Intern. J. Fuzzy Mathematical Archive
Vol. 8, No. 1, 2015, 51-57
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 11 August 2015
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

The Number of Fuzzy Subgroups of a finite Dihedral $D_{p^m q^n}$

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Received 31 July 2015; accepted 9 August 2015

Abstract. The main goal of this paper is to give formula for the number of fuzzy subgroups of a finite dihedral group $D_{p^m q^n}$ by using a recurrence relation

Keywords: Algebra; fuzzy subgroups; finite abelian p-groups; generating function; Dihedral group

AMS Mathematics Subject Classification (2010): 20N25, 20E15

1. Introduction

One of the famous problems in fuzzy subgroup theory is to find the number of fuzzy subgroups of dihedral group. Some authors discussed the case of cyclic groups and finite Abelian group [1-3, 8-9]. In [10] Tărnăuceanu obtain formula for the fuzzy subgroups of dihedral group D_{p^m} and $D_{p^m q}$ which was verified by Dabari, Saeedi and Farrokhi in [6]. In our earlier work [7], we have determined the number of subgroups of a finite abelian group of rank two. In this present paper, we establish a recurrence relation for the number of distinct fuzzy subgroups of dihedral group $D_{p^m q^n}$ and solve this recurrence relation by using the concept which was already used by Tărnăuceanu and Bentea [2].

2. Preliminaries

Let X be a fixed non-empty set. A *Fuzzy Sets* μ on X is a function from X to [0,1]. A *fuzzy subset* μ of a group G is called a *fuzzy subgroup* of G if

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\} \ \forall\, x, y \in G$

(ii) $\mu(x^{-1}) = \mu(x) \ \forall\, x \in G$

Clearly, $\mu(e) = \max \mu(G)$. For each $\alpha \in [0,1]$, the *level subset* corresponding to α is defined as $\mu_\alpha = \{x \in G : \mu(x) \geq \alpha\}$[4] . A fuzzy subset μ is a *fuzzy subgroup* of G if and only if its level subsets are subgroups of G [5].

Let $\sim$ be the natural equivalence relation on the set of all fuzzy subsets of G. Then $\mu \sim \rho$ iff $(\mu(x) > \mu(y) \Leftrightarrow \rho(x) > \rho(y) \ \forall x, y \in G)$. By the above equivalence relation, the fuzzy subgroups of G can be classified up to equivalence classes in such a way that two fuzzy subgroups μ *and* ρ of G are distinct if $\mu \nsim \rho$. Suppose that G is a finite group and $\mu: G \rightarrow [0,1]$ is a fuzzy subgroup of G. Let $\mu(G) = \{\alpha_1, \alpha_2, \alpha_3, .., \alpha_n\}$ and assume that $\alpha_1 > \alpha_2 > ... > \alpha_n$. Then μ provides the following chain of subgroups of G ending with G

$$\mu_{\alpha_1} \subseteq \mu_{\alpha_2} \subseteq \mu_{\alpha_3} \subseteq \cdots \subseteq \mu_{\alpha_n} = G \qquad (1)$$

Then for any $x \in G$ and $i = 1,2,3, \dots, n$, we have $\mu(x) = \alpha_i \iff x \in \mu_{\alpha_i} - \mu_{\alpha_{i-1}}$ with the assumption that $\mu_{\alpha_0} = \emptyset$. According to Volf [11], a necessary and sufficient condition for two fuzzy subgroups of G to be equivalent with respect to $\sim$ is that they are same level fuzzy subgroups, that is, they determine the same chain of subgroups of type (1). Hence, there exists a bisection between the equivalence classes of fuzzy subgroups of G and the set of chains of subgroups of the group G, which end in G. If F_G denotes the number of all distinct fuzzy subgroups of G, then F_G is the number of chains of subgroups of length one of G ending in G plus the number of chains of subgroups of length more than one of the group G, which end in G.

$F_G = 1 + \sum_{H<G} F_H$

$$F_G = 1 + \sum_{distinct\ H \in Iso(G)} (F_H \times n_H), \qquad (2)$$

where Iso(9) is the set of representative classes of subgroups of G and n_H denotes the size of the isomorphism class with representative *H*.

3. Dihedral group

The group D_n is known as Dihedral group of order 2n, this group is generated by two elements: a rotation y of order n and a reflection x of order 2. Under these notations, we have, $D_n =< x, y | x^2 = 1, y^n = 1, xy = y^{-1}x >$.

Theorem 3.1. [12] Every subgroup of D_n is cyclic or dihedral. A complete list of subgroups is as follows:-

(i) Cyclic subgroups $< y^d >$, where d|n

(ii) Dihedral subgroups $< y^d, xy^i >$, where d|n and $0 \le i \le d-1$

Here cyclic subgroup $< y^d > \sim Z_{n/d}$ and $< y^d, xy^i > \sim D_{n/d}$. As follows form Theorem 3.1, we have following results:

Theorem 3.2. Number of subgroups of group D_{p^n} are $\begin{cases} \mathbf{1} & \sim Z_{p^i} \text{ where } i = \overline{0,n} \\ p^{n-i} & \sim D_{p^i} \text{ where } i = \overline{0,n} \end{cases}$

Theorem 3.3. Number of subgroups of group $D_{p^m q^n}$ are

$$\begin{cases} \mathbf{1} & \sim Z_{p^i q^j} \text{ where } i = \overline{0,m} \text{ and } j = \overline{0,n} \\ \frac{p^m q^n}{p^i q^j} & \sim D_{p^i q^j} \text{ where } i = \overline{0,m} \text{ and } j = \overline{0,n} \end{cases}$$

Now we contract recurrence relations for number of fuzzy subgroups of group $Z_{p^m q^n}$, D_{p^m} and $D_{p^m q^n}$.

Theorem 3.4. The number of all distinct fuzzy subgroups of $D_{p^m q^n}$ is

$$\frac{2^m}{1-p}\Bigg[\Big[\sum_{a+2b+k+t=n}^{\infty}\Big[\binom{m}{k}\binom{m+t}{m} q^{k+t} p^{m+1} - 2\binom{m+1}{k}\binom{m+t}{m} +$$

$$\boldsymbol{p\binom{m}{k}\binom{m-1+t}{m-1}\Big] (-1)^{k+b+1} 2^{b+t} q^{b} \binom{a+b}{a,b} \left((q+2-\frac{1-q}{1-p})\right)^{a} - \sum_{a+2b+k+t=n-1}^{\infty} \Big[\binom{m}{k}\binom{m+t}{m} q^{k+t} p^{m+1} - 2\binom{m+1}{k}\binom{m+t}{m} + p\binom{m}{k}\binom{m-1+t}{m-1}\Big] (-1)^{k+b+1} 2^{b+t} q^{b+1} \binom{a+b}{a,b} \left((q+2-\frac{1-q}{1-p})\right)^{a}\Big]\Big]}$$

4. Construction of recurrence relations

Theorem 4.1. If $F(Z_{p^m q^n})$ denotes number of fuzzy subgroups of group $Z_{p^m q^n}$, then we can establish relation as

$$\boldsymbol{F(Z_{p^m q^n}) - 2F(Z_{p^m q^{n-1}}) - 2F(Z_{p^{m-1} q^n}) + 2F(Z_{p^{m-1} q^{n-1}}) = 0.}$$

Proof: We know that, the number of subgroups of group $Z_{p^m q^n}$ which are isomorphic to group $Z_{p^k q^h}$ is 1 for $0 \le h \le n$ and $0 \le k \le m$

By using (2), we have $F(Z_{p^m q^n}) = 1 + \sum_{k=0}^{m}\sum_{h=0}^{n-1} F(Z_{p^k q^h}) + \sum_{k=0}^{m-1} F(Z_{p^k q^n})$ (3)

Change n into n-1 in (3), we get

$$F(Z_{p^m q^{n-1}}) = 1 + \sum_{k=0}^{m}\sum_{h=0}^{n-2} F(Z_{p^k q^h}) + \sum_{k=0}^{m-1} F(Z_{p^k q^{n-1}}) \quad (4)$$

From (3) and (4), we get

$$F(Z_{p^m q^n}) - 2F(Z_{p^m q^{n-1}}) = \sum_{k=0}^{m-1} F(Z_{p^k q^{n-1}}) \quad (5)$$

Change m into m-1 in (5), we get

$$F(Z_{p^{m-1} q^n}) - 2F(Z_{p^{m-1} q^{n-1}}) = \sum_{k=0}^{m-2} F(Z_{p^k q^{n-1}}) \quad (6)$$

From (5) and (6), we get

$$F(Z_{p^m q^n}) - 2F(Z_{p^m q^{n-1}}) - 2F(Z_{p^{m-1} q^n}) + 2F(Z_{p^{m-1} q^{n-1}}) = 0 \quad (7)$$

Theorem 4.2. If $F(D_{p^m})$ denotes number of fuzzy subgroups of groupD_{p^m}, then we can establish relation as

$$\boldsymbol{F(D_{p^m}) - 2pF(D_{p^{m-1}}) = 2F(Z_{p^m}) - 2pF(Z_{p^{m-1}})}$$

Proof: By using (2) and Theorem 3.2, we have

$$F(D_{p^n}) = 1 + \sum_{i=0}^{n} F(Z_{p^i}) + \sum_{i=0}^{n-1} p^{n-i} F(D_{p^i}) \quad (8)$$

$$F(Z_{p^n}) = 1 + \sum_{i=0}^{n-1} F(Z_{p^i}) \quad (9)$$

(8)-(9), we get

$$F(D_{p^n}) - 2F(Z_{p^n}) = \sum_{i=0}^{n-1} p^{n-i} F(D_{p^i}) \quad (10)$$

Change n to n-1 in (10), we get

$$F(D_{p^{n-1}}) - 2F(Z_{p^{n-1}}) = \sum_{i=0}^{n-2} p^{n-i-1} F(D_{p^i}) \quad (11)$$

(10)-p (11), we get

$$F(D_{p^n}) - 2pF(D_{p^{n-1}}) = 2F(Z_{p^n}) - 2pF(Z_{p^{n-1}}) \quad (12)$$

Theorem 4.3. If $F(D_{p^m q^n})$ denotes number of fuzzy subgroups of group$D_{p^m q^n}$, then we can establish relation as

$$\boldsymbol{F(D_{p^m q^n}) - 2qF(D_{p^m q^{n-1}}) - 2pF(D_{p^{m-1} q^n}) + 2pqF(D_{p^{m-1} q^{n-1}}) = 2F(Z_{p^m q^n}) - 2qF(Z_{p^m q^{n-1}}) - 2pF(Z_{p^{m-1} q^n}) + 2pqF(Z_{p^{m-1} q^{n-1}}).}$$

Proof: By using (2) and Theorem 3.2, we have

$$F(D_{p^m q^n}) = 1 + \sum_{i=0}^{m}\sum_{j=0}^{n} F(Z_{p^i q^j}) + \sum_{i=0}^{m-1}\sum_{j=0}^{n} p^{m-i}q^{n-j}F(D_{p^i q^j}) + \sum_{j=0}^{n-1} q^{n-j}F(D_{p^m q^j}) \quad (13)$$

$$F(Z_{p^m q^n}) = 1 + \sum_{i=0}^{m-1}\sum_{j=0}^{n} F(Z_{p^i q^j}) + \sum_{j=0}^{n-1} F(Z_{p^m q^j}) \quad (14)$$

(13)-(14), we get

$$F(D_{p^m q^n}) - 2F(Z_{p^m q^n}) = \sum_{i=0}^{m-1}\sum_{j=0}^{n} p^{m-i}q^{n-j}F(D_{p^i q^j}) + \sum_{j=0}^{n-1} q^{n-j}F(D_{p^m q^j}) \quad (15)$$

Change n to n-1 in (15), we get

$$F(D_{p^m q^{n-1}}) - 2F(Z_{p^m q^{n-1}}) = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} p^{m-i}q^{n-j-1}F(D_{p^i q^j}) + \sum_{j=0}^{n-2} q^{n-j-1}F(D_{p^m q^j}) \quad (16)$$

(15)-q (16), we get

$$F(D_{p^m q^n}) - 2F(Z_{p^m q^n}) - 2qF(D_{p^m q^{n-1}}) + 2qF(Z_{p^m q^{n-1}}) = \sum_{i=0}^{m-1} p^{m-i}F(D_{p^i q^n}) \quad (17)$$

Change m to m-1 in (17), we get

$$F(D_{p^{m-1} q^n}) - 2F(Z_{p^{m-1} q^n}) - 2qF(D_{p^{m-1} q^{n-1}}) + 2qF(Z_{p^{m-1} q^{n-1}}) = \sum_{i=0}^{m-2} p^{m-1-i}F(D_{p^i q^n}) \quad (18)$$

(17)-p(18), we get

$$F(D_{p^m q^n}) - 2F(Z_{p^m q^n}) - 2qF(D_{p^m q^{n-1}}) + 2qF(Z_{p^m q^{n-1}}) - 2pF(D_{p^{m-1} q^n}) + 2pF(Z_{p^{m-1} q^n}) + 2pqF(D_{p^{m-1} q^{n-1}}) - 2pqF(Z_{p^{m-1} q^{n-1}}) = 0$$

$$F(D_{p^m q^n}) - 2qF(D_{p^m q^{n-1}}) - 2pF(D_{p^{m-1} q^n}) + 2pqF(D_{p^{m-1} q^{n-1}}) = 2F(Z_{p^m q^n}) - 2qF(Z_{p^m q^{n-1}}) - 2pF(Z_{p^{m-1} q^n}) + 2pqF(Z_{p^{m-1} q^{n-1}}) \quad (19)$$

Now we find solution of these recurrence relations which satisfies required initial condition.

5. Solution of recurrence relation

Multiply both sides of (19) by x^n and summation n from 1 to ∞, we get

$$\sum_{n=1}^{\infty} F(D_{p^m q^n})\, x^n - 2q\sum_{n=1}^{\infty} F(D_{p^m q^{n-1}})\, x^n - 2p\sum_{n=1}^{\infty} F(D_{p^{m-1} q^n})\, x^n + 2pq\sum_{n=1}^{\infty} F(D_{p^{m-1} q^{n-1}})\, x^n = 2\sum_{n=1}^{\infty} F(Z_{p^m q^n})\, x^n - 2q\sum_{n=1}^{\infty} F(Z_{p^m q^{n-1}})\, x^n - 2p\sum_{n=1}^{\infty} F(Z_{p^{m-1} q^n})\, x^n + 2pq\sum_{n=1}^{\infty} F(Z_{p^{m-1} q^{n-1}})\, x^n$$

Take $\sum_{n=0}^{\infty} F(D_{p^m q^n})\, x^n = B_m$ and $\sum_{n=0}^{\infty} F(Z_{p^m q^n})\, x^n = C_m$, we get

$$(B_m - F(D_{p^m})) - 2qxB_m - 2p(B_{m-1} - F(D_{p^{m-1}})) + 2pqxB_{m-1} = 2(C_m - F(Z_{p^m})) - 2qxC_m - 2p(C_{m-1} - F(Z_{p^{m-1}})) + 2pqxC_{m-1}$$

$$(1-2qx)B_m - 2p(1-qx)B_{m-1} - \left(F(D_{p^m}) - 2pF(D_{p^{m-1}})\right) = (2-2qx)C_m - 2p(1-qx)C_{m-1} - 2[F(Z_{p^m}) - pF(Z_{p^{m-1}})].$$

Use the concept $F(Z_{p^m}) = 2^m$ in above equation, we get

$(1-2qx)B_m - 2p(1-qx)B_{m-1} - (2^m(2-p)) = (2-2qx)C_m - 2p(1-qx)C_{m-1} - 2[2^m - p2^{m-1}]$ $(1-2qx)B_m - 2p(1-qx)B_{m-1}$
$= (2-2qx)C_m - 2p(1-qx)C_{m-1}$ (20)

Now we find value C_m, multiply both sides of (8) by x^n and summation n from 1 to ∞, we get

$(C_m - F(Z_{p^m})) - 2xC_m - 2(C_{m-1} - F(Z_{p^{m-1}})) + 2xC_{m-1} = 0$
$C_m(1-2x) - C_{m-1}(2-2x) = F(Z_{p^m}) - 2F(Z_{p^{m-1}}) = 2^m - 2^m = 0$
$C_m - \frac{(2-2x)}{(1-2x)}C_{m-1} = 0.$

The general solution of above equation is $C_m = A(\frac{2-2x}{1-2x})^m$
We know that $F(Z_{p^0 q^n}) = 2^n$
Then $C_0 = \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$
Hence, $C_m = \frac{1}{1-2x}(\frac{2-2x}{1-2x})^m$
Put the value of C_m, we get
$(2-2qx)C_m - 2p(1-qx)C_{m-1} = \left(\frac{2-2x}{1-2x}\right)^m \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-x)}\right)$
Hence, (20) becomes
$(1-2qx)B_m - 2p(1-qx)B_{m-1} = \left(\frac{2-2x}{1-2x}\right)^m \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-x)}\right)$
Then $B_m - \frac{2p(1-qx)}{(1-2qx)}B_{m-1} = \left(\frac{2-2x}{1-2x}\right)^m \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-x)(1-2qx)}\right)$ (21)
C.F of the recurrence relation (S) is $A(\frac{2p(1-qx)}{(1-2qx)})^m$
P.I of the recurrence relation (21) is $B\left(\frac{2-2x}{1-2x}\right)^m$
Now we find B
$B\left(\frac{2-2x}{1-2x}\right)^m - \frac{2p(1-qx)}{(1-2qx)}B\left(\frac{2-2x}{1-2x}\right)^{m-1} = \left(\frac{2-2x}{1-2x}\right)^m \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-x)(1-2qx)}\right)$
$B[1 - \frac{p(1-qx)(1-2x)}{(1-2qx)(1-x)}] = \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-x)(1-2qx)}\right)$
$B = \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-p+(pq+2p-1-2q)x+(2q-2pq)x^2)}\right)$

Then, the general solution of (21) is
$B_m = A(\frac{2p(1-qx)}{(1-2qx)})^m + \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-p+(pq-2q-1+2p)x+(2q-2pq)x^2)}\right)\left(\frac{2-2x}{1-2x}\right)^m$
Put m=0 in $\sum_{n=0}^{\infty} F(D_{q^n})\, x^n = B_0$
$B_0 = \sum_{n=0}^{\infty} \frac{2^n(q^{n+1}+q-2)}{q-1}x^n = \frac{1}{q-1}\left[\frac{q}{1-2qx} + \frac{q-2}{1-2x}\right] = \left[\frac{2(1-qx)}{(1-2qx)(1-2x)}\right]$
Put m=0 in (10), we get
$B_0 = A + \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-p+(-1-2q+pq+2p)x+(2q-2pq)x^2)}\right)$
Hence $\left[\frac{2(1-qx)}{(1-2qx)(1-2x)}\right] = A + \left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-p-(1+2q-pq-2p)x+(2q-2pq)x^2}\right)$

$A = \left[\frac{(-p)(1-qx)}{(1-2qx)(1-p+(pq-2q-1+2p)x+(2q-2pq)x^2)}\right]$

Hence

$B_m =$

$\left[\frac{(-p)(1-qx)}{(1-2qx)(1-p+(pq-2q-1+2p)x+(2q-2pq)x^2)}\right](\frac{2p(1-qx)}{(1-2qx)})^m +$

$\left(\frac{(1-qx)(2-2x-p+2px)}{(1-2x)(1-p+(pq-2q-1+2p)x+(2q-2pq)x^2)}\right)\left(\frac{2-2x}{1-2x}\right)^m$

$B_m = \frac{2^m(1-qx)}{(1-p+(pq-2q-1+2p)x+(2q-2pq)x^2)}\left[-p^{m+1}\frac{(1-qx)^m}{(1-2qx)^{m+1}} + 2\left(\frac{1-x}{1-2x}\right)^{m+1} - p\left(\frac{1-x}{1-2x}\right)^m\right]$

$B_m =$

$\frac{2^m(1-qx)}{1-p}\left[\sum_{a+b+k+t=0}^{\infty}\left[\binom{m}{k}\binom{m+t}{m}q^{k+t}p^{m+1} - 2\binom{m+1}{k}\binom{m+t}{m} +\right.\right.$

$\left.\left.p\binom{m}{k}\binom{m-1+t}{m-1}\right](-1)^{k+b+1}2^{b+t}q^b\binom{a+b}{a,b}\left((q+2-\frac{1-q}{1-p})\right)^a x^{k+t+a+2b}\right]$

But $B_m = \sum_{n=0}^{\infty} F\left(D_{p^m q^n}\right)x^n$, Hence $F\left(D_{p^m q^n}\right) = coefficient\ of\ x^n\ in\ B_m$

$F\left(D_{p^m q^n}\right) =$

$\frac{2^m}{1-p}\left[\left[\sum_{a+2b+k+t=n}^{\infty}\left[\binom{m}{k}\binom{m+t}{m}q^{k+t}p^{m+1} - 2\binom{m+1}{k}\binom{m+t}{m} +\right.\right.\right.$

$\left.p\binom{m}{k}\binom{m-1+t}{m-1}\right](-1)^{k+b+1}2^{b+t}q^b\binom{a+b}{a,b}\left((q+2-\frac{1-q}{1-p})\right)^a -$

$\sum_{a+2b+k+t=n-1}^{\infty}\left[\binom{m}{k}\binom{m+t}{m}q^{k+t}p^{m+1} - 2\binom{m+1}{k}\binom{m+t}{m} +\right.$

$\left.\left.\left.p\binom{m}{k}\binom{m-1+t}{m-1}\right](-1)^{k+b+1}2^{b+t}q^{b+1}\binom{a+b}{a,b}\left((q+2-\frac{1-q}{1-p})\right)^a\right]\right]$ (22)

Corollary [6, 10]. The number of all distinct fuzzy subgroups of $D_{p^m q^1}$ is $\frac{2^m}{(p-1)^2}\left[(m+2)p^{m+2}q - (m+3)p^{m+1}q + 2p^{m+2} - p^{m+1} + (m+2)p^2 - (3m+7)p + (2m+4)\right]$

Proof. Put n=1 in equation (22), we get desired result.

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