a letter such that  $\bar{a} \in \text{Ult}(\check{\Delta})$ . Then  $\check{\Delta}$  directs exactly one episturmian word starting with  $a$ .

*Note.* In [13], item (3) was stated in the more general case where  $\check{\Delta}$  is ultimately  $R$ -spinned. In this case,  $\check{\Delta}$  still directs exactly one episturmian word for each letter  $\bar{a}$  in  $\text{Ult}(\check{\Delta})$ , but contrary to what is written in [13], nothing can be said on its first letter.

As a consequence of the previous proposition and part (i) of Theorem 2.1, any  $L$ -spinned infinite word directs a unique epistandard word. The following important remark links the two parts of Theorem 2.1.

**Remark 2.3** [13]. If  $\mathbf{t}$  is an episturmian word directed by a spinned version  $\check{\Delta}$  of an  $L$ -spinned infinite word  $\Delta$ , then  $\mathbf{t}$  has exactly the same set of factors as the (unique) epistandard word  $\mathbf{s}$  directed by  $\Delta$ .

Moreover (with the same notation as in the previous remark):

**Remark 2.4.** The episturmian word  $\mathbf{t}$  is periodic if and only if the epistandard word  $\mathbf{s}$  is periodic, and this holds if and only if there is only one letter occurring infinitely often in  $\Delta$ , that is,  $|\text{Ult}(\Delta)| = 1$  (see [13], Prop. 2.9). More precisely, a

periodic episturmian word takes the form  $(\mu_{\check{w}}(x))^\omega$  for some finite spinned word  $\check{w}$  and letter  $x$ .

*Note.* Sturmian words are precisely the aperiodic episturmian words on a 2-letter alphabet.

When an episturmian word is aperiodic, we have the following fundamental link between the words  $(\mathbf{t}^{(n)})_{n \geq 0}$  and the spinned infinite word  $\check{\Delta}$  occurring in Theorem 2.1: if  $a_n$  is the first letter of  $\mathbf{t}^{(n)}$ , then  $\mu_{\check{x}_1 \dots \check{x}_n}(a_n)$  is a prefix of  $\mathbf{t}$  and the sequence  $(\mu_{\check{x}_1 \dots \check{x}_n}(a_n))_{n \geq 1}$  is not ultimately constant (since  $\check{\Delta}$  is not ultimately constant), then  $\mathbf{t} = \lim_{n \rightarrow \infty} \mu_{\check{x}_1 \dots \check{x}_n}(a_n)$ . This fact is a slight generalization of a result of Risley and Zamboni [30], Proposition. III.7, on *S-adic representations* for characteristic Arnoux-Rauzy sequences. See also the recent paper [4] for S-adic representations of Sturmian words. Note that *S-adic dynamical systems* were introduced by Ferenczi [6] as *minimal dynamical systems* (e.g., see [23]) generated by a finite number of substitutions. In the case of episturmian words, the notion itself is actually a reformulation of the well-known *Rauzy rules*, as studied in [25].

To anticipate next sections, let us also observe:

**Remark 2.5** [13]. If an aperiodic episturmian word is directed by two spinned words  $\Delta_1$  and  $\Delta_2$ , then  $\Delta_1$  and  $\Delta_2$  are spinned versions of the same  $L$ -spinned word.

This is no longer true for periodic episturmian words; for instance  $ab^\omega$  and  $\bar{b}a^\omega$  direct the same episturmian word  $(ab)^\omega = ababab \dots$ .

### 3. KNOWN RESULTS ON DIRECTIVE-EQUIVALENT WORDS

We have just seen an example of a periodic episturmian word that is directed by two different spinned infinite words. This situation holds also in the aperiodic case (see [13,15]). For example, the *Tribonacci word* (or *Rauzy word* [24]) is directed by  $(abc)^\omega$  and also by  $(abc)^n \bar{a}\bar{b}\bar{c}(a\bar{b}\bar{c})^\omega$  for each  $n \geq 0$ , as well as infinitely many other spinned words. More generally, by [13], any epistandard word has a unique  $L$ -spinned directive word but also has other directive words (see also [15] and Thm. 4.1).

We now consider in detail the following two questions. When do two different finite spinned words direct the same episturmian morphism? When do two different spinned infinite words direct the same unique episturmian word? We say that two different finite (resp. infinite) spinned words are *directive-equivalent* words if they direct the same episturmian morphism (resp. the same episturmian word).

In Section 3.1, we recall the characterizations of directive-equivalent finite spinned words. In Section 3.2, we recall known results about directive-equivalent infinite words. Section 4 will present a new characterization of these words.

### 3.1. FINITE DIRECTIVE-EQUIVALENT WORDS: PRESENTATION VERSUS BLOCK-EQUIVALENCE

Generalizing a study of the monoid of Sturmian morphisms by Séébold [31], the third author [26] answered the question: “When do two different spinned finite words direct the same episturmian morphism?” by giving a presentation of the monoid of episturmian morphisms. This result was reformulated in [27] using another set of generators and it was independently and differently treated in [15]. As a direct consequence, one can see that the monoid of pure epistandard morphisms is a free monoid and one can obtain the following presentation of the monoid of pure episturmian morphisms:

**Theorem 3.1** (direct consequence of [27], Prop. 6.5; reformulation of [15], Thm. 2.2). *The monoid of pure episturmian morphisms with  $\{L_\alpha, R_\alpha \mid \alpha \in \mathcal{A}\}$  as a set of generators has the following presentation:*

$$R_{a_1} R_{a_2} \dots R_{a_k} L_{a_1} = L_{a_1} L_{a_2} \dots L_{a_k} R_{a_1}$$

where  $k \geq 1$  is an integer and  $a_1, \dots, a_k \in \mathcal{A}$  with  $a_1 \neq a_i$  for all  $i$ ,  $2 \leq i \leq k$ .

This result means that two different compositions of morphisms in  $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$  yield the same pure episturmian morphism if and only if one composition can be deduced from the other one in a rewriting system, called the *block-equivalence* in [15]. Although Theorem 3.1 allows us to show that many properties of episturmian words are linked to properties of episturmian morphisms, it will be convenient for us to have in mind the block-equivalence that we now recall.

A word of the form  $xvx$ , where  $x \in \mathcal{A}$  and  $v \in (\mathcal{A} \setminus \{x\})^*$ , is called a ( $x$ -based) *block*. A ( $x$ -based) *block-transformation* is the replacement in a spinned word of an occurrence of  $xv\bar{x}$  (where  $xvx$  is a block) by  $\bar{x}v\bar{x}$  or vice-versa. Two finite spinned words  $w, w'$  are said to be *block-equivalent* if we can pass from one to the other by a (possibly empty) chain of block-transformations, in which case we write  $w \equiv w'$ . For example,  $\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c}$  and  $bab\bar{c}\bar{b}\bar{a}\bar{c}$  are block-equivalent because  $\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c} \rightarrow b\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c} \rightarrow bab\bar{c}\bar{b}\bar{a}\bar{c}$  and vice-versa. The block-equivalence is an equivalence relation over spinned words, and moreover one can observe that if  $w \equiv w'$  then  $w$  and  $w'$  are spinned versions of the same word over  $\mathcal{A}$ .

Theorem 3.1 can be reformulated in terms of block-equivalence:

**Theorem 3.1.** *Let  $w, w'$  be two spinned words over  $\mathcal{A} \cup \bar{\mathcal{A}}$ . Then  $\mu_w = \mu_{w'}$  if and only if  $w \equiv w'$ .*

### 3.2. INFINITE DIRECTIVE-EQUIVALENT WORDS: PREVIOUS RESULTS

The question: “When do two different spinned infinite words direct the same unique episturmian word?” was tackled by Justin and Pirillo in [15] for *bi-infinite episturmian words*, that is, episturmian words with letters indexed by  $\mathbb{Z}$  (and not by  $\mathbb{N}$  as considered until now). Let us recall relations between right-infinite



episturmian words and bi-infinite episturmian words (see [15], p. 332 and [9] for more details).

First we observe that a right-infinite episturmian word  $\mathbf{t}$  can be prolonged infinitely to the left with the same set of factors (see [15]). Note also that the definition of episturmian words considered in Section 2.2 (using reversal and special factors) can be extended to bi-infinite words. Furthermore, the characterization (Thm. 2.1) of right-infinite episturmian words by a sequence  $(\mathbf{t}^{(i)})_{i \geq 0}$  extends to bi-infinite episturmian words, with all the  $\mathbf{t}^{(i)}$  now bi-infinite episturmian words. That is, as for right-infinite episturmian words, we have bi-infinite words of the form  $\mathbf{l}^{(i)}.\mathbf{r}^{(i)}$  where  $\mathbf{l}^{(i)}$  is a left-infinite episturmian word and  $\mathbf{r}^{(i)}$  is a right-infinite episturmian word. Moreover, if the bi-infinite episturmian word  $\mathbf{b} = \mathbf{l}.\mathbf{r}$  is directed by  $\check{\Delta}$  with associated bi-infinite episturmian words  $\mathbf{b}^{(i)} = \mathbf{l}^{(i)}.\mathbf{r}^{(i)}$ , then  $\mathbf{r}$  is directed by  $\check{\Delta}$  with associated right-infinite episturmian words  $\mathbf{r}^{(i)}$ .

As a consequence of what precedes, Justin and Pirillo’s results about spinned words directing the same bi-infinite episturmian word are still valid for words directing the same (right-infinite) episturmian word. We summarize now these results, which will be helpful for the proof of our main theorem (Thm. 4.1, to follow).

First of all, Justin and Pirillo characterized pairs of words directing the same episturmian word in the case of *wavy* directive words, that is, spinned infinite words containing infinitely many  $L$ -spinned letters and infinitely many  $R$ -spinned letters. This characterization uses the following extension of the block-equivalence  $\equiv$  for infinite words.

Let  $\Delta_1, \Delta_2$  be spinned versions of  $\Delta$ . We write  $\Delta_1 \rightsquigarrow \Delta_2$  if there exist infinitely many prefixes  $f_i$  of  $\Delta_1$  and  $g_i$  of  $\Delta_2$  with the  $g_i$  of strictly increasing lengths, and such that, for all  $i$ ,  $|g_i| \leq |f_i|$  and  $f_i \equiv g_i c_i$  for a suitable spinned word  $c_i$ . Infinite words  $\Delta_1$  and  $\Delta_2$  are said to be *block-equivalent* (denoted by  $\Delta_1 \equiv \Delta_2$ ) if  $\Delta_1 \rightsquigarrow \Delta_2$  and  $\Delta_2 \rightsquigarrow \Delta_1$ .

**Theorem 3.2** ([15], Thm. 3.4, Cor. 3.5). *Let  $\Delta_1$  and  $\Delta_2$  be wavy spinned versions of  $\Delta \in A^\omega$  with  $|\text{Ult}(\Delta)| > 1$ . Then  $\Delta_1$  and  $\Delta_2$  direct the same (unique) episturmian word if and only if  $\Delta_1 \equiv \Delta_2$ .*

*Moreover when  $\Delta_1$  and  $\Delta_2$  do not have any common prefix modulo  $\equiv$ , and when there exists a letter  $x$  such that  $\Delta_1$  and  $\Delta_2$  begin with  $x$  and  $\bar{x}$  respectively, if  $\Delta_1 \equiv \Delta_2$ , then  $\Delta_1 = x \prod_{n \geq 1} v_n \check{x}_n$ ,  $\Delta_2 = \bar{x} \prod_{n \geq 1} \bar{v}_n \hat{x}_n$  for an  $L$ -spinned letter  $x$ , a sequence  $(v_n)_{n \geq 1}$  of  $x$ -free  $L$ -spinned words, and sequences of spinned letters  $(\check{x}_n)_{n \geq 1}$ ,  $(\hat{x}_n)_{n \geq 1}$  in  $\{x, \bar{x}\}$  such that  $(\check{x}_n)_{n \geq 1}$  contains infinitely many times the  $R$ -spinned letter  $\bar{x}$ , and  $(\hat{x}_n)_{n \geq 1}$  contains infinitely many times the  $L$ -spinned letter  $x$ .*

The relation  $\rightsquigarrow$  (and hence the block-equivalence  $\equiv$  for infinite words) is rather intricate to understand. So in some way the forms of  $\Delta_1$  and  $\Delta_2$  at the end of Theorem 3.2 are, although technical, easier to understand. Theorem 4.1, which refines the end of the previous result and proves the converse, describes all possible forms for pairs of directive-equivalent words without any use of notations  $\rightsquigarrow$  and  $\equiv$ .

When one of the two considered directive words is not wavy, Justin and Pirillo established:

**Proposition 3.3** ([15], Prop. 3.6). *Let  $\Delta_1$  and  $\Delta_2$  be spinned versions of the same word such that  $\Delta_1$  is wavy and letters of  $\Delta_2$  are ultimately of spin  $L$  (resp. ultimately of spin  $R$ ). If  $\Delta_1$  and  $\Delta_2$  are directive-equivalent, then  $\Delta_1 \rightsquigarrow \Delta_2$ . Moreover there exist spinned words  $w_1, w_2$ , an  $L$ -spinned letter  $x$ , and  $L$ -spinned  $x$ -free words  $(v_i)_{i \geq 1}$  such that  $\mu_{w_1} = \mu_{w_2}$ ,  $\Delta_1 = w_1 \bar{x} \prod_{i \geq 1} \bar{v}_i x$  and  $\Delta_2 = w_2 x \prod_{i \geq 1} v_i x$  (resp.  $\Delta_1 = w_1 x \prod_{i \geq 1} v_i \bar{x}$  and  $\Delta_2 = w_2 \bar{x} \prod_{i \geq 1} \bar{v}_i \bar{x}$ ).*

With the next two results, they considered the remaining cases of words directing aperiodic episturmian words. In the first one, the spins of the letters in each of the two directive words are ultimately  $L$  or ultimately  $R$ . The second result shows that if one of the directive words has the spins of its letters ultimately  $L$  (resp. ultimately  $R$ ), then the other directive word cannot have the spins of its letters ultimately  $R$  (resp. ultimately  $L$ ).

**Proposition 3.4** ([15], Prop. 3.7). *Let  $\Delta_1$  and  $\Delta_2$  be spinned versions of the same word  $\Delta \in \mathcal{A}^\omega$  with  $|\text{Ult}(\Delta)| > 1$ . If there exist spinned words  $w_1, w_2$  and an  $L$ -spinned infinite word  $\Delta'$  such that  $\Delta_1 = w_1 \Delta'$  and  $\Delta_2 = w_2 \Delta'$  (resp.  $\Delta_1 = w_1 \bar{\Delta}'$  and  $\Delta_2 = w_2 \bar{\Delta}'$ ), then  $\Delta_1, \Delta_2$  are directive-equivalent if and only if  $\mu_{w_1} = \mu_{w_2}$ .*

**Proposition 3.5** ([15], Prop. 3.9). *Let  $\Delta$  be an  $L$ -spinned infinite word. Then  $\Delta$  and  $\bar{\Delta}$  do not direct the same right-infinite episturmian word.*

Actually the previous statement is a corollary of Proposition 3.9 in [15] which considers more generally words directing episturmian words differing only by a shift.

Justin and Pirillo also discussed in [15] the periodic case and proved:

**Proposition 3.6** ([15], Prop. 3.10). *Suppose that  $\Delta_1 = \check{w}\check{y}a^\omega$  and  $\Delta_2 = \hat{w}\hat{y}\bar{a}^\omega$ , where  $\check{w}$  and  $\hat{w}$  (resp.  $\check{y}$  and  $\hat{y}$ ) are spinned versions of the same word and  $a$  is an  $L$ -spinned letter. Then  $\Delta_1$  and  $\Delta_2$  are directive-equivalent if and only if there exist sequences of letters  $(\check{a}_n)_{n \geq 1}$  and  $(\hat{a}_n)_{n \geq 1}$  such that  $\check{w}\check{y} \prod_{n \geq 1} \check{a}_n \equiv \hat{w}\hat{y} \prod_{n \geq 1} \hat{a}_n$ .*

We will see in Theorem 4.1 that other cases can occur for periodic episturmian words.

#### 4. DIRECTIVE-EQUIVALENT WORDS: A CHARACTERIZATION

As shown in the previous section, Justin and Pirillo provided quite complete results about directive-equivalent infinite words. Nevertheless they did not systematically provide the relative forms of two directive-equivalent words. The following characterization does it, moreover without the use of relations  $\rightsquigarrow$  and  $\equiv$ . This result also fully solves the periodic case, which was only partially solved in [15].

**Theorem 4.1.** *Given two spinned infinite words  $\Delta_1$  and  $\Delta_2$ , the following assertions are equivalent.*

- (i)  $\Delta_1$  and  $\Delta_2$  direct the same right-infinite episturmian word;
- (ii)  $\Delta_1$  and  $\Delta_2$  direct the same bi-infinite episturmian word;
- (iii) One of the following cases holds for some  $i, j$  such that  $\{i, j\} = \{1, 2\}$ :
  - (1)  $\Delta_i = \prod_{n \geq 1} v_n$ ,  $\Delta_j = \prod_{n \geq 1} z_n$  where  $(v_n)_{n \geq 1}, (z_n)_{n \geq 1}$  are spinned words such that  $\mu_{v_n} = \mu_{z_n}$  for all  $n \geq 1$ ;
  - (2)  $\Delta_i = wx \prod_{n \geq 1} v_n \check{x}_n$ ,  $\Delta_j = w' \bar{x} \prod_{n \geq 1} \bar{v}_n \hat{x}_n$  where  $w, w'$  are spinned words such that  $\mu_w = \mu_{w'}$ ,  $x$  is an  $L$ -spinned letter,  $(v_n)_{n \geq 1}$  is a sequence of non-empty  $x$ -free  $L$ -spinned words, and  $(\check{x}_n)_{n \geq 1}, (\hat{x}_n)_{n \geq 1}$  are sequences of non-empty spinned words over  $\{x, \bar{x}\}$  such that, for all  $n \geq 1$ ,  $|\check{x}_n| = |\hat{x}_n|$  and  $|\check{x}_n|_x = |\hat{x}_n|_x$ ;
  - (3)  $\Delta_1 = w\mathbf{x}$  and  $\Delta_2 = w'\mathbf{y}$  where  $w, w'$  are spinned words,  $x$  and  $y$  are letters, and  $\mathbf{x} \in \{x, \bar{x}\}^\omega$ ,  $\mathbf{y} \in \{y, \bar{y}\}^\omega$  are spinned infinite words such that  $\mu_w(x) = \mu_{w'}(y)$ .

*Note.* For  $a, b, c$  three different letters in  $\mathcal{A}$ , the spinned infinite words  $\Delta_1 = a(bc\bar{a})^\omega$  and  $\Delta_2 = \bar{a}(\bar{b}\bar{c}\bar{a})^\omega$  direct the same episturmian word that starts with the letter  $a$ . Indeed, these two directive words fulfill item (2) of Theorem 4.1 with  $w = w' = \varepsilon$ ,  $x = a$ , and for all  $n$ ,  $v_n = bc$  and  $\check{x}_n = \hat{x}_n = \bar{a}$ . Moreover the fact that  $\Delta_1$  starts with the  $L$ -spinned letter  $a$  shows that the word it directs starts with  $a$ . Similarly  $\Delta'_1 = \bar{a}b(c\bar{a}\bar{b})^\omega$  and  $\Delta'_2 = \bar{a}\bar{b}(\bar{c}\bar{a}\bar{b})^\omega$  direct the same episturmian word starting with the letter  $b$ . Since  $\Delta_2 = \Delta'_2$ , this shows that the relation “direct the same episturmian word” over spinned infinite word is not an equivalence relation.

Items 2 and 3 of Theorem 4.1 show that any episturmian word is directed by a spinned infinite word having infinitely many  $L$ -spinned letters, but also by a spinned word having both infinitely many  $L$ -spinned letters and infinitely many  $R$ -spinned letters (*i.e.*, a wavy word). To emphasize the importance of these facts, let us recall from Proposition 2.2 that if  $\check{\Delta}$  is a spinned infinite word over  $\mathcal{A} \cup \bar{\mathcal{A}}$  with infinitely many  $L$ -spinned letters, then there exists a unique episturmian word  $\mathbf{t}$  directed by  $\check{\Delta}$ . Unicity comes from the fact that the first letter of  $\mathbf{t}$  is fixed by the first  $L$ -spinned letter in  $\check{\Delta}$ .

Before proving Theorem 4.1, let us make two more remarks.

**Remark 4.2.** In items (1) and (2) of Theorem 4.1, the two considered directive words are spinned versions of the same  $L$ -spinned word. This does not hold in item (3), which deals only with periodic episturmian words. This is consistent with Remark 2.5. As an example of item (3), one can consider the word  $(ab)^\omega = L_a(b^\omega) = R_b(a^\omega)$  which, as already said at the end of Section 2.4, is directed by  $ab^\omega$  and by  $\bar{b}a^\omega$  ( $L_a(b) = ab = R_b(a)$ ). Note also that  $(ab)^\omega$  is directed by  $(\bar{a}\bar{b})^\omega$ , underlining the fact that  $x$  and  $y$  can be equal in item (3) of Theorem 4.1.

**Remark 4.3.** If an episturmian word  $\mathbf{t}$  has two directive words satisfying items (2) or (3), then  $\mathbf{t}$  has infinitely many directive words. Indeed, if item (2) is satisfied

and  $\bar{x}$  occurs in  $\check{x}_p$  ( $p \geq 1$ ), then by Theorem 3.1,  $x \left( \prod_{k=1}^{p-1} v_n \check{x}_n \right) v_p \check{x}'_p \bar{x} \equiv \bar{x} \left( \prod_{k=1}^{p-1} \bar{v}_n \check{x}_n \right) \bar{v}_p \check{x}'_p x$  where  $\check{x}'_p$  is such that  $\check{x}_p \equiv \bar{x} \check{x}'_p$ . Thus  $\mathbf{t}$  is also directed by  $w \bar{x} \left( \prod_{k=1}^{p-1} \bar{v}_n \check{x}_n \right) \bar{v}_p \check{x}'_p x \prod_{n \geq p+1} v_n \check{x}_n$ . Similarly if item (2) is satisfied and  $x$  occurs in  $\check{x}_p$  ( $p \geq 1$ ), then  $\mathbf{t}$  is also directed by  $w' x \left( \prod_{k=1}^{p-1} v_n \hat{x}_n \right) v_p \hat{x}'_p \bar{x} \prod_{n \geq p+1} \bar{v}_n \hat{x}_n$  where  $\hat{x}'_p$  is such that  $\hat{x}_p \equiv x \hat{x}'_p$ . If item (3) is satisfied, then  $\mathbf{t}$  is periodic and directed by  $w \mathbf{x}$  where  $\mathbf{x}$  is any spinned version of  $x^\omega$ .

The rest of this section is dedicated to the proof of Theorem 4.1.

*Proof of Theorem 4.1.* We have (i)  $\Leftrightarrow$  (ii) by the remarks on bi-infinite words at the beginning of Section 3.2.

(iii)  $\Rightarrow$  (i). Assume first that  $\Delta_1 = \prod_{n \geq 1} v_n$  and  $\Delta_2 = \prod_{n \geq 1} z_n$  for spinned words  $(v_n)_{n \geq 1}, (z_n)_{n \geq 1}$  such that  $\mu_{v_n} = \mu_{z_n}$  for all  $n \geq 1$ . From the latter equality and Theorem 3.1,  $\Delta_1$  has infinitely many  $L$ -spinned letters if and only if  $\Delta_2$  has infinitely many  $L$ -spinned letters.

Let us first consider the case when both  $\Delta_1$  and  $\Delta_2$  have infinitely many  $L$ -spinned letters. Without loss of generality we can assume that for all  $n \geq 1$ ,  $v_n$  and  $z_n$  contain at least one  $L$ -spinned letter. Now we need to define some more notations. Let  $\mathbf{t}_1$  and  $\mathbf{t}_2$  be the episturmian words directed by  $\Delta_1$  and  $\Delta_2$ , respectively (these episturmian words exist and are unique by Prop. 2.2). For  $n \geq 0$ , let  $\mathbf{t}_1^{(n)}$  and  $\mathbf{t}_2^{(n)}$  be the episturmian words as in (ii) of Theorem 2.1 and let  $a_n$  and  $b_n$  be their respective first letters. Finally, for  $n \geq 1$ , set  $p_n = \prod_{i=1}^n v_i$  and  $q_n = \prod_{i=1}^n z_i$ . The words  $\mu_{p_n}(a_{|p_n|})$  (resp.  $\mu_{q_n}(b_{|q_n|})$ ) are prefixes of  $\mathbf{t}_1$  (resp. of  $\mathbf{t}_2$ ). The letter  $a_{|p_n|}$  (resp.  $b_{|q_n|}$ ) is the first letter of  $\mu_{v_{n+1}}(\mathbf{t}_1^{(m)})$  (resp.  $\mu_{z_{n+1}}(\mathbf{t}_2^{(m)})$ ) with  $m = \sum_{i=1}^{n+1} |v_i| = \sum_{i=1}^{n+1} |z_i|$ . Since  $v_{n+1}$  (resp.  $z_{n+1}$ ) contains at least one  $L$ -spinned letter,  $a_{|p_n|}$  (resp.  $b_{|q_n|}$ ) is the first letter of  $\mu_{v_{n+1}}(w)$  (resp.  $\mu_{z_{n+1}}(w)$ ) for any word  $w$ . From  $\mu_{v_{n+1}} = \mu_{z_{n+1}}$ , we have  $a_{|p_n|} = b_{|q_n|}$  and so  $\mu_{p_n}(a_{|p_n|}) = \mu_{q_n}(b_{|q_n|})$  for all  $n \geq 1$ . If the sequence  $(\mu_{p_n}(a_{|p_n|}))_{n \geq 1}$  is not ultimately constant, then from  $\mathbf{t}_1 = \lim_{n \rightarrow \infty} \mu_{p_n}(a_{|p_n|})$  and  $\mathbf{t}_2 = \lim_{n \rightarrow \infty} \mu_{q_n}(b_{|q_n|})$ , we deduce that  $\mathbf{t}_1 = \mathbf{t}_2$ . If  $(\mu_{p_n}(a_{|p_n|}))_{n \geq 1}$  is ultimately constant, then necessarily there exists a letter  $a$  and an integer  $m$  such that for all  $n > m$ ,  $v_n$  and  $z_n$  belong to  $\{a\}^*$ . Then  $\mathbf{t}_1 = \mu_{v_1 \dots v_m}(a^\omega) = \mu_{z_1 \dots z_m}(a^\omega) = \mathbf{t}_2$ .

Now, with the same notations as in the above case, we consider the case when the letters of  $\Delta_1$  and  $\Delta_2$  are ultimately  $R$ -spinned. By Theorem 3.1, any equality  $\mu_v = \mu_z$  (for some different spinned words  $v$  and  $z$ ) implies that  $v$  and  $z$  both contain at least one  $L$ -spinned letter and one  $R$ -spinned letter. Hence, in our current case, there exists an integer  $m$  such that  $v_n = z_n$  for all  $n > m$ . Let  $\mathbf{t}$  be an episturmian word directed by  $\prod_{n > m} v_n = \prod_{n > m} z_n$  (such an episturmian word exists by Prop. 2.2). Then  $\mu_{p_m}(\mathbf{t}) = \mu_{q_m}(\mathbf{t})$  and  $\Delta_1$  and  $\Delta_2$  are directive-equivalent.

Now consider item (2) of part (iii). We assume that  $\Delta_1 = w x \prod_{n \geq 1} v_n \check{x}_n$  and  $\Delta_2 = w' \bar{x} \prod_{n \geq 1} \bar{v}_n \hat{x}_n$  where  $w, w'$  are spinned words such that  $\mu_w = \mu_{w'}$ ,

$x$  is an  $L$ -spinned letter,  $(v_n)_{n \geq 1}$  is a sequence of non-empty  $x$ -free  $L$ -spinned words, and  $(\check{x}_n)_{n \geq 1}$ ,  $(\hat{x}_n)_{n \geq 1}$  are non-empty spinned words over  $\{x, \bar{x}\}$  such that, for all  $n \geq 1$ ,  $|\check{x}_n| = |\hat{x}_n|$  and  $|\check{x}_n|_x = |\hat{x}_n|_x$ . By injectivity of the morphisms  $\mu_w = \mu_{w'}$ ,  $\Delta_1$  and  $\Delta_2$  are directive-equivalent if and only if  $w^{-1}\Delta_1$  and  $w'^{-1}\Delta_2$  are directive-equivalent. So, from now on, we assume without loss of generality that  $w = w' = \varepsilon$ .

By Proposition 2.2, there exist unique episturmian words  $\mathbf{t}_1$  and  $\mathbf{t}_2$  starting with  $x$  directed by the respective words  $\Delta_1$  and  $\Delta_2$  (observe that if  $\hat{x}_n \in \bar{x}^+$  for all  $n \geq 1$ , then  $\bar{x} \in \text{Ult}(\Delta_2)$ ). For  $i \geq 1$ , let  $\Delta_1^{(i)} = x \prod_{n \geq i} v_n \check{x}_n$  and  $\Delta_2^{(i)} = \bar{x} \prod_{n \geq i} \bar{v}_n \hat{x}_n$  and let  $\mathbf{t}_1^{[i]}$  and  $\mathbf{t}_2^{[i]}$  be the words beginning with  $x$  and directed by the respective words  $\Delta_1^{(i)}$  and  $\Delta_2^{(i)}$ . (The episturmian words  $\mathbf{t}_1^{[i]}$  and  $\mathbf{t}_2^{[i]}$  exist by Prop. 2.2.) For  $i \geq 1$  we also define  $\alpha_i := |\check{x}_i|_x = |\hat{x}_i|_x$  and  $\beta_i := |\check{x}_i|_{\bar{x}} = |\hat{x}_i|_{\bar{x}}$ .

Assume first that  $\alpha_i \neq 0$ . Then  $\bar{x}\bar{v}_i\hat{x}_i \equiv \bar{x}\bar{v}_i x x^{\alpha_i-1} \bar{x}^{\beta_i} \equiv x v_i \bar{x} x^{\alpha_i-1} \bar{x}^{\beta_i} \equiv x v_i \check{x}_i x^{\alpha_i-1} \bar{x}^{\beta_i} \bar{x}$  and  $x v_i \check{x}_i \equiv x v_i x^{\alpha_i-1} \bar{x}^{\beta_i} x$ . Let  $p_i = x v_i x^{\alpha_i-1} \bar{x}^{\beta_i}$ . From what precedes we deduce that  $\Delta_1^{(i)}$  and  $p_i \Delta_1^{(i+1)}$  are directive-equivalent, as  $\Delta_2^{(i)}$  and  $p_i \Delta_2^{(i+1)}$  are directive-equivalent. By the choice of words  $\mathbf{t}_1^{[i]}$  and  $\mathbf{t}_2^{[i]}$ , we deduce that  $\mathbf{t}_1^{[i]} = \mu_{p_i}(\mathbf{t}_1^{[i+1]})$  and  $\mathbf{t}_2^{[i]} = \mu_{p_i}(\mathbf{t}_2^{[i+1]})$  and each of these words starts with  $\mu_{p_i}(x)$ .

Now let us consider the case when  $\alpha_i = 0$ . Then  $\check{x}_i = \hat{x}_i = \bar{x}^{\beta_i}$ . We have  $x v_i \check{x}_i \equiv \bar{x} \bar{v}_i \bar{x}^{\beta_i-1} x$  and  $\bar{x} \bar{v}_i \hat{x}_i \equiv \bar{x} \bar{v}_i \bar{x}^{\beta_i-1} \bar{x}$ . Taking  $p_i = \bar{x} \bar{v}_i \bar{x}^{\beta_i-1}$ , we reach the same conclusion as in the case when  $\alpha_i \neq 0$ .

It follows from what precedes that  $\mathbf{t}_1$  and  $\mathbf{t}_2$  both start with  $\mu_{p_1 \dots p_i}(x)$  for all  $i \geq 1$ . Since  $v_i \neq \varepsilon$ , the sequence  $(\mu_{p_1 \dots p_i}(x))_{i \geq 1}$  is not ultimately constant; whence  $\mathbf{t}_1 = \mathbf{t}_2 = \lim_{i \rightarrow \infty} \mu_{p_1 \dots p_i}(x)$ .

Lastly, assume that  $\Delta_1 = w\mathbf{x}$  and  $\Delta_2 = w'\mathbf{y}$  for some spinned words  $w, w'$ , some letters  $x$  and  $y$ , and some spinned infinite words  $\mathbf{x} \in \{x, \bar{x}\}^\omega$ ,  $\mathbf{y} \in \{y, \bar{y}\}^\omega$  such that  $\mu_w(x) = \mu_{w'}(y)$ . The word  $\Delta_1$  (resp.  $\Delta_2$ ) directs the episturmian word  $\mu_w(x^\omega) = (\mu_w(x))^\omega$  (resp.  $\mu_{w'}(y^\omega) = (\mu_{w'}(y))^\omega$ ). Hence  $\Delta_1$  and  $\Delta_2$  are directive-equivalent.

(i)  $\Rightarrow$  (iii). Suppose  $\Delta_1$  and  $\Delta_2$  direct the same (right-infinite) episturmian word  $\mathbf{t}$ . Let us first assume that  $\mathbf{t}$  is aperiodic. Then, by Remark 2.5,  $\Delta_1$  and  $\Delta_2$  are spinned versions of the same infinite word  $\Delta \in \mathcal{A}^\omega$ . We now show that item (1) or item (2) holds using results of Justin and Pirillo in [15].

First consider the case when both  $\Delta_1$  and  $\Delta_2$  are wavy. Suppose there exist a sequence of prefixes  $(p_n)_{n \geq 0}$  of  $\Delta_1$  and a sequence of prefixes  $(p'_n)_{n \geq 0}$  of  $\Delta_2$  such that for all  $n \geq 0$ ,  $\mu_{p_n} = \mu_{p'_n}$ . Without loss of generality we can assume that  $p_0 = p'_0 = \varepsilon$  and the sequence  $(|p_n|)_{n \geq 0}$  is strictly increasing. For  $n \geq 1$ , let  $v_n, z_n$  be such that  $p_n = p_{n-1}v_n$ ,  $p'_n = p'_{n-1}z_n$ ; that is  $\Delta_1 = \prod_{n \geq 1} v_n$  and  $\Delta_2 = \prod_{n \geq 1} z_n$ . Let us prove by induction that  $\mu_{v_n} = \mu_{z_n}$  for all  $n \geq 1$ . First  $\mu_{v_1} = \mu_{p_1} = \mu_{p'_1} = \mu_{z_1}$ . For  $n \geq 2$ , since  $\mu_{p_n} = \mu_{p_{n-1}}\mu_{v_n}$ ,  $\mu_{p'_n} = \mu_{p'_{n-1}}\mu_{z_n}$ ,  $\mu_{p_n} = \mu_{p'_n}$  and  $\mu_{p_{n-1}} = \mu_{p'_{n-1}}$ , we have  $\mu_{p_{n-1}}\mu_{v_n} = \mu_{p_{n-1}}\mu_{z_n}$  and so  $\mu_{v_n} = \mu_{z_n}$  by left cancellativity of the monoid of episturmian morphisms. So item (1) is satisfied in this case.

Now assume that previous sequences  $(p_n)_{n \geq 0}$  and  $(p'_n)_{n \geq 0}$  do not exist. Let  $w$  and  $w'$  be the longest prefixes of the respective spinned words  $\Delta_1$  and  $\Delta_2$  such that  $\mu_w = \mu_{w'}$ . Further, let  $\Delta'_1$  and  $\Delta'_2$  be the spinned words such that  $\Delta_1 = w\Delta'_1$  and  $\Delta_2 = w'\Delta'_2$ . Then, by injectivity of  $\mu_w$ , the words  $\Delta'_1$  and  $\Delta'_2$  are directive-equivalent and have no prefixes with equal images by  $\mu$ .

By Theorem 3.2, there exists a letter  $x$  in  $\mathcal{A}$ , a sequence of non-empty  $x$ -free words  $(v_n)_{n \geq 1}$  over  $\mathcal{A}$ , and two sequences of non-empty words  $(\check{x}_n)_{n \geq 1}$ ,  $(\hat{x}_n)_{n \geq 1}$  over  $\{x, \bar{x}\}$  such that  $\Delta'_i = x \prod_{n \geq 1} v_n \check{x}_n$  and  $\Delta'_j = \bar{x} \prod_{n \geq 1} \bar{v}_n \hat{x}_n$  for some integers  $i, j$  such that  $\{i, j\} = \{1, 2\}$ . We have to prove that for all  $n \geq 1$ ,  $|\check{x}_n| = |\hat{x}_n|$  and  $|\check{x}_n|_x = |\hat{x}_n|_x$ . We use induction on  $n$  and prove also that for all  $n \geq 0$ , the words  $\Delta_i^{(n+1)} = x \prod_{m \geq n+1} v_m \check{x}_m$  and  $\Delta_j^{(n+1)} = \bar{x} \prod_{m \geq n+1} \bar{v}_m \hat{x}_m$  are directive-equivalent.

Let  $n \geq 1$  be an integer. By definition of  $\Delta_i^{(1)} = \Delta'_i$  and  $\Delta_j^{(1)} = \Delta'_j$  (when  $n = 1$ ) and by the induction hypothesis (when  $n \geq 2$ ), we know that the words  $\Delta_i^{(n)} = x \prod_{m \geq n} v_m \check{x}_m$  and  $\Delta_j^{(n)} = \bar{x} \prod_{m \geq n} \bar{v}_m \hat{x}_m$  are directive-equivalent.

Assume first that  $\hat{x}_n$  contains at least one occurrence of  $x$ . Then, with  $\alpha_n = |\hat{x}_n|_x$  and  $\beta_n = |\hat{x}_n|_{\bar{x}}$ , we have  $\bar{x} \bar{v}_n \hat{x}_n \equiv \bar{x} \bar{v}_n x x^{\alpha_n - 1} \bar{x}^{\beta_n} \equiv x v_n x^{\alpha_n - 1} \bar{x}^{\beta_n} \bar{x}$ . By injectivity of the morphism  $\mu_{xv_n}$  we deduce that the words  $\check{x}_n \prod_{m \geq n+1} v_m \check{x}_m = \check{x}_n v_{n+1} \check{x}_{n+1} \prod_{m \geq n+2} v_m \check{x}_m$  and  $x^{\alpha_n - 1} \bar{x}^{\beta_n} \bar{x} \prod_{m \geq n+1} \bar{v}_m \hat{x}_m = x^{\alpha_n - 1} \bar{x}^{\beta_n + 1} \bar{v}_{n+1} \hat{x}_{n+1} \prod_{m \geq n+2} \bar{v}_m \hat{x}_m$  direct the same episturmian word  $\mathbf{t}_n$ . The word  $v_{n+1}$  is not empty. Let  $c$  be its first letter, let  $D = c^{-1} v_{n+1} \check{x}_{n+1} \prod_{m \geq n+2} v_m \check{x}_m$  and let  $D' = (\bar{c})^{-1} \bar{v}_{n+1} \hat{x}_{n+1} \prod_{m \geq n+2} \bar{v}_m \hat{x}_m$ : the word  $\mathbf{t}_n$  is directed by  $\check{x}_n c D$  and by  $x^{\alpha_n - 1} \bar{x}^{\beta_n + 1} \bar{c} D'$ . Since  $\Delta_j$  is wavy,  $D'$  is also wavy. So  $x$  occurs in  $D'$  (among the  $\hat{x}_n$ ) and the word directed by  $D'$  starts with  $x$ . Consequently  $\mathbf{t}_n$  starts with  $\mu_{x^{\alpha_n - 1} \bar{x}^{\beta_n + 1} \bar{c}}(x) = x^{\alpha_n} c x^{\beta_n + 1}$ . The words  $v_m$  are non-empty, thus there exists a letter  $d \neq x$  that occurs in the word directed by  $D'$ . Consequently  $c x^{\alpha_n + \beta_n} d$  is the smallest factor of  $\mathbf{t}_n$  belonging to  $c\{x\}^*d$ . Since  $\mathbf{t}_n$  is also directed by  $\check{x}_n c D$ , it follows that  $\mathbf{t}_n$  starts with  $x^{|\check{x}_n|_x} c$  and the smallest factor of  $\mathbf{t}_n$  belonging to  $c\{x\}^*d$  is  $c x^{|\check{x}_n|_x} d$ . Hence  $|\check{x}_n|_x = \alpha_n = |\hat{x}_n|_x$  and  $|\check{x}_n| = \alpha_n + \beta_n = |\hat{x}_n|$ . Consequently  $\check{x}_n \equiv x^{\alpha_n} \bar{x}^{\beta_n} \equiv x^{\alpha_n - 1} \bar{x}^{\beta_n} x$ . The injectivity of the morphism  $\mu_{x^{\alpha_n - 1} \bar{x}^{\beta_n}}$  implies that  $\Delta_i^{(n+1)} = x \prod_{m \geq n+1} v_m \check{x}_m$  and  $\Delta_j^{(n+1)} = \bar{x} \prod_{m \geq n+1} \bar{v}_m \hat{x}_m$  are directive-equivalent.

When  $\check{x}_n$  contains at least one occurrence of  $\bar{x}$ , we similarly reach the same conclusion.

Now we show that it is impossible that  $\check{x}_n \in x^+$  and  $\hat{x}_n \in \bar{x}^+$ . Assume these relations hold and let  $k$  be the least integer strictly greater than  $n$  such that  $x \in \text{Alph}(\hat{x}_k)$  (such an integer exists since  $\Delta_j$  is wavy). Let  $\alpha_k = |\hat{x}_k|_x$  and  $\beta_k = |\hat{x}_k|_{\bar{x}}$ . Since all of the words  $\hat{x}_n, \dots, \hat{x}_{k-1}$  belong to  $\bar{x}^+$ , we have  $\bar{x} \bar{v}_n \hat{x}_n \bar{v}_{n+1} \dots \bar{v}_k \hat{x}_k \equiv \bar{x} \bar{v}_n \hat{x}_n \bar{v}_{n+1} \dots \bar{v}_k x^{\alpha_k} \bar{x}^{\beta_k} \equiv x v_n \hat{x}_n v_{n+1} \dots v_k x^{\alpha_k - 1} \bar{x}^{\beta_k}$ . Then by injectivity of the morphism  $\mu_{xv_n}$ , there exists an episturmian word directed by both  $\Delta = \check{x}_n \prod_{m \geq n+1} v_m \check{x}_m$  and  $\Delta' = \hat{x}_n v_{n+1} \dots v_k x^{\alpha_k - 1} \bar{x}^{\beta_k} \bar{x} \prod_{m \geq k+1} \bar{v}_m \hat{x}_m$ . But this is impossible since  $\Delta$  directs a word starting with  $x$  (recall that  $\check{x}_n \in x^+$ ) and  $\Delta'$  directs a word starting with the first letter of  $v_{n+1}$  (recall that  $\hat{x}_n \in \bar{x}^+$ ).

Let us now consider the case when one of the two words  $\Delta_1, \Delta_2$  is wavy and the other has all of its spins ultimately  $L$  or ultimately  $R$ . Then item (2) is verified by Proposition 3.3.

Suppose now that both  $\Delta_1$  and  $\Delta_2$  have all spins ultimately  $L$  (resp. ultimately  $R$ ). Then by Remark 2.5,  $\Delta_1$  and  $\Delta_2$  are spinned versions of the same word. Hence  $\Delta_1 = w\Delta$  and  $\Delta_2 = w'\Delta$  (resp.  $\Delta_1 = w\bar{\Delta}$  and  $\Delta_2 = w'\bar{\Delta}$ ) for some spinned words  $w, w'$  of the same length and an infinite  $L$ -spinned word  $\Delta$  (resp.  $R$ -spinned word  $\Delta$ ). Since  $\Delta_1$  and  $\Delta_2$  are directive-equivalent,  $\mu_w = \mu_{w'}$  by Proposition 3.4, and furthermore  $\Delta_1$  and  $\Delta_2$  have infinitely many prefixes whose images are equal by  $\mu$ . Therefore, as already seen, this situation satisfies item (1).

We have now ended the study of the aperiodic case, since by Proposition 3.5,  $\Delta_1$  and  $\Delta_2$  cannot direct the same aperiodic episturmian word if one of them has all spins ultimately  $L$  and the other has all spins ultimately  $R$ .

Finally we come to the periodic case:  $\Delta_1 = w\mathbf{x}$  and  $\Delta_2 = w'\mathbf{y}$  for some spinned words  $w, w'$ , letters  $x$  and  $y$ , and spinned infinite words  $\mathbf{x} \in \{x, \bar{x}\}^\omega, \mathbf{y} \in \{y, \bar{y}\}^\omega$ . In this case, the episturmian word directed by  $\Delta_1$  and  $\Delta_2$  is  $\mu_w(x)^\omega = \mu_{w'}(y)^\omega$ , which implies that  $\mu_w(x)^{|y|} = \mu_{w'}(y)^{|x|}$ . Then (see [19] for instance) there exists a primitive word  $z$  such that  $\mu_w(x)$  and  $\mu_{w'}(y)$  are powers of  $z$  (let us recall that a word  $w$  is primitive if it is not an integer power of a shorter word, i.e., if  $w = u^p$  with  $p \in \mathbb{N}$ , then  $p = 1$  and  $w = u$ ). One can quite easily verify that any episturmian morphism maps any primitive word to another primitive word (see also [13], Props. 2.8, 3.15). Since any letter constitutes a primitive word, both  $\mu_w(x)$  and  $\mu_{w'}(y)$  are primitive. Thus  $\mu_w(x) = z = \mu_{w'}(y)$ .  $\square$

### 5. NORMALIZED DIRECTIVE WORD OF AN EPISTURMIAN WORD

In the previous section we have seen that any episturmian word  $\mathbf{t}$  has a directive word with infinitely many  $L$ -spinned letters. To work on Sturmian words, Berthé *et al.* recently proved that it is always possible to choose a particular directive word.

**Theorem 5.1** [4]. *Any Sturmian word  $\mathbf{w}$  over  $\{a, b\}$  has a unique representation of the form*

$$\mathbf{w} = \lim_{n \rightarrow \infty} L_a^{d_1 - c_1} R_a^{c_1} L_b^{d_2 - c_2} R_b^{c_2} \dots L_a^{d_{2n-1} - c_{2n-1}} R_a^{c_{2n-1}} L_b^{d_{2n} - c_{2n}} R_b^{c_{2n}}(a)$$

where  $d_k \geq c_k \geq 0$  for all integer  $k \geq 1$ ,  $d_k \geq 1$  for  $k \geq 2$  and if  $c_k = d_k$  then  $c_{k-1} = 0$ .

In other words, any Sturmian word has a unique directive word over  $\{a, b, \bar{a}, \bar{b}\}$  containing infinitely many  $L$ -spinned letters but no factor of the form  $\bar{a}\bar{b}^n a$  or  $\bar{b}\bar{a}^n b$  with  $n$  an integer. Actually this result is quite natural if one thinks about the presentation of the monoid of Sturmian morphisms (see [31]). Using Theorems 3.1 and 4.1, we generalize Theorem 5.1 to episturmian words.

**Theorem 5.2.** *Any episturmian word  $\mathbf{t}$  has a spinned directive word containing infinitely many  $L$ -spinned letters, but no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$ . Such a directive word is unique if  $\mathbf{t}$  is aperiodic.*

The example given in Remark 4.2 shows that unicity does not necessarily hold for periodic episturmian words. A directive word of an aperiodic episturmian word  $\mathbf{t}$  satisfying the property given in Theorem 5.2 is called the *normalized directive word* of  $\mathbf{t}$ . We extend this definition to morphisms: a finite spinned word  $w$  is said to be a *normalized directive word* of the morphism  $\mu_w$  if  $w$  has no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$ .

One can observe that, by Theorem 3.1, for any morphism in  $L_a\mathcal{L}_{\mathcal{A}}^*R_a$ , we can find another decomposition of the morphism in the set  $R_a\mathcal{R}_{\mathcal{A}}^*L_a$ . Equivalently, for any spinned word in  $a\mathcal{A}^*\bar{a}$ , there exists a word  $w'$  in  $\bar{a}\bar{\mathcal{A}}^*a$  such that  $\mu_w = \mu_{w'}$ . This is the main idea used in the proof of the lemma below. The proof of Theorem 5.2 is based on an extension of this lemma to infinite words.

**Lemma 5.3.** *Any pure episturmian morphism has a unique normalized directive word.*

*Proof.*

*Existence of the normalized directive word.* Let  $w = (w_i)_{1 \leq i \leq |w|}$  be a spinned word over  $\mathcal{A} \cup \bar{\mathcal{A}}$ . We construct by induction on  $|w|$  a normalized directive word of  $\mu_w$ .

If  $|w| = 0$ , there is nothing to do:  $\varepsilon$  is a normalized directive word of the empty morphism. Assume we have constructed a normalized directive word  $w' = (w'_i)_{1 \leq i \leq k}$  of the morphism  $\mu_{w'} = \mu_w$ .

Let  $\bar{x}$  be a letter in  $\bar{\mathcal{A}}$ . Then, by normalization of  $w'$ , the word  $w'\bar{x}$  has no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$ . Moreover since  $\mu_w = \mu_{w'}$ , we have  $\mu_{w\bar{x}} = \mu_{w'\bar{x}}$ : the word  $w'\bar{x}$  is a normalized directive word of  $\mu_{w\bar{x}}$ .

Now let  $x$  be a letter in  $\mathcal{A}$ . The word  $w'x$  can have factors in  $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$ , but only as suffixes. If this does not hold, as in the previous case, the word  $w'x$  is a normalized directive word of  $\mu_{w'x}$ . Else  $w' = p\bar{x}\bar{u}_1\bar{x}\bar{u}_2 \dots \bar{x}\bar{u}_k$  for an integer  $k \geq 1$ , some  $L$ -spinned  $x$ -free words  $(u_i)_{1 \leq i \leq k}$  and a spinned word  $p$  having no suffix in  $\bar{x}\bar{\mathcal{A}}^*$ . The word  $w''\bar{x}$  where  $w'' = p\bar{x}u_1\bar{x}u_2 \dots \bar{x}u_k$  contains no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$ . Moreover Theorem 3.1 implies  $\mu_{w'x} = \mu_{w''\bar{x}}$ . Hence  $w''\bar{x}$  is a normalized directive word of  $\mu_{w'x}$ .

Let us make a remark on the inductive construction presented in this proof.

**Remark 5.4.** Let  $u, v, u', v'$  be four spinned words such that  $u'$  (resp.  $v'$ ) is the normalized directive word obtained by the above construction from  $u$  (resp.  $v$ ). If  $u$  is a prefix of  $v$  and if  $p$  is a prefix of  $u'$  ending by an  $L$ -spinned letter, then  $p$  is also a prefix of  $v'$ .

*Uniqueness.* Assume by way of contradiction that  $w$  and  $w'$  are two different spinned normalized words such that  $\mu_w = \mu_{w'}$ . By left cancellativity of the monoid of episturmian morphisms, we can assume that  $w$  and  $w'$  start with different letters. Moreover it follows from Theorem 3.1 that  $w$  and  $w'$  are spinned versions of the



same word. Without loss of generality, we can assume that  $w$  begins with a letter  $a \in \mathcal{A}$  and  $w'$  begins with  $\bar{a}$  and so for any word  $z$ ,  $\mu_w(z) = \mu_{w'}(z)$  begins with  $a$ . Hence  $w'$  must start with  $\bar{a}v\bar{a}$  for a word  $v \in \mathcal{A}^*$ . This contradicts its normalization.  $\square$

**Example 5.5.** Let  $f$  be the pure episturmian morphism with directive word  $\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c}\bar{a}$ . By Theorem 3.1,  $\mu_{\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}\bar{a}} = \mu_{\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}\bar{a}} = \mu_{ac\bar{b}ac\bar{a}}$  and hence  $\mu_{\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c}\bar{a}} = \mu_{\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c}\bar{a}}$  and  $\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c}\bar{a}$  is the normalized directive word of  $f$ .

*Proof of Theorem 5.2.*

*Existence of the normalized directive word.* Let  $\Delta = (w_i)_{i \geq 1}$  be a spinned directive word of an episturmian word  $\mathbf{t}$  (with  $w_i \in \mathcal{A} \cup \bar{\mathcal{A}}$ ). From Theorem 4.1, we can assume that  $\Delta$  has infinitely many  $L$ -spinned letters.

By Lemma 5.3, for any  $n \geq 1$ , the morphism  $\mu_{w_1 \dots w_n}$  has a unique normalized directive word  $(w_i^{(n)})_{1 \leq i \leq n}$ . (It follows from the proof of Lem. 5.3 that  $w_i$  and  $w_i^{(n)}$  are spinned versions of the same letter.)

Let  $p_n$  be the longest prefix of  $w_1^{(n)} \dots w_n^{(n)}$  that belongs to  $(\mathcal{A} \cup \bar{\mathcal{A}})^* \mathcal{A}$ . Let  $i_n \leq n$  be the integer such that  $p_n = w_1^{(n)} \dots w_{i_n}^{(n)}$ , and let  $\pi_n$  be the word  $\pi_n = \mu_{w_1^{(n)} \dots w_{i_n-1}^{(n)}}(w_{i_n}^{(n)})$ .

Since the morphisms  $\mu_{w_1 \dots w_n}$  and  $\mu_{w_1^{(n)} \dots w_n^{(n)}}$  are equal,  $\mathbf{t}$  has the directive word  $(w_1^{(n)}, \dots, w_n^{(n)}, w_{n+1}, w_{n+2}, \dots)$ , so  $\pi_n$  is a prefix of  $\mathbf{t}$ .

By Remark 5.4, for any  $n \geq 1$ ,  $p_n$  is a prefix of  $p_{n+1}$ , and since  $\Delta$  contains infinitely many  $L$ -spinned letters, for any  $n \geq 1$ , there exists an  $m > n$  such that  $|p_m| > |p_n|$ .

If  $|\text{Ult}(\Delta)| = 1$ , then there exists a letter  $a$  and an integer  $m$  such that  $\mathbf{t} = \mu_{p_m}(a^\omega)$  and  $p_m a^\omega$  is a normalized directive word of  $\mathbf{t}$ . If  $|\text{Ult}(\Delta)| > 1$ , the sequence  $(\pi_n)_{n \geq 1}$  is not ultimately constant, and  $\lim_{n \rightarrow \infty} \pi_n = \mathbf{t}$ . In this case  $\mathbf{t}$  is directed by the sequence  $\lim_{n \rightarrow \infty} p_n$  which is normalized by construction (indeed otherwise one of the prefixes  $p_n$  would not be normalized).

*Uniqueness of the normalized directive word.* Assume by way of contradiction that an aperiodic episturmian word  $\mathbf{t}$  has two different normalized spinned directive words  $\Delta_1 = (w_n)_{n \geq 1}$  and  $\Delta_2 = (w'_n)_{n \geq 1}$  (with  $w_n$  and  $w'_n \in \mathcal{A} \cup \bar{\mathcal{A}}$  for all  $n$ ). Let  $i \geq 1$  be the smallest integer such that  $w_i \neq w'_i$  (and for all  $j < i$ ,  $w_j = w'_j$ ). By Theorem 4.1,  $\Delta_1$  and  $\Delta_2$  are spinned versions of the same word (see Rem. 4.2). Thus, without loss of generality, we can assume that  $w_i = \bar{x}$  and  $w'_i = x$  for some letter  $x$ .

Let  $\mathbf{t}^{(i)}$  be the episturmian word with (normalized) directive word  $(w'_n)_{n \geq i}$  (by Prop. 2.2 this word is unique), then  $\mathbf{t}^{(i)}$  starts with  $x$  since  $w'_i = x$ . Since the word  $(w_n)_{n \geq 1}$  has infinitely many  $L$ -spinned letters, there exists an integer  $j > i$  such that  $w_j = y$  for a letter  $y \in \mathcal{A}$  and  $w_\ell \in \bar{\mathcal{A}}$  for each  $\ell$ ,  $i < \ell < j$ . Let  $\mathbf{t}^{(i)}$  be the

word with normalized directive word  $(w_n)_{n \geq i}$ , then  $\mathbf{t}^{(i)}$  has the word  $\mu_{w_i \dots w_{j-1}}(y)$  as prefix since  $w_j = y$  and so  $\mathbf{t}^{(i)}$  starts with  $y$  since  $w_i \dots w_{j-1} \in \bar{\mathcal{A}}^*$ . We have  $\mathbf{t} = \mu_{w_1 \dots w_{i-1}}(\mathbf{t}^{(i)}) = \mu_{w'_1 \dots w'_{i-1}}(\mathbf{t}'^{(i)})$ . By choice of  $i$ ,  $w_1 \dots w_{i-1} = w'_1 \dots w'_{i-1}$ . Consequently, since episturmian morphisms are injective on infinite words,  $\mathbf{t}^{(i)} = \mathbf{t}'^{(i)}$  and so  $x = y$ . But since  $w_i = \bar{x}$ ,  $w_{i+1} \dots w_{j-1} \in \bar{\mathcal{A}}^*$ , and  $w_j = x$ , we reach a contradiction to the normalization of  $(w_n)_{n \geq 1}$ .  $\square$

### 6. EPISTURMIAN WORDS HAVING A UNIQUE DIRECTIVE WORD

In Section 4 we have characterized pairs of words directing the same episturmian word. In Section 5 we proposed a way to uniquely define any episturmian word through a normalization of its directives words (as mentioned in the introduction; see [4,11,17,18] for some of its uses). Using these results we now characterize episturmian words having a unique directive word.

**Theorem 6.1.** *An episturmian word has a unique directive word if and only if its (normalized) directive word contains:*

- (1) *infinitely many L-spinned letters;*
- (2) *infinitely many R-spinned letters;*
- (3) *no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a} \bar{\mathcal{A}}^* a$ ;*
- (4) *and no factor in  $\bigcup_{a \in \mathcal{A}} a \mathcal{A}^* \bar{a}$ .*

*Such an episturmian word is necessarily aperiodic.*

*Proof.* Assume first that an episturmian word  $\mathbf{t}$  has a unique spinned directive word  $\Delta$ . By Theorem 5.2,  $\Delta$  is normalized and so contains infinitely many L-spinned letters and no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a} \bar{\mathcal{A}}^* a$ . By item (3) of Theorem 4.1 and by Remark 2.4,  $\mathbf{t}$  cannot be periodic. By item (2) of Theorem 4.1,  $\Delta$  also contains infinitely many R-spinned letters, and hence is wavy, i.e., contains infinitely many L-spinned letters and infinitely many R-spinned letters (recall from Sect. 3.2). Otherwise one can construct another directive word of  $\mathbf{t}$  – the fact that  $\mathbf{t}$  is aperiodic is important for having the  $(v_n)_{n \geq 1}$  non-empty in this construction. Finally Theorem 3.1 implies the non-existence of a factor in  $\bigcup_{a \in \mathcal{A}} a \mathcal{A}^* \bar{a}$  (otherwise, one can again construct another directive word for  $\mathbf{t}$ ).

Let us now prove that the four conditions (given in the statement of the theorem) are sufficient. Arguing by contradiction, we assume that an episturmian word  $\mathbf{t}$  is directed by two spinned infinite words  $\Delta_1$  and  $\Delta_2$ , both fulfilling the four given conditions. We observe that if  $\Delta_1$  or  $\Delta_2$  is ultimately written over  $\{x, \bar{x}\}$  for a letter  $x$  (which can occur only if  $\mathbf{t}$  is periodic), then at least one of the conditions is not fulfilled. Thus the two words  $\Delta_1$  and  $\Delta_2$  should verify one of the two first items in part (iii) of Theorem 4.1 (item (3) does not apply since  $\mathbf{t}$  is aperiodic). But the hypotheses on  $\Delta_1$  and  $\Delta_2$  imply that only item (1) can be verified so that  $\Delta_1 = \prod_{n \geq 1} v_n$ ,  $\Delta_2 = \prod_{n \geq 1} z_n$  for spinned words  $(v_n)_{n \geq 1}, (z_n)_{n \geq 1}$  such that  $\mu_{v_n} = \mu_{z_n}$  for all  $n \geq 1$ . Now by Theorem 3.1 and by the fact that words  $(v_n)_{n \geq 1}$  and  $(z_n)_{n \geq 1}$  have no factor in  $\bigcup_{a \in \mathcal{A}} \bar{a} \bar{\mathcal{A}}^* a$  nor  $\bigcup_{a \in \mathcal{A}} a \mathcal{A}^* \bar{a}$ , we must have  $v_n = z_n$  for all  $n \geq 1$ . Thus  $\Delta_1 = \Delta_2$ .  $\square$

As an example, a particular family of episturmian words having unique directive words consists of those directed by *regular wavy words*, *i.e.*, spinned infinite words having both infinitely many  $L$ -spinned letters and infinitely many  $R$ -spinned letters such that each letter occurs with the same spin everywhere in the directive word. More formally, a spinned version  $\check{w}$  of a finite or infinite word  $w$  is said to be *regular* if, for each letter  $x \in \text{Alph}(w)$ , all occurrences of  $\check{x}$  in  $\check{w}$  have the same spin ( $L$  or  $R$ ). For example,  $\bar{a}b\bar{a}a\bar{c}\bar{b}$  and  $(\bar{a}\bar{b}c)^\omega$  are regular, whereas  $\bar{a}b\bar{a}\bar{a}\bar{c}\bar{b}$  and  $(\bar{a}\bar{b}\bar{a})^\omega$  are not regular.

In the Sturmian case, we have:

**Proposition 6.2.** *Any Sturmian word has a unique spinned directive word or infinitely many spinned directive words. Moreover, a Sturmian word has a unique directive word if and only if its (normalized) directive word is regular wavy.*

*Proof.* Let  $\Delta$  be the normalized directive word of a Sturmian word  $\mathbf{t}$  over  $\{a, b\}$ . Then  $\Delta$  contains no factor belonging to  $\bar{a}\bar{b}^*a \cup \bar{b}\bar{a}^*b$  (where  $\alpha^* = \{\alpha\}^*$  for any letter  $\alpha$ ).

Assume first that  $\Delta$  contains infinitely many factors in  $ab^*\bar{a} \cup ba^*\bar{b}$ . Then  $\Delta = p\bigcup_{n \geq 1} x_n y_n$  for some spinned words  $p$  and  $(x_n, y_n)_{n \geq 1}$  such that, for all  $n \geq 1$ ,  $x_n \in ab^*\bar{a} \cup ba^*\bar{b}$  and  $y_n \in \{a, b, \bar{a}, \bar{b}\}^*$ . In this case,  $\Delta$  has infinitely many directive words; indeed, the spinned words  $(p[\bigcup_{n=1}^{k-1} x_n y_n] \bar{x}_k y_k \bigcup_{n \geq k+1} x_n y_n)_{k \geq 1}$  are (by Thm. 3.1) pairwise different directive words for  $\mathbf{t}$ .

Now assume that  $\Delta$  contains only finitely many factors in  $ab^*\bar{a} \cup ba^*\bar{b}$ . Since  $\Delta$  contains no factor in  $\bar{a}\bar{b}^*a \cup \bar{b}\bar{a}^*b$ , it is ultimately regular wavy. More precisely  $\Delta$  is regular wavy and either  $\Delta$  belongs to  $\{a, \bar{b}\}^\omega \cup \{\bar{a}, b\}^\omega$ , or  $\Delta$  belongs to one of the following sets of infinite words:  $S_1 = \{a, b, \bar{a}, \bar{b}\}^* a \{a, b\}^\omega$ ,  $S_2 = \{a, b, \bar{a}, \bar{b}\}^* b \{a, b\}^\omega$ ,  $S_3 = \{a, b, \bar{a}, \bar{b}\}^* \bar{a} \{a, b\}^\omega$  or  $S_4 = \{a, b, \bar{a}, \bar{b}\}^* \bar{b} \{a, b\}^\omega$ . Assume  $\Delta \in S_1$ . Since any Sturmian word is aperiodic,  $\Delta$  is not ultimately constant (see Rem. 2.4). Thus  $\Delta = pa\bigcup_{n \geq 1} x_n \bar{a}$  with  $x_n \in b^*$  for all  $n \geq 1$ . Once again in this case,  $\mathbf{t}$  has infinitely many directive words since the words  $(p[\bigcup_{n=1}^{k-1} \bar{a} x_n] a \bigcup_{n \geq k} x_n \bar{a})_{k \geq 1}$  are pairwise different directive words for  $\mathbf{t}$ . The cases when  $\Delta \in S_2$  or  $\Delta \in S_3$  or  $\Delta \in S_4$  are similar.

We end with the case when  $\Delta$  is regular wavy. In this case,  $\Delta$  contains infinitely many  $L$ -spinned letters, infinitely many  $R$ -spinned letters, no factor in  $ab^*\bar{a} \cup ba^*\bar{b}$ , and no factor in  $\bar{a}\bar{b}^*a \cup \bar{b}\bar{a}^*b$ . Hence by Theorem 6.1,  $\mathbf{t}$  has a unique directive word.  $\square$

Proposition 6.2 shows a great difference between Sturmian words and episturmian words constructed over alphabets with at least three letters. Indeed, when considering words over a ternary alphabet, one can find episturmian words having exactly  $m$  directive words for any  $m \geq 1$ . For instance, the episturmian word  $\mathbf{t}$  directed by  $\Delta = a(b\bar{a})^{m-1}b\bar{c}(ab\bar{c})^\omega$  has exactly  $m$  directive words, namely  $(\bar{a}\bar{b})^i a (b\bar{a})^j b\bar{c}(ab\bar{c})^\omega$  with  $i + j = m - 1$ . Notice that the suffix  $b\bar{c}(ab\bar{c})^\omega$  of  $\Delta$  is regular wavy, and the other  $m - 1$  spinned versions of  $\Delta$  that also direct  $\mathbf{t}$  arise from the  $m - 1$  words that are block-equivalent to the prefix  $a(b\bar{a})^{m-1}$ .

**Remark 6.3.** One of the referees suggested to give the name “anti-normalized” to any spinned infinite word containing infinitely many  $R$ -spinned letters, but no factor in  $\bigcup_{a \in \mathcal{A}} a\mathcal{A}^*\bar{a}$ . He/She noticed that one can prove “anti-normalized” versions of Theorem 5.2 and Lemma 5.3, and consequently Theorem 6.1 can be rephrased as: “An episturmian word has a unique directive word if and only if its normalized and anti-normalized spinned directive words coincide”. Nevertheless, anti-normalized spinned words should be handled with care since any such spinned word could (when letters are ultimately  $R$ -spinned) direct several episturmian words by the third item of Proposition 2.2.

*Acknowledgements.* The authors thank Eddy Godelle for his remarks and his suggestion for improvement of the proof of Theorem 5.2.

## REFERENCES

- [1] J.-P. Allouche and J. Shallit, *Automatic sequences: Theory, Applications, Generalizations*. Cambridge University Press (2003).
- [2] P. Arnoux and G. Rauzy, Représentation géométrique de suites de complexités  $2n + 1$ . *Bull. Soc. Math. France* **119** (1991) 199–215.
- [3] J. Berstel, Sturmian and episturmian words (a survey of some recent results), in *Proceedings of CAI 2007. Lect. Notes Comput. Sci.*, Vol. **4728**. Springer-Verlag (2007).
- [4] V. Berthé, C. Holton and L.Q. Zamboni, Initial powers of Sturmian sequences. *Acta Arith.* **122** (2006) 315–347.
- [5] X. Droubay, J. Justin and G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy. *Theoret. Comput. Sci.* **255** (2001) 539–553.
- [6] S. Ferenczi, Complexity of sequences and dynamical systems. *Discrete Math.* **206** (1999) 145–154.
- [7] A. Glen, *On Sturmian and episturmian words, and related topics*. Ph.D. thesis, The University of Adelaide, Australia (2006).
- [8] A. Glen, A characterization of fine words over a finite alphabet. *Theoret. Comput. Sci.* **391** (2008) 51–60.
- [9] A. Glen and J. Justin, *Episturmian words: a survey*. *RAIRO-Theor. Inf. Appl.* (submitted). e-print [arxiv:0801.1655](https://arxiv.org/abs/0801.1655) (2007).
- [10] A. Glen, J. Justin and G. Pirillo, Characterizations of finite and infinite episturmian words via lexicographic orderings. *Eur. J. Combin.* **29** (2008) 45–58.
- [11] A. Glen, F. Levé and G. Richomme, Quasiperiodic and Lyndon episturmian words. *Theoret. Comput. Sci.* DOI: [10.1016/j.tcs.2008.09.056](https://doi.org/10.1016/j.tcs.2008.09.056).
- [12] E. Godelle, Représentation par des transvections des groupes d’artin-tits. *Group Geom. Dyn.* **1** (2007) 111–133.
- [13] J. Justin and G. Pirillo, Episturmian words and episturmian morphisms. *Theoret. Comput. Sci.* **276** (2002) 281–313.
- [14] J. Justin and G. Pirillo, On a characteristic property of Arnoux-Rauzy sequences. *RAIRO-Theor. Inf. Appl.* **36** (2003) 385–388.
- [15] J. Justin and G. Pirillo, Episturmian words: shifts, morphisms and numeration systems. *Int. J. Found. Comput. Sci.* **15** (2004) 329–348.
- [16] F. Levé and G. Richomme, Quasiperiodic infinite words: some answers. *Bull. Eur. Assoc. Theor. Comput. Sci.* **84** (2004) 128–138.
- [17] F. Levé and G. Richomme, Quasiperiodic episturmian words, in *Proceedings of the 6th International Conference on Words*, Marseille, France (2007).

- [18] F. Levé and G. Richomme, Quasiperiodic Sturmian words and morphisms. *Theoret. Comput. Sci.* **372** (2007) 15–25.
- [19] M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Vol. **17**. Addison-Wesley (1983).
- [20] M. Lothaire, *Algebraic Combinatorics on Words*, Encyclopedia of Mathematics and its Applications, Vol. **90**, Cambridge University Press (2002).
- [21] M. Morse and G. Hedlund, Symbolic Dynamics II. Sturmian trajectories. *Amer. J. Math.* **61** (1940) 1–42.
- [22] G. Paquin and L. Vuillon, A characterization of balanced episturmian sequences. *Electron. J. Combin.* **14** (2007) 33.
- [23] N. Pytheas Fogg, Substitutions in dynamics, arithmetics and combinatorics. *Lect. Notes Math.*, Vol. **1794**. Springer (2002).
- [24] G. Rauzy, Nombres algébriques et substitutions. *Bull. Soc. Math. France* **110** (1982) 147–178.
- [25] G. Rauzy, Mots infinis en arithmétique, in *Automata on Infinite words*, edited by M. Nivat, D. Perrin. *Lect. Notes Comput. Sci.*, Vol. **192**. Springer-Verlag, Berlin (1985).
- [26] G. Richomme, Conjugacy and episturmian morphisms. *Theoret. Comput. Sci.* **302** (2003) 1–34.
- [27] G. Richomme, Lyndon morphisms. *Bull. Belg. Math. Soc. Simon Stevin* **10** (2003) 761–785.
- [28] G. Richomme, Conjugacy of morphisms and Lyndon decomposition of standard Sturmian words. *Theoret. Comput. Sci.* **380** (2007) 393–400.
- [29] G. Richomme, *A local balance property of episturmian words*, in *Proc. DLT '07. Lect. Notes Comput. Sci.*, Vol. **4588**. Springer, Berlin (2007) 371–381.
- [30] R. Risley and L. Zamboni, A generalization of Sturmian sequences: combinatorial structure and transcendence. *Acta Arith.* **95** (2000) 167–184.
- [31] P. Séébold, Fibonacci morphisms and Sturmian words. *Theoret. Comput. Sci.* **88** (1991) 365–384.
- [32] Z.-X. Wen and Y. Zhang, Some remarks on invertible substitutions on three letter alphabet. *Chinese Sci. Bull.* **44** (1999) 1755–1760.

Communicated by J. Berstel.

Received February 26, 2008. Accepted October 17, 2008.