Axiomatic Cost and Surplus-Sharing

by

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Abstract

The equitable division of a joint cost (or a jointly produced output) among agents with different shares or types of output (or input) commodities, is a central theme of the theory of cooperative games with transferable utility. Ever since Shapley's seminal contribution in 1953, this question has generated some of the deepest axiomatic results of modern microeconomic theory.

More recently, the simpler problem of rationing a single commodity according to a profile of claims (reflecting individual needs, or demands, or liabilities) has been another fertile ground for axiomatic analysis. This rationing model is often called the bankruptcy problem in the literature.

This Chapter reviews the normative literature on these two models, and emphasizes their deep structural link via the Additivity axiom for cost sharing: individual cost shares depend additively upon the cost function. Loosely speaking, an additive cost sharing method can be written as the integral of a rationing method, and this representation defines a linear isomorphism between additive cost sharing methods and rationing methods.

The simple proportionality rule in rationing thus corresponds to average cost pricing and to the Aumann-Shapley pricing method (respectively in the case of homogeneous or heterogeneous output commodities). The uniform rationing rule, equalizing individual shares subject to the claim being an upperbound, corresponds to serial cost sharing. And random priority rationing corresponds to the Shapley-Shubik method, applying the Shapley formula to the Stand Alone costs.

Several open problems are included. The axiomatic discussion of non-additive methods to share joint costs appears to be a promising direction for future research.

JEL : C71, D62, D63

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Contents

Introduction

The oldest formal principle of distributive justice is, without a doubt, Aristotle's celebrated maxim:

Equals should be treated equally, and unequals, unequally in proportion to relevant similarities and differences

(in the modern rendition by the social psychology literature, see, e.g., Deutsch [1985])

Inspired by the axiomatic approach to the theory of cooperative games (initiated in 1953 by Shapley's seminal contribution–Shapley [1953]), a considerable research effort explores the logical limits of the old maxim, within a small number of simple models of fair division. All such models stage a given production technology and a given set of users of the technology. Individual users influence the production plan in different ways, either by demanding different quantities of output, or contributing different quantities of input, or both. When individual demands (or contributions) are homogeneous (total demand is simply the sum of individual demands) and the technology has constant returns to scale, the fair distribution of inputs and outputs among users can and should simply follow Aristotle's proportionality principle. The logical challenge is to deal with variable returns of the technology and heterogeneity of the individual demands/contributions.

This survey of the theory of cooperative production is organized around three basic models, to which most of the literature is devoted. First we discuss the *rationing model* (Part 1), where a given amount of resources (e.g., money) must be divided among beneficiaries with unequal claims on the resources. In this very bare model, the only available information about the technology is a single point of the production set. Then we look at a *one input - one output* technology (Part 2), where all users consume (possibly different amounts of) a homogeneous output commodity, and contribute (possibly different amounts of) a homogeneous input commodity: a typical problem specifies a list of individual demands of output (resp. input contributions) and the entire production function (with variable returns) and asks to divide fairly the corresponding input cost (resp. output produced). In the third model we assume a technology with a single homogeneous input and one heterogeneous output per agent, (resp. one homogeneous output and one heterogeneous input per agent), and we speak of the *heterogeneous* goods model (Part 3). The formal definition of the cooperative production problem is the same as in the homogeneous good models of Part 2: it consists of a list of heterogeneous demands and the entire production function. The question is to divide fairly the corresponding total input cost (resp. to divide the output produced, given the list of heterogeneous input contributions and the technology).

One important feature in the model is whether the input or output commodities come in indivisible units or are infinitely divisible: we speak of a discrete or real variable, respectively. Both versions are meaningful and important in applications: the goods on demand may be cars (discrete) or length of a runway (real); individual contributions may come in days on the job (discrete) or cash (real) and so on. Each one of the three models involves two kinds of variables: the exogeneously given claims/demands (rationing model, Part 1), demands of output or contributions of input (Parts 2 and 3) on the one hand, and the endogeneously determined shares of the resources (Part 1) or cost shares or output shares (Parts 2 and 3) on the other hand. Each kind of variable can be either discrete (*d*) or real (*r*). In the rationing model (Part 1) the main model is of the "*rr*" type (the exogeneous and endogeneous variables are both real) but we also discuss the *dd*-model (both kinds of variables are discrete). In the cost and surplus sharing models both the *rr*-model and the *dr*-model (exogeneous variable is discrete, endogeneous one is real) play an important role; for instance the classical model of cooperative games is of type *dr*, but the theory of Aumann-Shapley pricing happens in the *rr*-world.

Besides the issue of realism, the choice between a discrete or a real model involves a familiar trade-off. A discrete model is mathematically much simpler, as it typically involves no topological difficulty. For instance, in the *dd*-model the set of possible (rationing or cost sharing) methods is essentially finite; in the *dr* model, a typical cost sharing method is a linear operator on a finite dimensional space (see Part 3); in the *rr* method a cost sharing model is a linear operator on a functional space. On the other hand, in the discrete model even the basic proportionality principle mentioned at the outset is hard to write, it can be approximated at best.

Some comments about the type of axioms we impose are in order. A few of them convey a simple idea of equity: of this type are Equal Treatment of Equals and the crucial Dummy axiom in the heterogeneous goods model expressing some notion of reward. There are also some incentive compatibility requirements, such as No Advantageous Reallocation in Parts 1 and 2 or Demand Monotonicity in Part 3. For the latter axiom, the equity and incentive compatibility interpretations coexist and reinforce each other.

Yet the main axiomatic tools throughout the survey are driven neither by equity nor incentives. They are properties of *structural invariance* expressing the commutativity of the allocation method with respect to certain variations in the cost or surplus sharing problem under scrutiny. For instance Consistency, the leading axiom in Part 1, requires the method to commute with a variation in the society of agents concerned. Additivity, by far the most important axiom in Parts 2 and 3, is commutativity of the cost sharing method with respect to the sum of cost functions. Scale Invariance and Unit Invariance (Part 3) are about changing the unit in which a particular good is measured. And so on.

The structural invariance axioms are the powertools of the mathematical analysis, the backbone of the most interesting characterization results. The most spectacular example is the Additivity axiom in Parts 2 and 3. The entire set $\mathscr{R}$ of rationing methods studied in Part 1 is shown to be linearly isomorphic to the set of additive cost sharing methods in the homogeneous good model (Theorem 2.2) and isomorphic to the extreme points of the (convex) set of additive cost sharing methods in the heterogeneous goods model (Theorems 3.1 and 3.3). This double isomorphism allows us to follow the "same" allocation method in the three different models: the proportional rationing method (Part 1) becomes average cost sharing in Part 2 and the Aumann-Shapley method in Part 3; the uniform gains method in Part 1 becomes serial cost sharing in Parts 2, 3; priority rationing (Part 1) becomes incremental cost sharing in Parts 2,3. New methods emerge as well: uniform losses rationing suggests the dual serial cost sharing method.

The main lesson to be learned from this overview is that the powerful structural invariance axioms are double-edged swords. For instance Additivity with respect to cost functions implies an isomorphism between cost sharing and rationing methods, but it also severely limits the choices open to the mechanism designer. When a structural invariance axiom such as Additivity conflicts with a set of reasonable equity and/or incentives requirements, we feel that the invariance axiom must be the first to go. This opens up the question of finding a less restrictive version of the invariance axiom, for which the impossibility result becomes a limited possibility result. A good example is the new and yet hardly explored space of nonadditive cost sharing methods: Section 3.7.

Relation to other chapters in the Handbook

In the current chapter, we view the users of the technology as entirely passive: they have inelastic demands of output, or input contributions, or claims. The axiomatic analysis is supposed to enlighten a benevolent dictator on the possible interpretations of fairness when dividing cost or output, or whatever resources must be split among the participants. Another approach sees the users as rational microeconomic agents, endowed with classical preferences and choosing independently and strategically the amount of output they want to consume or of input they choose to contribute. Any given division method (whether or not it is fair in the sense of this chapter) yields a specific noncooperative demand game (resp. input contribution game) where a user's cost share depends on the entire profile of demands (resp. his output share depends on the profile of contributions). In this view a division method is a *decentralization* device: each user knows his own preferences but may be completely unaware of other users' preferences. The general results of Chapter 5, Vol. 1 on mechanism design and of Chapter 23, Vol. 2 on strategyproofness become then relevant. In particular the incentives properties of the uniform gains rationing method (and more generally of the fixed path methods: see Section 1.8) are reinterpreted in Chapter 23, Vol. 2 in the context of the fair division problem with singlepeaked preferences: there they mean that the direct revelation of preferences is a strategyproof mechanism. Similarly, serial cost sharing (and more generally the fixed path generated methods in Part 3) gives rise to a strategyproof social choice function whenever the cost function is supermodular.

The second important link is with Chapter 26, Vol. 2, on fair allocation and 20, Vol. 2, on fair compensation. A third way to look at the cooperative production problem is as a special instance of the social choice problem in a particular economic environment. The social planner takes into account the technology (production set) and the whole profile of individual preferences, then selects a *first best* (Pareto efficient) allocation that he deems optimal. One way to do so is by defining a full fledged social preference over the set of feasible allocations in the economy: Chapter 16, Vol. 2, explains why this approach will lead, in most models, to a conceptual dead-end in classical Arrowian fashion. An alternative route is to simply select one (efficient) allocation by means of fairness axioms: this is the route taken in Chapter 26, Vol. 2, for a general family of allocation problems that includes cooperative production; this is also the approach taken in Chapter 20, Vol. 2, for a family of models very close to our homogeneous goods problems. The main difference is that in the first best approach the social choice function selects the shares of output as well as the shares of input from the profile of individual preferences: therefore the profile of cost shares, say, depends on more than the profile of demands and the cost function, and a formula such as the Shapley-Shubik method is generally not relevant.

1. Rationing

1.1. The problem and some examples

A *rationing problem* is a triple (N, t, x) where N is a finite set of agents, the nonnegative real number t represents the amount of resources to be divided, the vector $x = (x_i)_{i \in N}$ specifies for each agent i a *claim* x_i, and these numbers are such that

$$0 \le x_i \text{ for all } i \quad ; \quad 0 \le t \le \sum_{i \in N} x_i$$

A solution to the rationing problem is a vector $y = (y_i)_{i \in N}$, specifying a share y_i for each agent i and such that

$$0 \le y_i \le x_i \text{ for all } i \quad ; \quad \sum_{i \in N} y_i = t$$

The crucial inequality $y_i \le x_i$ may not be meaningful if claims are subjective evaluations of needs (or responsibility): an agent may underestimate his "objective" need (or responsibility), prompting the social planner to violate the above inequality. Our model ignores this possibility: thus it is the most convincing when the claims x_i are "objectively" measured, as in the case of a contractual debt.

Several of the axiomatic properties of rationing methods pertain to *variations in the population (also called society)* N of concerned agents. See the merging properties in Section 1.2 and the consistency property playing the leading role from Section 1.3 onward. Therefore the formal model must specify the set $\mathcal{N}$ of potential agents from which a certain subset N is selected to generate an actual problem. In general, $\mathcal{N}$ could be finite or infinite, although the "real" society N is always finite. One exception is the discussion of symmetric and consistent rationing methods in Section 1.4: there we must assume that the set $\mathcal{N}$ is infinite.

It is neither easy nor necessary at this stage to interpret a rationing problem directly as a model of cooperative production. The link of the rationing model with cooperative production will become apparent in Part 2, when we discuss the implications of the powerful Additivity axiom in the homogeneous goods model of cost and surplus sharing: see Theorem 2.2.

Inheritance problems provide the oldest example on record of the rationing problem (see O'Neill [1982] and Rabinovich [1973] borrowing examples from the Babylonian Talmud): here *t* is the liquidation value of the bankrupt firm and x_i is the debt owed to creditor *i* (see Aumann and Maschler [1985]).

Taxation is another important example: now *t* is total tax to be levied and x_i is agent *i*'s fiscal liability (see Young [1988] [1990]). Note that in a taxation example, the resources to be divided are a “bad,” whereas they are a “good” in a inheritance or bankruptcy story. A microeconomic example similar to taxation is the cost sharing of an indivisible public good: *t* is the cost of the good and x_i is the benefit to agent *i*.

Rationing occurs in markets where the price of a commodity is fixed (for instance, at zero): *t* is the available supply and x_i is agent *i*'s demand of good *i* (Benassy [1982] Drèze [1975]). Medical triage is an example: *t* measures the available medical resources and x_i is the quantity needed by agent *i* for full treatment (Winslow [1992]). Rationing food among refugees is similar: x_i measures a nutritional need, and *t* the nutritional value of the available food. The supply chain problem is a management example: the central supplier collects orders from its retailers and can not meet all demands at once (Cachon and Larivière [1996]).

Often the resources to be divided come in indivisible units: organs for transplants, seats in crowded airplanes or in popular sports events, visas to potential immigrants (Elster [1992]) as well as cars allocated by General Motors to its car dealers. In this case x_i and *t* are integers, e.g., in the case of organs or visas x_i can only be 0 or 1. Important examples where x_i and *t* are integers come from queuing and scheduling: a server can process one job per unit of time and agent *i* requests x_i jobs; at any time *t* such that $t \leq \sum x_i$, the service protocol solves a rationing problem.

A *rationing method r* associates to any rationing problem (N,t,x) a solution $y = r(N,t,x)$. We study rationing methods with the axiomatic methodology. In the main model, that we call the *rr*-model, all variables t, x_i, y_i vary over the nonnegative real line, and correspond to divisible resources, claims, demands, etc. Sections 1.2 to 1.8 are devoted to the *rr*-model. In Section 1.9 we study the *dd*-model where t, x_i, y_i are all nonnegative integers.

We denote by $\mathscr{R}$ the set of rationing methods with a given potential population $\mathscr{N}$. When required for clarity, we indicate whether claims and shares are discrete or real variables, e.g., $\mathscr{R}_{dd}$ means that claims are integer valued and shares are real valued.

All rationing methods discussed below satisfy the following property

Resource Monotonicity (RM)

$$\{t \leq t'\} \Rightarrow \{r(N,t,x) \leq r(N,t',x)\} \textit{ for all } N,t,t' \textit{ and } x$$

This is a mild and compelling requirement: when resources (whether desirable or not) increase, no one should see his share reduced. In most of the results below, Resource Monotonicity needs not be assumed and follows from the other axioms (e.g., Upper Composition (1.4) implies *RM*); exceptions are Theorems 1.4 and 1.6.

With the exception of Section 1.7, all methods are equitable in the sense that they do not discriminate a priori between the agents. This corresponds to the two familiar axioms:

Equal Treatment of Equals:

$$x_i = x_j \Rightarrow y_i = y_j \text{ for all } N,x,t \text{ and all } i,\, j \in N$$

Symmetry:

$$y = r(N,t,x) \text{ is a symmetric function of the variables } x_i, i \in N$$

Note that Symmetry implies Equal Treatment of Equals.

An important operation is the duality operator transforming gains into losses. If *r* is a rationing method, its dual method *r** is defined as

$$r^*(N,t,x) = x - r(N,x_N - t,x) \text{ for all } N,t,x$$

(where we use the notation $x_N = \sum_{i \in N} x_i$). Given *x*, the method *r** allocates *t* units of "gains" exactly as *r* allocates the corresponding losses $(x_N - t)$.

Overview of Part 1. The proportional rationing method is characterized in Section 1.2 by the property that it treats claims as anonymous transferable "bonds". In Section 1.3 we discuss two important methods equalizing respectively the gains and losses on individual claims and introduce the Upper and Lower Composition axioms. The celebrated Contested Garment method inspired by a bargaining interpretation of the rationing model is the subject of Section 1.4. The next two Sections focus on the structural invariance axiom called Consistency, leading to the characterization of parametric methods in Section 1.5, and of the equal sacrifice methods in Section 1.6. Section 1.7 discusses the rich family of asymmetric methods meeting Consistency Upper and Lower Composition. Fixed path methods, discussed in Section 1.8, are another family of asymmetric methods, that play a crucial role in Part 3. The probabilistic rationing of indivisible goods is the discrete variant of the rationing model: see Section 1.9. Finally, Section 1.10 discusses the variant of the rationing problem where the available resources may exceed the sum of individual claims.

1.2. The proportional method

With the exception of Section 1.9, all variables t, x_i and y_i are real numbers: we are in the *rr*-model. The *proportional rationing* method is defined as follows:

$$y = pr(N,t,x) = \frac{t}{x_N}\cdot x \text{ whenever } x_N > 0$$

(whenever $x_N = 0$, all rationing methods select $y = 0$)

Several related characterizations of the proportional rationing method pertain to the possibility of merging a subset of agents into a single agent with the combined demand, or, conversely, to split one single agent into several smaller agents. These results are all related to the fact that proportional rationing "discounts" each unit of claim/demand by the same factor, irrespective of who presents this unit of claim/demand (whether the agent has a large or small global demand is irrelevant). Hence the proportional method is compelling when claims are transferable like anonymous bonds. Any other method is vulnerable to manipulations by transferring claims across agents or changing their identity by adding "artificial" agents.

For a given set *N* of agents and a subset *S*, $S \subseteq N$, we denote by $N^{[S]}$ the set with $(|N|-|S|+1)$ agents where all agents in *S* have been "merged" into a single agent denoted *S**. For instance:

$$N = \{1,2,3,4,5\}, S = \{2,4,5\} \Rightarrow N^{[S]} = \{1, S^*, 3\}$$

For any *x* in $\mathbb{R}_+^N$ we denote $x_S = \sum_{i\in S} x_i$, $x_{[S]} =$ projection of *x* on $\mathbb{R}_+^S$; and $x^{[S]} \in \mathbb{R}_+^{N^{[S]}}$ is defined by $x_i^{[S]} = x_i$ if $i \notin S$, $x_{S^*}^{[S]} = x_S$. Now we consider four independence properties of increasing notational complexity; yet, by Theorem 1 below, they are logically equivalent in $\mathscr{R}$.

No Advantageous Reallocation (NAR)

For all $N, S,$ all t and all x, x' : $x^{[S]} = x'^{[S]} \Rightarrow r_S(N,t,x) = r_S(N,t,x')$ (1.1)

This says that by reallocating individual demands among the agents in *S*, the total share of this coalition is unchanged, thus preventing such maneuver to be profitable.

Irrelevance of Reallocations (IR)

For all $N, S,$ all t and all x, x' : $x^{[S]} = x'^{[S]} \Rightarrow \{r_j(N,t,x) = r_j(N,t,x')$ for all $j \in N \setminus S\}$

Reallocations of demands do not affect agents outside the scope of the reallocation.

Independence of Merging and Splitting (IMS)

For all $N, S,$ all t and all x : $r(N,t,x)^{[S]} = r(N^{[S]}, t, x^{[S]})$

The merging operation is the move from *N* to $N^{[S]}$; splitting is the converse transformation. By repeated applications of Independence of Merging and Splitting we get the following property. Assume $(N_k)_{k \in M}$ is a partition of *N* and let $x \to x^*$ be the "merging" mapping from $\mathbb{R}_+^N$ into $\mathbb{R}_+^M$ given by

$$x_k^* = x_{N_k} \text{ for all } k \in M$$

Then $r_{N_k}(N,t,x) = r_k(M,t,x^*)$ for all $k \in M$. The next property provides a more precise decomposition of the rationing method by means of a partition.

Decomposition (DEC)

For any N and any partition $(N_{k'})_{k' \in M}$ of $N,$ for all $t,$ all x and all k :
$r(N,t,x)_{[N_k]} = r(N_k, t_k, x_{[N_k]})$ where $t_k = r_k(M,t,x^*)$

We compute first the shares of the members (N_k) of the partition, and then allocate each share within the relevant coalition.

Theorem 1.1. *Assume N contains three agents or more. The proportional method meets all four properties NAR, IR, IMS, and DEC. Conversely the proportional method is the* ***only*** *rationing method meeting* ***any one*** *of the four above properties.*

The characterizations gathered in Theorem 1.1 are inspired from similar results by Banker [1981], O'Neill [1982], Moulin [1987] and Chun [1988]. See also Chun [1999] and de Frutos [1999]. Remarkably, the symmetry properties (such as Equal Treatment of Equals) are not used.

In Section 1.4, another characterization of the proportional rationing method is based on the fact that it is *self-dual* $(r = r^*)$, that is to say it allocates gains and losses in exactly the same way (Proposition 1.6).

1.3. Uniform Gains and Uniform Losses

This pair of rationing methods are as important and (almost) as simple as the proportional method. They aim at equalizing, respectively, the actual "gains" y_i and the net losses $(x_i - y_i)$ across agents, under the feasibility constraints of a rationing method:

The *Uniform Gains* method *ug*:

$$y_i = ug_i(N,t,x) = \min\{\lambda, x_i\} \quad \text{where } \lambda \text{ is the solution of } \sum_N \min\{\lambda, x_i\} = t$$

The *Uniform Losses* method *ul:*

$$y_i = ul_i(N,t,x) = (x_i - \mu)_+ \quad \text{where } \mu \text{ is the solution of } \sum_N (x_i - \mu)_+ = t$$

(where $(z)_+ = \max\{z, 0\}$). In the literature, these two methods are often called Constrained Equal Awards, and Constrained Equal Losses.

For a given rationing problem (N,t,x), let us denote by *Y* the set of feasible solutions:

$$Y(N,t,x) = \{y \in \mathbb{R}_+^N | 0 \le y_i \le x_i \text{ and } \sum_N y_i = t\}$$

One checks easily that $ug(N,t,x)$ is the unique solution maximizing over $Y(N,t,x)$ the "leximin" ordering; that is, it lexicographically maximizes the smallest coordinate y_i, then the next smallest coordinate and so on. Similarly, $ul(N,t,x)$ is the unique maximizer of the "leximin" ordering applied to the vector of losses $(x_i - y_i)$.

The pair $\{ug, ul\}$ is a dual pair: $ul = ug^*$ and $ug = ul^*$. This important fact allows a parallel axiomatic treatment of these two methods.

Both methods *ug*, *ul*, as well as *pr* and all other symmetric methods discussed below, respect the natural order of gains and losses. That is, they meet the following two axioms

$$\textit{Ranking}: \quad x_i \le x_j \Rightarrow y_i \le y_j \qquad (1.2)$$

$$\textit{Ranking*}: \quad x_i \le x_j \Rightarrow (x_i - y_i) \le (x_j - y_j) \qquad (1.3)$$

The two axioms above are *dual*, namely a rationing method *r* satisfies one axiom if and only if the dual method *r** satisfies the dual axiom.

Although both methods *ug*, *ul* agree on the ranking of absolute gains and losses, they differ sharply in the ranking of relative gains and losses. Consider the following two dual axioms:

$$\textit{Progressivity:} \quad 0 < x_i \le x_j \Rightarrow \frac{y_j}{x_j} \le \frac{y_i}{x_i}$$

$$\textit{Regressivity:} \quad 0 < x_i \le x_j \Rightarrow \frac{y_i}{x_i} \le \frac{y_j}{x_j}$$

Proposition 1.1. *The uniform gains method is Progressive, but not Regressive. It is the most progressive method among those satisfying Ranking.*

The uniform losses method is Regressive, but not Progressive. It is the most regressive method among those satisfying Ranking.*

(The precise definition of "the most progressive" is left to the reader.)

Our next pair of dual axioms plays a very important role throughout this Part. They are structural invariance properties (see Introduction) allowing to decompose the computation of shares when the available resources are estimated from above or from below:

Upper Composition (UC):

$$\text{For all } N, x \text{ and } t, t' \ : \ \{0 \le t \le t' \le x_N\} \Rightarrow \{r(N,t,x) = r(N,t,r(N,t',x))\} \qquad (1.4)$$

Lower Composition (LC):

$$\text{For all } N, x \text{ and } t, t' : \{0 \le t' \le t \le x_N\} \Rightarrow \{r(N,t,x) = r(N,t',x) + r(N,t-t',x-r(N,t',x))\} \qquad (1.5)$$

If we allocate first the resources t', and later it appears that the available resources are actually lower, namely *t*, Upper Composition allows to simply take the optimistic shares $r(N,t',x)$ as the initial demands from which to further ration until *t*. We may forget about the initial demands *x* once we know an upper bound of the actual resources. Note that *UC* implies Resource Monotonicity.

Dually, if we know a lower bound t' of the actual resources *t*, Lower Composition allows to distribute the pessimistic shares $r(N,t',x)$, subtract these shares from the initial demands and distribute the balance $(t-t')$ according to the reduced claims $x - r(N,t',x)$.

Proposition 1.2. *The three methods pr, ug and ul meet the two axioms Upper Composition and Lower Composition.*

The family of methods meeting UC and LC is large: in Section 1.7 we describe a rich set of such methods, and we show – with the help of additional requirements – that our three basic methods *pr*, *ug* and *ul* play a central role within this family: Corollary to Theorem 1.5. For the time being, we state two pairs of dual characterizations of *ug* and *ul*. They are technically simple, but their interpretation is quite interesting. In the following statement, we omit the variable N that plays no role.

Independence of Claim Truncation (ICT)

For all $N,t,x: r(t,x) = r(t, x \wedge t)$ where $(x \wedge t)_i = \min\{x_i, t\}$

The part of one's claim that is not feasible has no influence on the allocation of the resources:

Composition from Minimal Rights (CMR)

For all $N,t,x: r(t,x) = m(t,x) + r\ (t - m_N(t,x), x - m(t,x))$, where $m_i(t,x) = (t - x_{N\setminus i})_+$

Agent i's minimal claim $m_i(t,x)$ is this part of the resources that he will receive, even in the most pessimistic case where the claims of all other agents are met in full. CMR is the special case of LC where $t' = m_N(t,x)$.

Proposition 1.3. *(Dagan [1996], Herrero and Villar [2000]). The Uniform Gains method is characterized by the two properties Lower Composition and Independence of Claim Truncation. The Uniform Losses method is characterized by Upper Composition and Composition for Minimal Rights.*

A different approach uses a priori bounds on individual shares, namely bounds that do not depend on the size of other agent's claims. We denote by $|N| = n$ the cardinality of N.

$$\textit{Lower Bound:}\ \text{for all } N,t,x, \text{ and all } i \quad : \quad y_i = r_i(N,t,x) \geq \min\{x_i, \frac{t}{n}\}$$

$$\textit{Upper Bound:}\ \text{for all } N,t,x, \text{ and all } i \quad : \quad y_i = r_i(N,t,x) \leq \{\frac{t}{n} + (x_i - \frac{x_N}{n})\}_+$$

It is plain that *ug* meets the Lower but not the Upper Bound, whereas *ul* meets the Upper but not the Lower Bound. Lower Bound says that agent *i* is guaranteed a fair share of the resources *unless* he demands no more than the fair share, in which case his demand is met in full.

Dually, Upper Bound states that agent i's loss $x_i - y_i$ is not smaller than the average deficit $x_N - t$, unless his claim is smaller than the average deficit, in which case he gets no resources.

The Lower Bound has a lot of bite when *t* is small; if $t \le n \cdot \min_i \{x_i\}$, Lower Bound forces equal gains: $y_i = t / n$ for all i. Similarly, if *t* is close enough to x_N, Upper Bound forces equal losses:

$$\{x_N - n \cdot \min_i \{x_i\} \le t \le x_N\} \Rightarrow \{x_i - y_i = x_j - y_j \text{ all } i, j\}$$

Note that for $|N| = 2$, Lower Bound characterizes the *ug* method, and (by duality) Upper Bound characterizes the *ul* method. This simple fact does not extend to the case $|N| \ge 3$; however, we can still characterize the *ug* method if we bring Lower Composition to the rescue. Consider the following very mild requirement:

Zero Consistency:

$$\text{For all } N, t, x \text{ and all } i \quad : \quad \{x_i = 0\} \Rightarrow \{r(N,t,x)_{[N \setminus i]} = r(N \setminus i, t, x_{[N \setminus i]})\} \qquad (1.6)$$

It is hard to imagine under what circumstances the presence of a null demand agent (who therefore receives nothing) could influence the allocation of resources among the other, active agents.

Proposition 1.4. *The Uniform Gains method is characterized by the following three properties: Lower Bound, Lower Composition and Zero-Consistency.*

The Uniform Losses method is characterized by the three properties, Upper Bound, Upper Composition and Zero-Consistency.

1.4. The Contested Garment method and Self Duality

The contested garment method is a rationing method for *two* agents only, in the vein of the familiar "split the difference" principle for two person bargaining. The interpretation of x_i as the verifiable claim of agent *i* (as opposed to a vague demand) is required for the application of the *cg* method and its *n*-person extensions. The method is inspired by the following two quotes from the Babylonian Talmud (see O'Neill [1982], Aumann and Maschler [1985]): "R. Tahifa, the Palestinian, recited in the presence of R. Abbahu: two [people] cling to a garment; [the decision is that] one take as much as his grasp reaches and the other take as much as his grasp reaches and the rest is divided equally between them." “Two hold a garment . . . if one of them says, 'It is all mine' and the other says, 'Half of it is mine,' . . . the former then receives three quarters and the latter receives one quarter.”

Consider a two person rationing problem (t, x_1, x_2). We can interpret agent *i*'s "grasp" optimistically as $\min\{x_i, t\}$ (in case his own claim takes absolute priority over the other claim) or pessimistically as $(t - x_j)_+$ (if the other agent gets his full claim). Then we split the remaining deficit (case of optimistic claims) or surplus (case of pessimistic claims). Both computations yield the same method:

$$y_1 = \min\{x_1, t\} + \frac{1}{2}(t - \min\{x_1, t\} - \min\{x_2, t\}) \quad \text{(optimistic grasp)}$$

$$= (\mathrm{t - x}_2)_+ + \frac{1}{2}(t - (t - x_1)_+ - (t - x_2)_+) \qquad \text{(pessimistic grasp)}$$

A more transparent reading of this formula in the case $x_1 \leq x_2$ is:

$$\begin{aligned} &\text{if } t \leq \min\{x_1, x_2\}: && y_1 = y_2 = \frac{1}{2}t \\ &\text{if } x_1 \leq t \leq x_2: && y_1 = \frac{x_1}{2};\, y_2 = t - \frac{x_1}{2} \\ &\text{if } \max\{x_1, x_2\} \leq t: && y_1 = \frac{1}{2}(t + x_1 - x_2);\, y_2 = \frac{1}{2}(t + x_2 - x_1) \end{aligned} \tag{1.7}$$

The Contested Garment method is self-dual, $r^* = r$, namely it allocates gains and losses in exactly the same way. This property follows at once from the optimistic and pessimistic formulas above:

$$r_1^*(t, x) = x_1 - \frac{1}{2}(x_N - t - (x_2 - t)_+ + (x_1 - t)_+) = \frac{1}{2}(t + x_1 + (t - x_1)_+ - x_2 - (t - x_2)_+)$$

and the identity $z + (z - t)_+ = \min\{z, t\}$.

The Contested Garment method is *self-dual*, $r^* = r$, namely it allocates gains and losses in exactly the same way. This property follows at once from the optimistic and pessimistic formulas above:

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and the identity $z + (z - t)_+ = \min\{z, t\}$.

On the other hand, the Contested Garment method fails Upper Composition and (by duality) Lower Composition: property (1.4) fails for $x_1 = 10$, $x_2 = 20$, $t = 15$ and $t' = 18$.

Note that *cg* coincides with *ug* for small *t*, i.e., $t \le \min\{x_1, x_2\}$, and with *ul* for large *t*, i.e., $\max\ \{x_1, x_2\} \le t$. More importantly, *cg* shares the two invariance properties used above to capture *ug* and *ul*:

Proposition 1.5. *(Dagan [1996])*

The Contested Garment method is characterized by Self-Duality and Independence of Claim Truncation; or by Self-Duality and Composition from Minimal Rights; or by Equal Treatment of Equals, Independence of Claim Truncation, and Composition from Minimal Rights.

Compare this result, for two-person problems, with the following compact characterization of the proportional method, for problems of arbitrary size.

Proposition 1.6. *(Young [1988])*

The proportional method is characterized by Self Duality and Upper Composition; or by Self Duality and Lower Composition.

Two natural extensions of the *cg* method for an arbitrary number of agents have been proposed. The first one relies on the observation that for $n = 2$, the *cg* method is the average of the two priority methods. The *12-priority* method is the rationing method (denoted *prio*(12)) that gives absolute priority to agent 1 over agent 2, hence:

$$\text{if } t \text{ is such that } t \le x_1 \quad : \quad y = (t, 0)$$
$$\text{if } t \text{ is such that } x_1 \le t \le x_1 + x_2 \quad : \quad y = (x_1, t - x_1)$$

Define symmetrically the *21-priority* method *prio*(21) and notice that formula (1.7) defining *cg* can be written as:

$$cg = \frac{1}{2}\, prio(12) + \frac{1}{2}\, prio(21)$$

Hence the first generalization of *cg* as the *Random Priority* method, namely the arithmetic average of the priority methods over all orderings of *N*. Let σ be an ordering of *N* as $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, namely σ_1 is the highest priority agent and so on. We define $y = prio(\sigma)(N, t, x)$ as follows:

$$\text{if } k \text{ is an integer such that: } \sum_{i=1}^{k} x_{\sigma_i} \le t \le \sum_{i=1}^{k+1} x_{\sigma_i}$$

$$y_{\sigma_j} = x_{\sigma_j} \text{ for } j = 1, \ldots, k$$

$$y_{\sigma_{k+1}} = t - (\sum_1^k x_{\sigma_i}) \quad (1.8)$$

$$y_{\sigma_j} = 0 \text{ for } j = k+2, \ldots, n$$

Random Priority method:

$$y = \frac{1}{n!} \sum_{\sigma} prio(\sigma)(N, t, x) \text{ where the sum bears on all orderings of } N \quad (1.9)$$

The second natural extension of *cg* to any *n* uses an explicit mixture of the uniform gains and uniform losses methods. This is the *Talmudic* rationing method due to Aumann and Maschler [1985] (who argue convincingly that its intuition was present already in the ancient Talmudic literature)

Talmudic method:

$$y = tal(N, t, x) = ug(N, \min\{t, \frac{x_N}{2}\}, \frac{x}{2}) + ul(N, (t - \frac{x_N}{2})_+, \frac{x}{2}) \quad (1.10)$$

The Talmudic method halves each claim and follows Uniform Gains until each half claim is met. It then applies the Uniform Losses method to the remaining half claims. The Talmudic method coincides with *cg* in the case of two agents – yet another equivalent formulation of *cg*.

Both the Talmudic and Random Priority methods are self-dual. Both coincide with Uniform Gains whenever $t \le \min_i\{x_i\}$ and with Uniform Losses whenever $t \ge \max_i\{x_{N \setminus i}\}$.

The next result shows a remarkable relation between the two methods, Random Priority and Talmudic, and the two most important value solutions for cooperative games, namely the Shapley value and the nucleolus. These solutions are defined in Section 3.2 and 3.6 respectively. Fix a rationing problem (N, t, x) and define two (dual) cooperative games, generalizing the bargaining interpretation of the contested garment:

$$\text{for all } S \subseteq N \quad : \quad v(S) = \min\{x_S, t\} \text{ (optimistic grasp)}$$

$$w(S) = (t - x_{N \setminus S})_+ \text{ (pessimistic grasp)}$$

Note that $v(N) = w(N) = t$.

Theorem 1.2. (O'Neill [1982], Aumann and Maschler [1985])

i) The Random Priority method allocates the resources according to the Shapley value of the above games.

ii) The Talmudic method allocates the resources according to the nucleolus of the above games.

1.5. Consistent and symmetric methods

The axiom of Consistency has played a major role in the recent microeconomic literature on distributive justice, see Chapter 26, Vol. 2. See also the surveys by Thomson [1990] and Maschler [1990]. Consistency in the rationing problem is both very natural and extremely powerful, as demonstrated by the results of this and the next subParts.

Consistency (CSY):

For all $N, S, S \subseteq N,$ all $t,$ all $x : r(N \setminus S, t - r_S(N,t,x)\,,\, x_{[N \setminus S]}) = r(N,t,x)_{[N \setminus S]}$ (1.11)

Equivalently, Consistency can be defined by looking at coalitions *S* with a single agent *i*:

$$r(N \setminus i\,,\, t - r_i(N,t,x), x_{[N \setminus i]}) = r(N,t,x)_{[N \setminus i]}$$

The axiom says that upon removing one (or several) agent from the society *N*, and taking away the resources allocated to this agent (or agents) within N, the allocation of shares within the reduced society remains the same. In other words, changing the status of an agent from "active participant" to "passive expense of resources" does not alter the overall distribution; removing one agent and his share of resources is of no consequence to other agents. Thus Consistency is a decomposition property with respect to changes in the set of relevant agents.

Note that Consistency is a self-dual axiom: a rationing method is consistent if and only if its dual method is consistent as well.

In this Section we discuss symmetric methods only. In this family a powerful characterization result of (essentially) all consistent methods is available.

Our first result says that Consistency allows us to extend in at most one way a two person symmetric rationing method.

Proposition 1.7. *Let* $r(\{1,2\})(t,(x_1,x_2))$ *be a rationing method* ***defined for two person problems only****. Assume that* $r(\{1,2\})$ *is symmetric and resource monotonic. Then there is at* ***most one*** *consistent rationing method r (defined for all finite societies N) that coincides with r({1,2}) for all two- person problems. Moreover, r is symmetric and resource monotonic.*

The discussion of parametric methods below establishes that *pr, ug, ul* as well as the Talmudic rationing method are consistent (of course this claim can be checked directly). Therefore Proposition 1.6 has the following corollaries:

i) the Talmudic method is the only consistent extension of the contested garment method (for two-person problems) to an arbitrary number of agents,

ii) the Uniform Gains method is the only consistent method satisfying Lower Bound $(y_i \geq \min\{x_i, \frac{t}{2}\})$ for two agents problems, (and a dual statement holds for Uniform Losses by Proposition 1.4).

Proposition 1.7 begs the question: what symmetric two person rationing methods can be extended to a (symmetric) consistent method for an arbitrary number of agents? A general answer is given by Dagan and Volij [1997] and Kaminski [2000]: a certain binary relation associated with the two person method must be transitive. Theorem 1.3 below gives a much more transparent answer under one additional mild requirement:

Continuity

$$r(N,t,x) \text{ is continuous in } (t,x), \text{ for all } N \qquad (1.12)$$

We define now the family of *parametric* rationing methods. They are the key to Theorem 1.3. Let $f(\lambda, z)$ be a real valued function of two real variables, with $0 \leq \lambda \leq \Lambda$ and $z \geq 0$; the Upper Bound Λ may be finite or infinite. We assume:

$$f(0,z) = 0 \quad ; \quad f(\Lambda\,; z) = z \quad f(\lambda, z) \text{ is nondecreasing and continuous in } \lambda \text{ over } [0,\Lambda] \quad (1.13)$$

To any such function *f* we associate a unique rationing method *r* as follows

$$\text{For all } N,t,x \quad : \quad r_i(N,t,x) = f(\lambda, x_i) \text{ where } \lambda \text{ is a solution of } \sum_{i\in N} f(\lambda, x_i) = t$$

(this equation may have an interval of solutions λ but they all give the same shares to every agent). We call *r* the parametric method associated with *f*. By construction a parametric method is symmetric; clearly, it is consistent as well.

The three basic methods *pr*, *ug* and *ul* are parametric, for the following functions *f*:

Proportional: $f(\lambda, z) = \lambda.z$ and $\Lambda = 1$

Uniform Gains: $f(\lambda, z) = \min\{\lambda, z\}$ and $\Lambda = +\infty$

Uniform Losses: $f(\lambda, z) = (z - \frac{1}{\lambda})_+$ and $\Lambda = +\infty$

Among the two extensions of *cg* discussed in Section 1.4, the Random Priority method is *not* consistent, whereas the Talmudic method *is* consistent. To check the former claim, take $N = \{1,2,3\}$, $t = 10, x = (6,8,10)$ and compute the shares allocated under Random Priority:

$$y_1 = \frac{1}{3}\cdot 6 + \frac{1}{6}2 = 2\frac{1}{3}\,;\; y_2 = \frac{1}{3}8 + \frac{1}{6}4 = 3\frac{1}{3}\,;\; y_3 = \frac{1}{3}10 + \frac{1}{6}4 + \frac{1}{6}2 = 4\frac{1}{3}$$

Next remove agent 3 and his share $4\,{}^1\!/_3$, which leaves us with the reduced problem: $N \setminus \{3\}, t' = 5\,{}^2\!/_3, x' = (6,8)$. Now the shares under Random Priority are: $y'_1 = y'_2 = 2\,{}^5\!/_6$.

To check the latter claim, we show that the Talmudic method is parametric. Set $\Lambda = 2$ and define $f(\lambda, z)$ as follows:

$$\begin{aligned} f(\lambda,z) &= \frac{\lambda}{1-\lambda} && \text{for } 0 \le \lambda \le \frac{z}{z+2} \\ &= \frac{z}{2} && \text{for } \frac{z}{z+2} \le \lambda \le \frac{z+4}{z+2} \\ &= z - \frac{2-\lambda}{\lambda-1} && \text{for } \frac{z+4}{z+2} \le \lambda \le 2 \end{aligned}$$

The next result establishes that parametric methods capture, essentially, all consistent and symmetric rationing methods.

Theorem 1.3. (Young [1987]) *A parametric method is a consistent and symmetric rationing method. Conversely, a rationing method satisfying Equal Treatment of Equals, Consistency and Continuity can be represented as a parametric method where* $f(\lambda, z)$ *is continuous in both variables.*

Note that in the converse statement, it is enough to assume pairwise consistency, namely the restriction of property (1.11) to subsets *S* containing two agents. On the other hand, the converse statement holds only if we assume that the size of the set *N* can be arbitrarily large, that is to say, the set $\mathcal{N}$ of potential agents must be infinite. This is an important limitation of Theorem 1.3 as well as of Theorem 1.4, in the next Section that does not apply to Theorem 1.5 in Section 1.7.

The class of parametric methods is very rich. Chun, Schummer and Thomson [1998], for instance, discuss a method of egalitarian inspiration much different from any of the methods discussed in this survey.

1.6. Equal Sacrifice Methods

The *equal sacrifice* methods are an important subset of the parametric ones. They appear early on in the discussion of equitable taxation schedules (see Mill [1859] and the discussion in Young [1990]).

Fix a real valued function $u(z)$ of the nonnegative real variable *z*, and suppose that *u* is continuous and *strictly* increasing. Think of *u* as a reference utility function. Loosely speaking, the equal sacrifice rationing method associated with *u* is defined by solving for all N, t, x the following system of equations:

$$u(x_i) - u(y_i) = u(x_j) - u(y_j) \text{ for all } i, j \in N \text{ and } \sum_{i \in N} y_i = t \qquad (1.14)$$

Because *u* is strictly increasing, the above system has at most one solution. Assume for a moment that such a solution exists. Then at the allocation *y*, each and every agent contributes an equal "sacrifice," namely the same net utility loss measured along the reference utility scale *u*. This is especially appealing in the context of taxation. Let x_i be agent *i*'s taxable income, y_i be his after tax income, and $(x_N - t)$ be the total tax to be levied. Then the system (1.14) distributes taxes so as to equalize the net sacrifice measured along the scale *u*. Concavity of *u* – decreasing marginal utility – means that a dollar taken from the rich translates into a lesser sacrifice than a dollar taken from the poor. Hence the choice of *u* allows the social planner to adjust the progressivity of taxation while following the normatively transparent principle of equal sacrifice.

Here is a precise definition of the equal sacrifice methods.

Proposition 1.8. *Fix u, a continuous and strictly increasing real valued function defined on the nonnegative real line. For any rationing problem* (N, t, x) *the following system has a unique solution y, and y is a solution to the rationing problem:*

$$\sum_{i \in N} y_i = t \text{ and for all } i : \{y_i > 0 \ \Rightarrow \ u(x_i) - u(y_i) = \max_j \{u(x_j) - u(y_j)\} \qquad (1.15)$$

This rationing method satisfies Symmetry, Ranking, Consistency and Upper Composition.

All equal sacrifice methods are clearly consistent, but, in general an equal sacrifice method fails Lower Composition. The only exceptions are the Proportional and Uniform Losses methods. Moreover, an equal sacrifice method meets Ranking* (1.3) if and only if the utility function *u* is concave.

We turn to some examples of equal sacrifice methods. The simplest ones involve power utility functions:

$u_0(z) = Log\ z$ yields the proportional method

$u^1(z) = z$ yields the Uniform Losses method

Interestingly, the Uniform Gains solution is not an equal sacrifice method, but it is the limit of power methods. Consider the family of utility functions u_p:

$$u_p(z) = -\frac{1}{z^p} \quad \text{where } 0 < p < +\infty \tag{1.16}$$

For p close to zero the corresponding method approaches the proportional method, whereas for p arbitrarily large it approaches the Uniform Gains method. Let us compute for instance the method corresponding to u_1; the system (1.14) always has a unique solution, and yields explicitly the parametric representation:

$$\frac{1}{y_i} - \frac{1}{x_i} = \frac{1}{y_j} - \frac{1}{x_j} \text{ all } i, j \Leftrightarrow y_i = \frac{\lambda \cdot x_i}{\lambda + x_i} \text{ all } i$$

Next consider the family of utilities u^q

$$u^q(z) = z^q \text{ where } 0 < q < +\infty \tag{1.17}$$

For q close to zero, the corresponding method approaches *pr*, for $q = 1$ it is the method *ul*, and for q arbitrary large it approaches the "hyperregressive" method that gives full priority to the agents with the largest x_i. In the case of two agents, this method is defined as

$$\begin{aligned} r(t, x_1, x_2) &= prio(12)(t, x_1, x_2) && \text{if } x_2 < x_1 \\ &= prio(21)(t, x_1, x_2) && \text{if } x_1 < x_2 \\ &= \left(\frac{t}{2}, \frac{t}{2}\right) && \text{if } x_1 = x_2 \end{aligned}$$

Note finally that for $q \le 1, u^q$ is concave and the corresponding method meets Ranking*.

We state next a partial converse of proposition 1.7. It uses three additional axioms:

Strict Monotonicity: for all N, t, t', x : $\{t < t'\} \Rightarrow \{y_i < y'_i \text{ for all } i\}$

Strict Ranking: for all N, t, x and all i, j : $\{x_i < x_j\} \Rightarrow \{y_i < y_j\}$

Scale Invariance: for all N, t, x and $\alpha \ge 0$: $r(N, \alpha \cdot t, \alpha \cdot x) = \alpha \cdot r(N, t, x)$

Strict Monotonicity and Strict Ranking are demanding properties; for instance both *ug* and *ul* (as well as *cg*) fail both requirements. They are intuitively reasonable and yet they cut a subset of rationing methods that is not topologically closed, an unpalatable feature.

Scale Invariance, on the other hand, is an impeccable invariance axiom insisting that the choice of the unit to measure both the demands/claims/taxable income and the available resources, should be of no consequence whatsoever. It is satisfied by all rationing methods discussed so far.

Theorem 1.4. Young [1988]

i) A rationing method satisfying Consistency, Upper Composition, Strict Monotonicity and Strict Ranking must be an equal sacrifice method, defined by system (1.14).

ii) A rationing method satisfying Consistency, Upper Composition, Strict Monotonicity, Strict Ranking and Scale Invariance, must be an equal sacrifice method derived from a power function u_p, $0 < p < \infty$, *((1.16)), or must be the proportional method.*

An important open question. Replace in statement *i)* Strict Monotonicity and Strict Ranking by Monotonicity and Ranking: now all equal sacrifice methods (given by (1.15)) with a concave utility, as well as Uniform Gains, are available. Is this all? Similarly, if in statement *ii)* we weaken the same two axioms in the same way, all methods derived from the power functions u_p ((1.16)) as well as u^q ((1.17)) for $q \leq 1$, and *ug* meet these requirements. Is this all?

Young [1990] offers an empirical "verification" of Theorem 1.4, by showing a number of actual tax schedules that fit well within the family of equal sacrifice methods constructed from the power functions u_p.

1.7. Asymmetric methods: combining the invariance axioms

When the recipients of the resources have different *exogeneous* rights, in addition to their possibly different demands, the symmetry axiom must be abandoned. In bankruptcies and inheritances, creditors or heirs often have different status implying some priorities between their claims, irrespective of their sizes. For instance the federal government's claim on the assets of a bankrupt firm has absolute priority over the claims of the trustees, who have priority over those of the shareholders and so on.

The most asymmetric rationing methods are the priority methods $prio(\sigma)$ ((1.8)). In order to define a *consistent* priority method we must introduce the set $\mathcal{N}$ from which the agents can be drawn. This set can be finite or infinite. Recall that in the previous Part about symmetric methods, $\mathcal{N}$ was any countably infinite set. In the current Part, by contrast, we can accommodate the case of a finite set $\mathcal{N}$.

We denote by σ an ordering (complete, transitive, antisymmetric relation) of $\mathcal{N}$ and for any *finite* subset *N* of $\mathcal{N}$, we also write σ for the induced ordering on *N*. Any finite set *N* is ordered by σ as $N = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and for any rationing problem (N, t, x), we define the allocation $prio(\sigma)(N, t, x)$ exactly as in (1.8). Note that the dual of $prio(\sigma)$ is the priority method with the opposite ordering of $\mathcal{N}$.

The following fact is obvious: for any ordering σ, the priority method prio (σ) meets Consistency, Upper and Lower Composition, and Scale Invariance.

Thus our four powerful invariance axioms are met by the three basic symmetric methods *pr*, *ug* and *ul* as well as by the most asymmetric ones, the priority methods. Theorem 1.5 below describes the relatively simple family of methods satisfying all four axioms: they "connect" the three symmetric methods to the priority ones in interesting ways.

We define the *composition* of rationing methods. Given are $\mathcal{N}$ and a partition $\mathcal{N} = \cup \mathcal{N}_\alpha$ where the parameter α varies in $\mathcal{A}$. For every α we are also given a rationing method on $\mathcal{N}_\alpha$ denoted r^α; moreover $\tilde{r}$ is a rationing method on $\mathcal{A}$. The *composition* of these methods is denoted $\tilde{r}[r^\alpha, \alpha \in \mathcal{A}] = r$. For any problem (N, t, x), with a finite society *N*, $N \subseteq \mathcal{N}$, we define $N_\alpha = N \cap \mathcal{N}_\alpha$ and *A* is the finite subset of $\mathcal{A}$ containing α if and only if N_α is nonempty. The shares $y = r(N, t, x)$ are computed in two steps: first we split *t* among the subsets N_α (i.e., among the "agents" of *A*) according to $\tilde{r}$, then the share z_α allocated to N_α is divided among the agents in N_α according to r^α:

$$z_\alpha = \tilde{r}_\alpha(A, t, (x_{N_\beta})) \text{ for } \alpha \in A \quad ; \quad y_i = r_i^\alpha(N_\alpha, z_\alpha x_{[N_\alpha]}) \text{ for } i \in N_\alpha$$

Thus, the operation of composition generates "two tiered" rationing methods that may apply different equity principles for the aggregate problem (on $\mathcal{A}$) and for any of the decentralized problems (on N_α). Note that the Decentralization property (Section 1.2) says precisely that a certain method is preserved by "self-composition".

We say that the composition operation respects property Q if, whenever all methods $\tilde{r}, r_\alpha, \alpha \in \mathcal{A}$, meet Q, so does the method $\tilde{r}[r_\alpha, \alpha \in \mathcal{A}]$.

Proposition 1.9.

i) The composition of rationing methods respects the following properties: Resource Monotonicity, Upper and Lower Composition, and Scale Invariance.

ii) The composition operation does **not** *respect the Consistency property, or Equal Treatment of Equals.*

iii) If each method $r^\alpha, \alpha \in \mathcal{A}$*, is consistent, and* σ *is an ordering of* $\mathcal{A}$*, the composition* $prio(\sigma)[r^\alpha, \alpha \in \mathcal{A}]$ *is consistent as well.*

Proposition 1.9 shows that the three invariance axioms *UC, LC* and *SI*, are met by a rich family of rationing methods, obtained by composing such methods as *pr, ug, ul* (as well as their asymmetric versions g^w and l^w, to be defined shortly) in an arbitrary number of tiers. There are many more methods in this family, as discussed in Moulin and Shenker [1999].

When we impose *CSY* as well, the set of available methods becomes much simpler, although it still allows a great deal of flexibility. The following asymmetric versions of *ug* and *ul* play a key role in the characterization result.

For any set of *positive* weights w_i, one for each $i \in \mathcal{N}$, we define the *weighted gains method* g^w as follows:

$$\text{for all } N, t, x: y_i = g_i^w(N,t,x) = \min\{\lambda w_i, x_i\} \text{ where } \lambda \text{ solves } \sum_{\mathrm{N}} \min\{\lambda w_i, x_i\} = t$$

Its dual method is the *weighted losses* method l^w:

$$\text{for all } N, t, x: y_i = l_i^w(N,t,x) = \max\{x_i - \mu w_i, 0\} \text{ where } \mu \text{ solves } \sum_{N} \max\{x_i - \mu w_i, 0\} = t$$

The Uniform Gains and Uniform Losses methods are the two particular methods corresponding to uniform weights $(w_i = 1 \text{ for all } i)$. Note that when the weights of the different agents are very unequal, the methods g^w and l^w become arbitrarily close to any priority method: it will be enough to guarantee that if agent *i* is higher than agent *j* in the priority ordering, his weight becomes infinitely bigger than agent *j*'s weight.

Clearly, the methods g^w and l^w meet all four invariance axioms *CSY*, *UC*, *LC* and *SI*. In view of Proposition 1.9, we can construct many rationing methods meeting the four invariance axioms as follows. Partition arbitrarily the set $\mathcal{N}$ in "priority classes" and order these classes. In each priority class, use *either* the proportional, *or* a weighted gains, *or* a weighted losses method. An example is provided by the American bankruptcy law, which arranges the creditors in priority classes and uses the proportional method within each class (Kaminski [2000]).

In order to state the last theorem in this Part, we need two more definitions. We say that the rationing method *r gives priority to agent i over agent j* if *j* does not get anything unless *i*'s demand is met in full: $y_j > 0 \Rightarrow y_i = x_i$ (for all N, t, x). We say that a rationing method is *irreducible* if for any pair i, j, *r* does not give priority to *i* over *j*. For instance *pr*, g^w and l^w (for any *w*) are all irreducible (recall that we require positive weights w_i).

Theorem 1.5. (Moulin [2000])

i) Let r be a rationing method meeting Consistency, Upper and Lower Composition and Scale Invariance. Then there is a partition $\mathcal{N} = \cup \mathcal{N}_\alpha$, *an ordering* σ *of* $\mathcal{A}$, *and for each* α *an irreducible method* r^α *meeting CSY, UC, LC and SI such that:*

$$r = prio(\sigma)[r^\alpha, \alpha \in \mathcal{A}]$$

ii) Let r be an irreducible method meeting Consistency, Upper and Lower Composition and Scale Invariance. If $\mathcal{N}$ *contains at least three agents, then r is either the proportional method, or a weighted gains method, or a weighted losses method.*

In Moulin [2000], the somewhat involved family of irreducible methods for the case $|\mathcal{N}| = 2$ is described in full.

Within the family uncovered in Theorem 1.5, our three basic rationing methods are the only symmetric methods (except in the case $|\mathcal{N}| = 2$).

Corollary to Theorem 1.5. *Assume* $\mathcal{N}$ *contains at least three agents. Then there are exactly* ***three*** *rationing methods satisfying Equal Treatment of Equals and the four invariance axioms: they are the Proportional, Uniform Gains and Uniform Losses methods.*

A much needed next step in the theory of rationing methods is an asymmetric version of Theorem 1.3: what is the set of methods consistent and continuous? Naumova [2000] offers an asymmetric generalization of Theorem 1.4, where the utility functions measuring sacrifice are personalized.

Another interesting open question (discussed in Moulin [2000]) is to generalize Theorem 1.5 (or its Corollary) by dropping one of the four invariance axioms. For instance a method meeting Consistency, Scale Invariance and Upper Composition is *priority to higher demands*: given the profile of demands *x*, this method gives priority to *i* over *j* if and only if $x_i > x_j$, and treats equal demands equally (thus it is symmetric as well); it emerged in Section 1.5 as the limit as some equal sacrifice methods (see the discussion of power methods (1.17)). Its dual method, *priority to lower demands*, meets all four axioms in Theorem 1.5 except Upper Composition. The characterization of all rationing methods meeting Consistency, Scale Invariance and one of the composition axioms is wide open.

1.8. Fixed path methods

This important family of rationing methods contains asymmetric variants of the uniform gains method as well as the priority methods. The fixed path methods play an important role in Part 3 when we discuss Demand Monotonicity (Sections 3.4 and 3.6). They emerge also in the model of fair division under single-peaked preferences (briefly discussed in Section 1.10), where they are a key example of strategy-proof methods. In this Section we merely define these methods and check their invariance properties.

It is necessary to place an exogeneous bound on individual demands. This bound may be finite or infinite. We call it the *capacity* of agent *i* and write X_i where $X_i \leq +\infty$ (real or infinite). A rationing problem (N,t,x) must now satisfy $0 \leq x_i \leq X_i$ for all *i*. We always assume that x_i is finite for all *i*.

A fixed path method is defined from a family of monotone paths $\gamma(N)$, one for each possible society *N*. The path $\gamma(N)$ is a *nondecreasing* mapping from $[0, X_N]$ into $[0, X_{[N]}]$ such that

$$\text{for all } t, 0 \leq t \leq X_N \quad : \quad \sum_N \gamma_i(N,t) = t, \quad 0 \leq \gamma_i(N,t) \leq X_i \text{ for all } i$$

$$\lim_{t \to X_N} \gamma_i(N,t) = X_i \text{ for all } i$$

Note that γ must be continuous in *t*. If X_i is finite for all *i*, the limit property holds true because $\gamma(N, X_N) = X_{[N]}$.

The fixed path method r^γ is now defined as follows:

$r_i^\gamma(N,t,x) = \min\{\gamma_i(N,s), x_i\}$ for all *i*, where *s* is a solution of $\sum_N \min\{\gamma_i(N,s), x_i\} = t_i$ (1.19)

If we take $x = X$ ($x = X_{[N]}$) in the above equation, we find

$$\gamma(N,t) = r^\gamma(N,t,X) \tag{1.20}$$

Examples of fixed path methods include the uniform gains method (for the path $ug(N,t,X)$) any weighted gains method, and any priority method $prio(\sigma)$. Note that a priority method can be represented as a fixed path method only if all capacities X_i are finite (with the possible exception of the capacity of the last agent in the priority ordering). The path $t \to prio(N,t,X)$ follows the edges of the cube $[0, X]$ in the order specified by σ.

If $X_i = X_j$ for all i, j, uniform gains is a symmetric fixed path method. It is the only fixed path method meeting Equal Treatment of Equals: indeed the path $r(N,t,X)$ must be diagonal by ETE, so the claim follows from (1.20) and (1.19).

The set of fixed path methods is not stable by duality: for instance uniform losses is not such a method. It contains no self-dual method.

Proposition 1.10.

i) All fixed path methods meet Upper Composition. They generally fail Lower Composition.

ii) A fixed path method is consistent if and only if the associated paths $N \to \gamma(N)$ commute with the projection operator:

$$\text{for all } N,S,S \subseteq N : \gamma^{(N)}{}_{[S]} = \gamma(S) \text{ namely } \gamma(N,t)_{[S]} = \gamma(S, \gamma_S(N,t)) \text{ for all } t \tag{1.21}$$

Note that all the methods obtained by a priority composition of weighted gains methods (see Proposition 1.9) are fixed path methods and satisfy Lower Composition. I conjecture that there is no other fixed path method meeting LC.

The property (1.21) in statement *ii)* is especially easy to read when the maximal set $\mathcal{N}$ of potential agents is finite. The single path $\gamma(\mathcal{N})$ from 0 to $X_{[\mathcal{N}]}$ generates the entire family of paths $\gamma(N)$ by simple projection on *N*. In this case we can really speak of a one path method.

1.9. Rationing indivisible goods

We modify the rationing model assuming that the commodity being distributed comes in indivisible units. Examples include cars, appliances, seats for a concert or in a plane, organs for transplant, etc.

The formal model is identical, except that all the variables t, x_i, y_i are nonnegative integers. The definitions of a rationing problem, a solution, and a rationing method are unchanged. The set of such methods is denoted $\mathcal{R}_{dd}$. The duality operation is unchanged.

It is convenient to think of a rationing method as a *scheduling* algorithm. Fix *N* and *x* and restrict attention to *resource monotonic* rationing methods. The path $t \to r(N,t,x)$ is described as a sequence $\{i_1,\ldots,i_K\}$ in N, where $K = x_N$ and i_1 is the agent receiving the first unit ($r(N,1,x)$ gives the unit to i_1), i_2 is the agent receiving the second unit and so on. In the sequence $\{i_1,\ldots,i_K\}$, agent *i* appears exactly x_i times, for all *i*.

The definitions of Consistency, Upper and Lower Composition, are all unchanged. Note that Consistency has a particularly simple formulation in terms of the sequence $\{i_1,\ldots,i_k\}$ describing the path $t \to r(N,t,x)$. The axiom says that by simply dropping all occurrences of a certain agent *i* in this sequence, we obtain the sequence describing the path $t \to r(N \setminus i, t, x_{[N \setminus i]})$.

Symmetry is lost when we allocate indivisible goods, as long as the allocation is deterministic. If we now think of the division of resources as a random variable, we can restore this basic equity property, at least in the ex ante sense. It turns out that the *probabilistic* rationing of indivisible goods arise naturally in the discussion of additive cost sharing methods in Part 3 – an entirely deterministic model –.

A probabilistic rationing method associates to every deterministic rationing model (N,t,x) (where t and x_i are integers) a random variable Y such that, with probability one, $0 \le Y_i \le x_i$ for all *i* and $Y_N = t$. The three basic methods *pr*, *ug* and *ul* have a canonical probabilistic analog.

To define the proportional method, fix the profile of claims x_i and throw x_i balls of color *i* in an urn, for each $i \in N$; drawing from the urn *t* times, independently and without replacement – and with uniform probability – generates the random variable $Y = r(N,t,x)$ of the random proportional method. Clearly, the expected value of Y_i is agent *i*'s proportional share $t.(x_i / x_N)$.

The random proportional method meets Consistency, Upper and Lower Composition, as well as Equal Treatment of Equals (ex ante). Conversely, the method is characterized by ETE, UC and LC: Moulin [1999b].

The probabilistic analog of uniform gains is called Fair Queuing (Shenker [1995], Demers et al. [1990]). Given a profile of claims x_i, this method gives away one unit to each agent in round robin fashion, selecting randomly and with uniform probability the ordering in which they receive each unit; an agent drops out only when his claim is not met in full. The expected value of agent *i*'s share after *t* units have been distributed is exactly his uniform gains share in the deterministic problem (N, t, x).

The Fair Queuing method meets Consistency and Upper Composition, but fails Lower Composition. Moulin and Stong [2000] show that this method is characterized by the combination of CSY, UC, and a strong form of Equal Treatment of Equals: two agents with identical claims have equal expected shares, and their actual (ex post) shares never differ by more than one unit.

The dual method, Fair Queuing* allocates each unit with equal probability among the agents with the highest *remaining* claim, i.e., their initial claim net of the units received in earlier rounds.

The characterization results in the probabilistic model of rationing are generally sharper than in the classical model. Moulin and Stong [2000] provide very complete descriptions of the set of methods meeting UC and LC, or CSY and UC (or CSY and LC).

1.10. Two variants of the rationing model

a) Surplus sharing

In a *surplus sharing problem* (N, t, x), the resources *t* must be divided according to the profile of claims *x* and we assume $t \geq x_N$: the resources exceed the sum of individual claims. One interpretation is that x_i is the amount of investment contributed by agent *i* to a joint venture, and *t* is the total return, allowing a profit $t - x_N$. Alternatively, the resources being distributed are undesirable (a tax, a workload) and agent *i*'s claim x_i entitles him to receive no more than a share x_i of the total liability. These claims are not compatible.

A *solution y* to the surplus sharing problem allocates a share y_i to agent *i* in such a way that $0 \le x_i \le y_i$ and $y_N = t$. A *surplus sharing method d* associates a solution $y = d(N,t,x)$ to every surplus sharing problem (N,t,x).

The Proportional surplus sharing method is given by the same formula as in the rationing case. Uniform Gains is defined as follows:

$$y_i = ug_i(N,t,x) = \max\ \{\lambda, x_i\} \quad \text{where } \lambda \text{ is the solution of } \sum_N \max\ \{\lambda, x_i\} = t$$

The counterpart of the Uniform Losses rationing method simply divides the surplus equally, and for this reason we call it the *egalitarian* method:

$$y_i = eg_i(N,t,x) = x_i + \frac{1}{n}(t - x_N)$$

In the surplus sharing model there is no duality operation, hence no analog to the Contested Garment method.

Consistency and Scale Invariance have the same definition but there is only one *Composition* axiom:

$$\text{for all } N,t,t',x \quad : \quad x_N \le t' \le t \Rightarrow d(N,t,x) = d(N,t,d(N,t',x)) \qquad (1.22)$$

Several axiomatic results about rationing have a direct counterpart in the surplus sharing model, and several new results emerge as well. For instance, the proportional method is characterized, as in Theorem 1.1, by Independence of Merging (or Splitting), or by Decomposition. On the other hand, many surplus sharing methods meet No Advantageous Reallocation, including the egalitarian method.

Theorems 1.3 about parametric methods and Theorems 1.4 about equal sacrifice methods are readily adapted to the surplus sharing context: Young [1987], Moulin [1987].

The following result is the counterpart of Theorem 1.5 and its Corollary. The asymmetric generalizations of the egalitarian method divide the surplus in proportion to a set of fixed shares $w_i, w_i \ge 0$ for all *i* and $w_N = 1$:

$$y_i = r_i^w(N,t,x) = x_i + w_i.(t - x_N)$$

The proportional method and the fixed share method r^w meet No Advantageous Reallocation, Consistency, Composition and Scale Invariance. Conversely, these four axioms characterize this family of surplus sharing methods. If we add Equal Treatment of Equals to the list of requirements, only the Proportional and the Egalitarian methods are left. See Moulin [1987].

b) Fair division with single-peaked preferences

Think of a context where the size of agent *i*'s claim/demand x_i is private information, so that agent *i* may choose to misrepresent its actual value if this proves beneficial. We make the following assumption on individual preferences over shares: given that his (real) claim/demand is x_i, agent *i* strictly prefers y_i to y_i' if $y_i' < y_i \leq x_i$ but strictly prefers y_i' to y_i if $x_i \leq y_i' < y_i$. This is the familiar assumption of singlepeakedness. It is a realistic assumption in the rationing problem if the resources being distributed are not freely disposable: think of food that must be eaten in one day, or of a share in a risky venture. For examples and discussion of this assumption see Sprumont [1991] or Barbera, Jackson and Neme [1997].

A fair division method works as follows in this context. The mechanism elicits the peaks of individual preferences (corresponding to the claims x_i in the rationing or surplus sharing models) and each peak x_i can be anywhere in the fixed interval $[0, X_i]$. For a given amount of resources *t*, the sum of individual claims x_N may be smaller or larger than *t*. Thus the allocation problem may be a rationing problem or a surplus sharing problem and an allocation method is a pair of one rationing and one surplus sharing method.

Incentive compatibility of this mechanism is the strategy-proofness property: reporting one's true peak is optimal for every agent, irrespective of other agents' reports.

The key observation is that Uniform Gains (used both for the rationing and the surplus sharing cases) is a strategy-proof method, and so are all the fixed paths methods, where a different path can be used for the rationing and for the surplus sharing cases. Conversely, Uniform Gains is characterized by Strategy-proofness, Efficiency and Equal Treatment of Equals: Sprumont [1991], see also Ching [1994]. Similarly, the consistent fixed path methods are characterized by Strategy-proofness, Efficiency, Consistency and Resource Monotonicity: Moulin [1999a], see also Barbera, Jackson and Neme [1997].

There is also a sizable literature looking at the fair division problem with singlepeaked preferences from an equity angle, and where axioms such as No Envy or Population Monotonicity play a big role: see Thomson [1994a, b], [1995], [1997], Schummer and Thomson [1997] and references there. Once again Uniform Gains stands out as the method of choice.

2. Sharing variable returns

2.1. The problem and some examples

A (one-dimensional) cost sharing *problem* is a triple (N, C, x) where *N* is a finite set of agents, *C* is a continuous nondecreasing *cost function* from $\mathbb{R}_+$ into $\mathbb{R}_+$ such that $C(0) = 0$, and $x = (x_i)_{i \in N}$ specifies for each agent *i* a *demand* x_i, $x_i \geq 0$.

A solution to the cost sharing problem (N, C, x) is a vector $y = (y_i)_{i \in N}$ specifying a cost share for every agent and such that

$$y_i \geq 0 \text{ for all } i, \quad ; \quad \sum_{i \in N} y_i = C(\sum_{i \in N} x_i) \qquad (2.1)$$

A surplus sharing problem is the same mathematical object as a cost sharing problem but its interpretation is different: the given function is denoted *F* (to avoid confusion) and is now a production function; if total input contribution is *z*, total output is $F(z)$; next x_i is agent *i*'s input contribution and y_i is agent *i*'s share of the total output $F(x_N)$. The whole axiomatic discussion is unaffected by the choice of one or the other context, although certain axioms are not equally natural in both contexts. With the exception of a few examples, we use the cost sharing interpretation and terminology throughout Parts 2 and 3.

A cost sharing *method* (resp. a surplus sharing *method*) is a mapping φ associating to any cost sharing (resp. surplus sharing) problem a solution $y = \varphi(N, C, x)$. We denote by $\mathscr{M}$ the set of cost sharing methods thus defined.

Note that variable population axioms play no role in this Section (see comment *b*) in Section 2.5). Therefore, omit *N* in the variables of φ: we write $y = \varphi(C, x)$.

The question addressed in this Section is the equitable division of cost (or surplus) shares when the returns of the technology vary. In other words our initial postulate is that constant returns pose no equity issue whatsoever: costs (or surplus) shares must simply be proportional to individual demands of output (resp. contributions of input). This corresponds to the following axiom on the cost sharing method φ.

Constant Returns

$$\{C(z) = \lambda . z \text{ for all } z \geq 0\} \Rightarrow \{\varphi(N, C, x) = \lambda . x\} \text{ for all } N, \text{ all } \lambda \geq 0 \text{ all } C, \text{ all } x \qquad (2.2)$$

A simple example of a cost sharing problem with increasing returns (decreasing average cost) is discount pricing. The agents in *N* are grouping their order of wine (there is only one quality of wine). Wine can be bought at the local store at price p_1 or at a lower price p_2 from a discount retailer located far away. In the latter case a fixed transportation cost c_0 (independent of the shipment size) must be added. Hence the cost function:

$$C(z) = \min\{p_1 \cdot z, c_0 + p_2 \cdot z\} \qquad (2.3)$$

If the total demand x_N justifies buying from the discount retailer (if $x_N > c_0 / (p_1 - p_2)$) how should total cost be split among the buyers? With several suppliers, the cost function C takes the form of a concave, increasing and piecewise linear function starting at $C(0) = 0$.

Our second example is a cost sharing problem with decreasing returns (increasing average cost). Think of *N* as encompassing all the consumers of a certain good (*N* is a monopsonist for this good) competitively supplied. Thus the demand *z* is met at price $S^{-1}(z)$, where the supply function $p \to S(p)$ is increasing; the resulting cost function $C(z) = z \cdot S^{-1}(z)$ has decreasing returns.

In the surplus sharing context, we find symmetrical examples displaying increasing or decreasing returns technologies. For instance, the agent in *N* may be monopolizing the supply of a certain good for which the demand is competitive. The market absorbs *z* units of output at price $D(z)$ where *D* is decreasing; hence the revenue function $F(z) = z \cdot D(z)$ has decreasing returns.

A simple example with increasing returns involves fixed costs (as in example (2.3)). The agents can use a technology with constant returns r_1 and no fixed input cost, or they can pay a fixed input cost c_0 and benefit from higher returns r_2:

$$F(z) = \max\{r_1 \cdot z \, , r_2 \cdot (z - c_0)\} \qquad (2.4)$$

A brief overview of Part 2 follows. In Section 2.2, the average cost sharing method is characterized in precisely the same way as proportional rationing in Section 1.2. Serial cost sharing is introduced in Section 2.3: together with average cost sharing, it plays the key role in the current model. In Section 2.4 the property of Additivity (of cost shares with respect to the addition of cost functions) is defined and the main theorem derived: the set of rationing methods is isomorphic to that of additive cost sharing methods; in particular serial cost sharing corresponds to the uniform gains rationing method. Some variants and open questions are gathered in Section 2.5.

2.2. Average cost method

The simplest cost sharing method divides total cost in proportion to individual demands. It is denoted *ac*:

$$y = ac(C, x) = \frac{C(x_N)}{x_N} \cdot x \qquad (2.5)$$

(of course, if $x_N = 0$ we must have $y = 0$). The average cost method entirely ignores the returns of the technology between 0 and the total demand x_N. From all the methods discussed in this Section, it is the most informationally economical. This is convenient from an implementation viewpoint, but has no normative appeal per se.

A first type of axiomatic justification for this method mimics those of the proportional rationing method in Section 1.2. The axioms of No Advantageous Reallocations (NAR), Irrelevance of Reallocations (*IR*), and Independence of Merging and Splitting (*IMS*) are transported word for word from that context to that of cost sharing methods by simply replacing the resources *t* in rationing by the cost function *C*. Theorem 1.1 has the following counterpart.

Theorem 2.1. *Assume N contains three agents or more. The average cost method meets the three properties NAR, IR and IMS, as well as the following property:*

No Charge for Null Demand

$$\{x_i = 0\} \Rightarrow \{y_i = \varphi_i(N, C, x) = 0\} \quad \text{for all } C, x \text{ and all } i \qquad (2.6)$$

Conversely, the average cost method is the ***only*** *cost sharing method charging nothing for a null demand and meeting* ***any one*** *of NAR, IR or IMS.*

The interpretation of (2.6) in the case of cost sharing is that no one should have to pay anything for no output; in the case of output sharing, it is sometime referred to as “No Free Lunch”: you don't receive any output if you did not participate in the production process by contributing some money or some labor. All methods discussed in Sections 2.2 to 2.4 satisfy (2.6). In Section 2.5 we give some arguments against this axiom and offer a method that violates it.

The interpretation of the three axioms *NAR*, *IR* and *IMS* is the same as in the case of rationing: one does not need to monitor the "identity" of the various units of demands (whether a certain unit comes from an agent with a large or small demand is irrelevant). Any unit of demand is treated anonymously and therefore there is no benefit in passing them around.

2.3. Serial cost sharing

The average cost (average returns) method entirely ignores the variation of the returns between 0 and x_N. When those returns vary widely and when individual demands are of very different size as well, this result in an unpalatable distribution of costs (or output). Consider the (decreasing returns) cost function

$$C(z) = (z - 10)_+ \text{ where, as usual, } (a)_+ = \max\{a, 0\} \qquad (2.7)$$

The first 10 units are free, and additional demands cost 1 per unit. Say $N = \{1,2,3\}$ and consider the profile of demands $x = (3,5,7)$. The average cost method gives $y = (1, 1\tfrac{2}{3}, 2\tfrac{1}{3})$. Is it fair that agent 1 pays anything, when he could argue that his fair share of the 10 free units is $3\tfrac{1}{3}$ and that he is not consuming that much? The point is that agent 1 is charged the high average cost that he did not cause in the first place: as $C(3x_1) = 0$, if no one else asks more than he does, no one has to pay; hence he should not be held responsible for costs that only arise because other agents demand more than he does.

Notice that, viewed in the light of output sharing, the argument is less convincing: here $F(z) = (z - 10)_+$ is a production function requiring a fixed cost of 10 before output can be collected (a particular example of (2.4)). Agent 1's contribution of 3 units of input is useful, even if applied to pay the fixed cost; other agents should give him *some* share of the output.

Next we look at the (increasing returns) cost function (a special case of the discount pricing example (2.3)):

$$C(z) = \min\{z, 9 + \frac{z}{10}\} \qquad (2.8)$$

with $x = (3,5,7)$. Average cost yields $y = (2.1, 3.5, 4.9)$ so agent 1 ends up paying less than his Stand Alone cost $C(x_1) = 3$. Note that the first 10 units cost 1 apiece, and that the price drops to .1 for each additional unit. This time, agents 2 and 3 protest that they were the ones responsible for reaching the low marginal cost, because $3x_1 < 10$, so agent 1 should not get any benefit from that; his fair share of the cost is 3 because the returns are constant up to the level $3x_1$.

Notice that the argument is even stronger in the output sharing context. The production function has a high return of 3 up to 10 units of input, after which the return drops to $1/10$. Agent 1 is entitled to a fair share of the "good returns": as his "demand" falls below this fair share 10/3, he should receive 9 units of output, a far cry from what the average returns method offers him.

The above discussion suggests the following upper and lower bounds on cost shares, depending on the variation of marginal costs/returns. The set N is fixed and $\#(N) = n$.

Increasing marginal costs bounds (IMC bounds)

$$\text{if } C \text{ is convex} \quad : \quad C(x_i) \le y_i = \varphi_i(C,x) \le \frac{C(nx_i)}{n} \text{ for all } i, \text{ all } x \qquad (2.9)$$

Decreasing marginal cost bounds (DMC bounds)

$$\text{if } C \text{ is concave:} \quad \frac{C(nx_i)}{n} \le y_i = \varphi_i(C,x) \le C(x_i) \text{ for all } i, \text{ all } x \qquad (2.10)$$

We let the reader check that each one of the announced bounds is compatible with budget balance in the corresponding domain of cost functions. For instance a convex cost function such that $C(0) = 0$ is subadditive hence the left-hand inequality in the IMC bound is feasible. And so on.

Consider a convex cost function. The *Stand Alone lower bound* $y_i \ge C(x_i)$ says simply that no agent can benefit from the presence of other users of the technology. This is compelling when marginal costs increase because the consumption of any user creates a negative externality on that of any other user. Indeed, most cost sharing methods discussed in Part 2 meet the Stand Alone lower bound when C is convex, and the Stand Alone upper bound when C is concave (case where any user creates a positive externality on any other user). This is true for all additive methods: Corollary 1 to Theorem 2.1.

By contrast, the two remaining inequalities in (2.9) (2.10) *fail* for the average cost method, as shown by the numerical examples above.

Consider again a convex cost function and the *Unanimity Upper Bound* $y_i \le C(nx_i)/n$. This says that an agent's cost share cannot exceed her share when all agents demand the same amount as she does (and are treated equally). Given that marginal costs increase, this conveys the idea that agent i is entitled to a fair share of the "good" marginal costs, namely those of the first nx_i units. Think of the scheduling example: we are saying that all agents have an equal right to the best (i.e., the earliest) slots in the queue. If x_i is much smaller than the other demands, this bound has a lot of bite.

A symmetrical interpretation holds for the Unanimity Lower Bound ($y_i \geq C(nx_i)/n$) when C is concave: in the output sharing context, it says that agent *i* is entitled to a fair share of the good marginal returns; in the cost sharing context, that she should accept her fair share of responsibility for the "bad" marginal cost. See Moulin [1992] for a general discussion of the notion of unanimity bounds.

The serial cost sharing formula (Shenker [1995], Moulin and Shenker [1992]) is directly inspired by the unanimity bounds. Fix *C* and a profile of demands *x*. We start by relabeling the agents by increasing demands: $x_1 \leq x_2 \leq \ldots \leq x_n$. First we split equally the cost of the first nx_1 units among all agents. Now agent 1 is served (and pays $C(nx_1)/n$) and we split equally the cost of additional units between the remaining agents $\{2,3,\ldots,n\}$, until agent 2 is served, and so on. Formally we define a sequence $x^i, i=1,\ldots,n$ as follows:

$$x^1 = nx_1\,;\ x^2 = x_1 + (n-1)x_2\,;\ \ldots;\ x^i = (n-i+1)x_i + \sum_{j=1}^{i-1} x_j\,;\ \ldots;\ x^n = x_N \qquad (2.11)$$

Note that the sequence x^i is nondecreasing. The serial cost shares are now:

$$y_1 = \frac{C(x^1)}{n};\ y_2 = y_1 + \frac{C(x^2)-C(x^1)}{n-1};\ \ldots;\ y_i = y_{i-1} + \frac{C(x^i)-C(x^{i-1})}{n-i+1};\ \ldots \qquad (2.12)$$

or equivalently:

$$y_1 = \frac{C(x^1)}{n};\ y_2 = \frac{C(x^2)}{n-1} - \frac{C(x^1)}{n(n-1)};\ \ldots;\ y_i = \frac{C(x^i)}{n-i+1} - \sum_{j=1}^{i-1} \frac{C(x^j)}{(n-j+1)(n-j)} \qquad (2.13)$$

In the cases $n=2$ and $n=3$ the general formulas (2.12), (2.13) are simple:

$$n=2,\ x_1 \leq x_2 \quad : \quad y_1 = \frac{1}{2}C(2x_1)\,;\ y_2 = C(x_1+x_2) - \frac{1}{2}C(2x_1)$$

$$n=3\,;\ x_1 \leq x_2 \leq x_3 \quad : \quad y_1 = \frac{1}{3}C(3x_1)\,;\ y_2 = \frac{1}{2}C(x_1+2x_2) - \frac{1}{6}C(3x_1)\,;$$

$$y_3 = C(x_N) - \frac{1}{2}C(x_1+2x_2) - \frac{1}{6}C(3x_1)$$

For instance, in the numerical examples discussed above:

$$\{C(z) = (z-10)_+,\ x = (3, 5, 7)\} \Rightarrow y = (0, 1.5, 3.5)$$

$$\{C(z) = \min\{z, 9 + \frac{z}{10}\},\ x = (3, 5, 7)\} \Rightarrow y = (3, 3.65, 3.85)$$

Recall from the discussion after (2.7), that the serial cost share $y_1 = 0$ is plausible in the cost sharing interpretation, less so in a surplus sharing story. Similarly in the case of the cost function (2.8), the serial cost share $y_1 = 3$ denies any cost saving to agent 1, despite the fact that his presence increases the cost savings of the other two agents: this is clearly an extreme interpretation of fairness in this example.

In the examples, the agent with the smallest demand prefers his serial cost share to his average cost share in the example with increasing marginal cost and his preferences are reversed in the example with decreasing marginal cost. The preferences of the agent with the largest demand are diametrically opposed. This is a general fact.

Kolpin [1998] proposes further interpretations of the serial formula in terms of linear pricing.

We conclude Section 2.3 by generalizing the decentralized bounds (2.9), (2.10) for the serial cost shares to a cost function with arbitrary returns. That is, we give an upper and a lower bound on $y_i = \varphi_i(C, x)$ that only depend upon *C*, x_i and *n*, the number of users. This is important for an uninformed agent, who cannot assess the size of other agents' demands.

Proposition 2.1. *The serial cost sharing method meets the Increasing Marginal Costs bounds ((2.9)) and the Decreasing Marginal Costs bounds ((2.10)). Moreover, for* ***any non decreasing*** *cost function C (such that* $C(0) = 0$ *), it satisfies the following* ***Universal Bounds****:*

$$\frac{1}{n} C(x_i) \le y_i = \varphi_i(C, x) \le C(nx_i) \qquad (2.14)$$

It is easy to check that the average cost method fails both universal bounds. Take the cost function (2.7) and $x = (3, 5, 7)$: the upper bound is violated for agent 1. Take the cost function (2.8) and $x(3, 20, 27)$: the lower bound is violated for agent 1.

The universal bounds are deceptively mild: they eliminate many appealing cost sharing methods. Among the additive methods analyzed in Section 2.4, the universal lower bound is met by many methods besides serial cost sharing. For instance the Shapley-Shubik cost sharing method (see Section 2.4) meets this bound, and so does any convex combination of serial and Shapley-Shubik. On the other hand the universal upper bound essentially characterizes serial cost sharing: Theorem 2.3 below.

2.4. Additive cost sharing

In the rationing problem, the requirement that the solution *y* depends linearly upon the resources *t* is enough to single out the proportional rationing method: Chun [1988]. By contrast, in the cost sharing problem with homogeneous goods, there is a rich family of cost sharing methods where the solution $y = \varphi(C, x)$ depends additively upon the function *C*. Theorem 2.2 below establishes a linear isomorphism between this family and the set of (resource monotonic) rationing methods. Thus Additivity leaves a lot of maneuvering room to the mechanism designer.

With a slight abuse of notation we denote by $\mathscr{R}$ the set of monotonic rationing methods (note that all rationing methods discussed in Part 1 are monotonic). An element *r* of $\mathscr{R}$ defines for all $x \in R_+^N$ a monotonic (hence continuous) path $t \to r(t, x)$ from 0 to *x*:

$$0 \leq r(t, x) \leq x, \; r_N(t, x) = t \;\; \text{for all } t, 0 \leq t \leq x_N$$

$$t \leq t' \Rightarrow r(t, x) \leq r(t', x) \;\; \text{for all } t, t', 0 \leq t' \leq x_N$$

The domain $\mathscr{D}$ of cost functions consists of all the functions *C* that can be written as the difference of two convex functions: this domain contains all the twice continuously differentiable functions, as well as all the piecewise linear functions. Naturally, we also require each function *C* in $\mathscr{D}$ to be non decreasing and such that $C(0) = 0$.

We denote by $\mathscr{M}$ the set of cost sharing methods: an element φ of $\mathscr{M}$ associates a solution $\varphi(C, x)$ to every cost sharing problem (C, x) where $C \in \mathscr{D}$ and $x \in R_+^N$. In addition to Constant Returns (2.2), we consider the following powerful axiom:

Additivity(*ADD*)

$$\varphi(C^1 + C^2; x) = \varphi(C^1; x) + \varphi(C^2; x) \quad \text{for all } C^1, C^2 \in \mathscr{D}, \text{ all } x \tag{2.15}$$

This property allows to decompose the computation of cost shares whenever the cost function itself can be additively decomposed. This commutativity brings a sharp representation result: the additive cost sharing methods are isomorphic to rationing methods.

We denote by Γ_t the cost function $\Gamma_t(z) = \min\{z, t\}$ (easier to interpret as a production function: returns are one until the level t, then drop to zero). Finally we denote by $\mathscr{M}$ $(P, Q, ...)$ the subset of cost sharing methods meeting the properties $P, Q, ...$

Theorem 2.2. (Moulin and Shenker [1994]). *Consider the following two mappings, from* $\mathscr{R}$ *into* $\mathscr{M}$ *(CR,ADD) and from* $\mathscr{M}$ *(CR,ADD) into* $\mathscr{R}$ *:*

$$r \to \varphi: \; \varphi(C,x) = \int_0^{x_N} C'(t)dr(t,x) \text{ for all } C \in \mathscr{D}, \text{all } x \tag{2.16}$$

$$\varphi \to r: \; r(t,x) = \varphi(\Gamma_t, x) \text{ for all } t, x \tag{2.17}$$

These two mappings define a linear isomorphism from $\mathscr{R}$ *into* $\mathscr{M}$ *(CR,ADD) and back.*

Corollary to Theorem 2.2. *All cost sharing methods in* $\mathscr{M}$ *(CR, ADD) meet the following properties:*

i) *No charge for null demand:* $x_i = 0 \Rightarrow y_i = 0$

ii) *Stand Alone lower (upper) bound under increasing (decreasing) marginal costs:*

$$C \text{ convex} \Rightarrow y_S \geq C(x_S) \text{ for all } C, x, \text{ all } S \subseteq N$$

$$C \text{ concave} \Rightarrow y_S \leq C(x_S) \text{ for all } C, x, \text{all } S \subseteq N$$

Theorem 2.2 establishes a precise isomorphism between monotonic rationing methods and additive cost sharing methods. In particular, the key methods on both sides are matched, and many of the normative requirements in one model have a counterpart in the other one. Below is a list of rationing methods and cost sharing methods matched by the linear isomorphism.

a) proportional rationing $\leftrightarrow$ *average cost sharing*

In $\mathscr{R}$, the proportional method gives to every dollar of claim the same right to the resources t ; similarly in $\mathscr{M}$, average cost sharing gives to every unit of demand the same responsibility in total cost (every unit of input is entitled to the same output share).

b) uniform gains rationing $\leftrightarrow$ *serial cost sharing*

Fix N and x, and label the agent so that $x_1 \leq x_2 \leq ... \leq x_n$. Agent i's share $y_i = ug_i(t,x)$ where t varies in $[0, x_N]$ is easily computed, with the help of the sequence x^i given by (2.11):

$$\text{if } 0 \leq t \leq x^1 : y_i = \frac{1}{n} t$$

$$\text{if } x^1 \leq t \leq x^2 : y_i = x_1 + \frac{1}{n-1}(t - x^1)$$

...

$$\text{if } x^{j-1} \leq t \leq x^j : y_i = x_{j-1} + \frac{1}{n-j+1}(t - x^{j-1})$$

...

$$x^{i-1} \le t \le x^i : y_i = x_{i-1} + \frac{1}{n-i+1}(t - x^{i-1})$$

$$x^i \le t \qquad : y_i = x_i$$

Therefore the cost sharing method associated with uniform gains by (2.16) is precisely given by (2.12), as claimed.

In $\mathscr{R}$ the uniform gains method gives an equal claim to all agents on the first units of resources until their claim is met. Similarly, serial cost sharing makes all individual demands pay an equal share of the first units produced until their demand is met.

c) priority rationing $\leftrightarrow$ *incremental cost sharing*

To an ordering σ of the agents in $N-$ a mapping from $\{1,....,n\}$ into $N-$ we associate the following incremental cost sharing method:

$$y_{\sigma_1} = C(x_{\sigma_1});\ y_{\sigma_2} = C(x_{\sigma_1} + x_{\sigma_2}) - C(x_{\sigma_1});...;y_{\sigma_i} = C(\sum_{j=1}^{i} x_{\sigma_j}) - C(\sum_{j=1}^{i-1} x_{\sigma_j}) \qquad (2.18)$$

It corresponds to the priority rationing method $prio(\sigma)$ (see formula (1.8)).

d) random priority ordering $\leftrightarrow$ *Shapley-Shubik cost sharing*

The averaging operation is preserved by the linear isomorphism, therefore the Random Priority rationing method ((1.9)) is associated with the arithmetic average of all incremental cost sharing methods. This method, originally proposed by Shubik [1962], distributes costs according to the Shapley value of the Stand Alone cost game:

$$y_i = \sum_{0 \le s \le n-1} \frac{s!(N-s-1)!}{n!} \sum_{\substack{S \subseteq N \setminus i \\ |S| = s}} (C(x_{S \cup i}) - C(x_S)) \qquad (2.19)$$

In the case of two agents, this gives the cost sharing method corresponding to the contested garment rationing method:

$$y_i = \frac{1}{2}\{C(x_1 + x_2) + C(x_i) - C(x_j)\} \text{ where } \{i, j\} = \{1, 2\} \qquad (2.20)$$

The Shapley-Shubik method plays an important role in the model with heterogeneous goods (Part 3); its characterization there (Corollary 1 to Theorem 3.4) is quite convincing. Contrast this with the lack of normative arguments in favor of Random Priority rationing, or in favor of the Shapley-Shubik method in the current model with one homogeneous good.

The isomorphism in Theorem 2.2 also suggests new cost sharing methods corresponding to simple rationing methods. For instance Uniform Losses gives rise to a “dual” serial method, where all users pay an equal share of the last units produced (instead of the first units, in the case of serial cost sharing) until the smallest demand is satisfied, after which the remaining users share equally the cost of the next highest units, and so on. In the case of two agents with $x_1 \leq x_2$, this gives the following cost shares:

$$y_1 = \frac{1}{2}(C(x_1 + x_2) - C(x_2 - x_1)); \;\; y_2 = \frac{1}{2}(C(x_1 + x_2) - C(x_2 - x_1))$$

The Talmudic rationing method (1.10) leads to a somewhat exotic method, except in the case of two agents, where it coincides with the Shapley-Shubik method. For $n = 3$ and $x_1 \leq x_2 \leq x_3$, the method associated with Talmudic rationing gives the following cost shares:

$$y_1 = \frac{1}{3}C(x_1 + x_2 + x_3) - \frac{1}{3}C(x_2 + x_3 - \frac{x_1}{2}) + \frac{1}{3}C(\frac{3x_1}{2})$$

$$y_2 = \frac{1}{3}C(x_1 + x_2 + x_3) - \frac{1}{6}C(x_2 + x_3 - \frac{x_1}{2}) - \frac{1}{6}C(\frac{3x_1}{2}) + \frac{1}{2}C(\frac{x_1}{2} + x_2) - \frac{1}{2}C(\frac{x_1}{2} + x_3)$$

$$y_3 = \frac{1}{3}C(x_1 + x_2 + x_3) + \frac{1}{6}C(x_2 + x_3 - \frac{x_1}{2}) - \frac{1}{6}C(\frac{3x_1}{2}) - \frac{1}{2}C(\frac{x_1}{2} + x_2) + \frac{1}{2}C(\frac{x_1}{2} + x_3)$$

We conclude this subSection by a characterization of serial cost sharing within the set $\mathcal{M}(CR, ADD)$. All methods in this set meet the Stand Alone bounds when the cost function is either convex or concave (Corollary to Theorem 2.2), but they typically fail the Universal Bounds (2.14). The Shapley-Shubik method meets the lower bound (because in the sum (2.19) the term with $S = \emptyset$ has weight $1/n$) but fails the upper bound (even for $n = 2$).

The universal upper bound is a key ingredient in the characterization of the serial method; yet it is not sufficient to single out this method in $\mathcal{M}(CR, ADD)$.

Consider the counterpart of zero-consistency for rationing methods (property (1.6)):

$$\{x_i = 0\} \Rightarrow \{\varphi_i(N, C, x) = 0 \text{ and } \varphi(N, C, x)_{[N \setminus i]} = \varphi(N \setminus i, C, x_{[N \setminus i]})\} \text{ for all } N, C, x, \; i \quad (2.21)$$

Within $\mathcal{M}(CR, ADD)$ this property is isomorphic to the axiom (1.6). It is a very mild requirement, met by all cost sharing methods discussed in Part 2 (with the exception of some methods allowing for negative cost shares: see Section 2.5, point a).

In order to pin down the serial method, we strengthen Zero Consistency by allowing the removal of a *non paying* agent, provided we make sure to serve his demand (that could be strictly positive). Given a method in $\mathcal{M}$, the property is stated as follows:

$$\{\varphi_i(N,C,x)=0\} \Rightarrow \{\varphi(N,C,x)_{[N\setminus i]} = \varphi(N\setminus i, \tilde{C}, x_{[N\setminus i]})$$

$$\text{where } \tilde{C}(z) = C(z+x_i)\} \ \text{ for all } N,C,x \text{ and } i \tag{2.22}$$

(Note that $\tilde{C}$ will have a jump at 0 if $C(x_i)$ is positive; but the universal lower bound guarantees $C(x_i)=0$).

The last ingredient is the unobjectionable equity requirement called *Ranking:* $x_i \le x_j \Rightarrow y_i \le y_j$. Note that the Ranking axiom (1.2) for rationing methods conveys the same idea (is even written in the same way) but is not equivalent via the linear isomorphism. If the rationing method *r* meets Ranking, the corresponding cost sharing method may not do so.

Theorem 2.3. (Moulin and Shenker [1994])

Serial cost sharing is characterized by the combination of the five axioms Constant Returns, Additivity, Universal Bounds (2.14), Ranking, and property (2.22).

2.5. Variants of the model and further axioms

a) Distributivity (Moulin and Shenker [1999])

The Distributivity axiom expresses the *commutativity* of the computation of cost shares with respect to the *composition* of cost functions:

$$\textit{Distributivity}(\textit{DIS})$$

$$\varphi(C^1 \circ C^2, x) = \varphi(C^1, \varphi(C^2, x)) \text{ for all } C^1, C^2 \in \mathcal{D}, \text{and all } x$$

The addition of cost (or production) functions corresponds to technologies operating in parallel: a given demand of output (resp. a contribution of output) yields two types of costs, e.g., advertising costs and production costs, (resp. enters two production functions). Their composition corresponds to technologies running sequentially: $x \to y_2 = C^2(x) \to y_1 = C^1(y_2)$, the input of C^2 is the output of C^1 (and a similar interpretation if C^1, C^2 represent production functions). Both axioms, Additivity and Distributivity, allow us to decompose the computation of cost shares if the cost or production function itself is decomposed.

One consequence of Distributivity (with no counterpart in the case of Additivity) is *reversibility of fairness:*

$$y = \varphi(C,x) \Leftrightarrow x = \varphi(C^{-1}, y) \text{ for all } x, y$$

Given a pair *(x, y)* with one profile of outputs and one profile of inputs, we can either take the profile of demands as given and check that *y* is the corresponding fair profile of costs shares for the given cost function, or we can take the vector *y* as given and check that *x* is fair for the given production function. These two tests are equivalent for a distributive method.

Distributive methods include average cost sharing, serial cost sharing, as well as any incremental method. Yet the Shapley-Shubik method (2.19) is not distributive, and in fact a proper convex combination (with fixed coefficients) of distributive methods is *never* distributive!

Moulin and Shenker [1999] characterize the rich family of additive and distributive methods (meeting Constant Returns). In this family, average cost sharing is the only self-dual method (a result related to Proposition 1.6), and serial cost sharing the only method meeting the universal lower bound (or upperbound) (2.14).

b) Negative cost shares and the decreasing serial method

In some contexts it makes sense to allow negative cost shares $(y_i < 0)$ or to charge for a null demand $(x_i = 0 \text{ and } y_i > 0)$.

Suppose marginal costs increase (as in the monopsonist example of Section 2.1). Then an agent who demand little or nothing (who refrains from demanding much) is helping the agents with a large demand, so we may want to compensate him by giving him some money (paid for by other agents). Symmetrically, consider an output sharing problem and suppose marginal returns decrease. Think of the “tragedy of the commons” story: input is fishing effort and output is the total catch in the common property lake. Then an agent who refrains from adding more input may argue that she deserves a share of total catch (and end up with $x_i = 0$ and $y_i > 0$). Note that the dual of the two stories above, where we switch from cost sharing to output sharing or vice versa do not ring as plausible. To punish an agent who does not work if the production function in convex $(y_i < 0 \text{ for } x_i \text{ small})$, or to charge one who does not demand any output if the cost function is concave, crosses the line of acceptable coercion by the mechanism designer!

The decreasing serial cost sharing method (De Frutos [1998], Suh [1997]) follows exactly the formulas ((2.11), (2.12), (2.13)) except that individual demands are arranged in decreasing order: $x_1 \geq x_2 \geq ... \geq x_n$ (so that the sequence x^i is decreasing, too). With two agents and $x_1 \geq x_2$:

$$y_1 = \frac{1}{2}C(2x_1); y_2 = C(x_1 + x_2) - \frac{1}{2}C(2x_1)$$

If C is strictly concave, y_i is positive whenever x_i is zero and x_j is positive; on the other hand, no agent receives a negative cost share (this is clear in the case $n = 2$ and can be checked in general on (2.11), (2.12)). If C is strictly convex, y_i is negative whenever x_i is zero and x_j is positive.

The decreasing serial cost sharing method fails both universal bounds (2.14) and has not received an axiomatic characterization at the time of this writing.

Hougarth and Thorlund-Petersen [1998] proposes an interesting mixture of the increasing and decreasing serial methods, arguing that we should keep the former if C is convex and the latter if C is concave. Their method is not additive with respect to cost functions.

c) Consistency?

The Consistency axiom played a key role for the analysis of rationing methods, but it is absent from that of cost sharing methods. Using the linear isomorphism between rationing and cost sharing methods, one would like to characterize the subset of $\mathcal{M}\ (CR, ADD)$ associated with consistent rationing methods. This may even suggest an appropriate definition of Consistency for general cost sharing methods. A definition of Consistency is offered by Tijs and Koster [1998]: it suffers from the same drawback as the definition discussed in Remark 3.1 below, namely it does not work in a domain of non decreasing cost functions.

A related and equally natural question is to characterize the subset of $\mathcal{M}\ (CR, ADD)$ associated with the (symmetric) parametric methods (Section 1.5). Both questions are wide open.

$$y_1 = \frac{1}{2}C(2x_1); y_2 = C(x_1 + x_2) - \frac{1}{2}C(2x_1)$$

If C is strictly concave, y_i is positive whenever x_i is zero and x_j is positive; on the other hand, no agent receives a negative cost share (this is clear in the case $n = 2$ and can be checked in general on (2.11), (2.12)). If C is strictly convex, y_i is negative whenever x_i is zero and x_j is positive.

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3. Heterogeneous outputs or inputs

3.1. The problem

In the cost sharing version of the more general model now under scrutiny, each agent i demands a different good, and the technology specifies the total cost $C(x_1, x_2, \ldots, x_n)$. In the output sharing version, each agent i contributes the amount x_i of an "input i" and total output $F(x_1, \ldots, x_n)$ must be shared among the participants. Thus we identify "good i" and "agent i".

Examples of such cost sharing problems include sharing the cost of a network connecting geographically dispersed users (so the heterogeneity of demand comes from the heterogeneity of space, as in road networks), or of a telecommunication network in which the users need different service (e.g., different bandwidth, or different degrees of reliability in service, or they use the network at different times of the day). Another example is the cost sharing of a large project (dam, space station) between various beneficiaries (e.g., power company, farmers, tourism industry, in the dam example: see Straffin and Heaney [1981]).

Examples of both cost sharing and output sharing are commonplace in the accounting literature (see Thomas [1977]). The various divisions of the firm contribute heterogeneous inputs to a common project, say the launching of a new product: how should the revenue of the project be distributed among them? The cost sharing issue arises when the divisions share a common service, such as the central administration unit.

The main simplifying assumption of the current model is that each agent demands exactly one output good (or contributes exactly one input). On the other hand, the domain of cost (or production) functions is very general: $C(0) = 0$ and C nondecreasing in each x_i, are the only restriction we impose when the variables are discrete (Sections 3.2, 3.3); when x_i is a real number, we add some regularity conditions.

The mathematical complexity of the models raises significantly above that in Parts 1 and 2. We look first at the case of binary demands (each x_i is 0 or 1) in Sections 3.2, i.e., the classical theory of values for cooperative games with transferable utility. We consider variable demands of indivisible goods (each x_i is an integer) in Sections 3.3 and 3.4, and finally variable demands of divisible goods in Sections 3.5 and 3.6.

In Sections 3.2 to 3.6, we look at additive methods only, as we did in most of Section 2. We extend the isomorphism between rationing methods and additive cost sharing methods (Theorem 2.2): in the case of heterogeneous goods, the set of rationing methods is identified with the *extreme points* of the set of additive methods meeting the Dummy axiom (Theorems 3.1 and 3.3).

The Shapley-Shubik cost sharing method, and its asymmetric counterparts, the random order values, emerge forcefully from the axiomatic discussion. Shapley's characterization result in the context of binary demands (Proposition 3.1) now has company in the variable demand model, whether demands are integer valued or real valued (see Corollary 2 to Theorem 3.1 and Corollaries 1 and 3 to Theorem 3.4).

The two other prominent methods are the Aumann-Shapley pricing method, extending average cost sharing to the context of heterogeneous goods, and the additive extension of serial cost sharing: they are discussed in Sections 3.3 to 3.6 and characterized in Section 3.6 (Corollaries 2 and 3 to Theorem 3.4).

Up to 1995, the literature on cost sharing with variable demands was unanimously arguing for the Aumann-Shapley method. The initial axiomatic characterization by Billera and Heath [1982] and Mirman and Tauman [1982] (see also Billera, Heath and Raanan [1978]) was refined in several ways (Tauman [1988] is a good survey). One version of this result is in Corollary 2 to

Theorem 3.4. Moulin [1995a] spells out a critique of the Aumann-Shapley method based on the properties of Demand Monotonicity and Ranking. The former says that the cost share of an agent should not decrease when his demand of output increases, *ceteris paribus*. The latter says that, when all goods enter symmetrically in the cost function, the ranking of individual cost shares is the same as that of individual demands.

Both properties *DM* and *RKG* are compelling when each good is identified with a different agent. They are less compelling if the demand of good *i* aggregates many small individual demands, which is the standard interpretation in the literature on the Aumann-Shapley method. In this survey we stick to the first interpretation and emphasize the critique of the AS method. In turn this pushes the Shapley-Shubik and serial methods to the forefront.

Additivity of cost shares with respect to the cost function, the main assumption throughout Parts 2 and 3, is a powerful mathematical tool, yet not a compelling normative requirement. Additivity narrows down the set of cost sharing methods drastically, thus bringing a number of impossibility statements when we require other properties with more normative appeal: an example is the combination of Demand Monotonicity and Average Cost for Homogeneous Goods (see Proposition 3.3 and Corollary 2 to Theorem 3.4). When the impossibility hurts, the first axiom to go should be Additivity. The literature on nonadditive methods is reviewed in Section 3.7: it contains very few papers but its potential for growth is huge.

3.2. Binary demands: the Shapley value

This is the model of the classical cooperative games with transferable utility where the only restriction is our assumption that the cost function is nondecreasing.

A binary cost sharing problem is a triple (N, C, x) where *N* is a finite set of agents, *C* is a nondecreasing function from $\{0,1\}^N$ into $\mathbb{R}_+$ such that $C(0) = 0$, and $x = (x_i)_{i \in N}$ is a profile of demands, where each $x_i = 0$ or 1.

For convenience, we denote the vector of demands *x* as a, possibly empty, subset *S* of *N*: $x_i = 1$ *iff* $i \in S$. Thus the cost function *C* associates to each coalition *S*, $S \subseteq N$, a number $C(S)$, interpreted as the cost of serving all agents in *S* and only them. Our assumptions on *C* are:

$$C(\varnothing) = 0 \quad ; \quad S \subseteq T \Rightarrow C(S) \leq C(T) \text{ for all } S, T \subseteq N$$

A solution to the binary cost sharing problem (N, C, S) is a profile of cost shares $y = (y_i)_{i\in N}$, where each y_i is a real number and:

$$y_i \geq 0 \text{ for all } i, \sum_{i\in N} y_i = C(S)$$

A binary cost sharing method is a mapping φ associating to any problem (N, C, S) a solution $y = \varphi(N, C, S)$.

The idea of sharing costs in proportion to demands reduces in this model to dividing equally $C(S)$ among all agents in *S* (and charging nothing to those outside *S*). However this method violates the basic principle of *reward,* namely that cost shares should reflect responsibilities in generating the costs. A minimal requirement to that effect is that an agent who "obviously" is not generating any cost should pay nothing. The Dummy axiom conveys just that idea.

We use the notation $\partial_i C(S) = C(S) - C(S \setminus i)$ for the marginal cost (saving) of subtracting agent *i* from coalition *S*. Of course, $\partial_i C(S) = 0$ if $i \notin S$.

Dummy (DUM)

$$\{\partial_i C(T) = 0 \text{ for all } T \subseteq N\} \Rightarrow \{y_i = \varphi_i(N, C, S) = 0\} \quad \text{for all } N, S, i \text{ and } C \qquad (3.1)$$

An agent is called a dummy for the cost function *C* if it costs nothing to serve her, irrespective of the number of other users being served. The egalitarian method ($y_i = C(S)/\#(S)$ if $i \in S$, $y_i = 0$ otherwise) charges a dummy agent as any other, therefore it violates Dummy.

Additivity (ADD)

$$\varphi(N, C^1 + C^2, S) = \varphi(N, C^1, S) + \varphi(N, C^2, S) \quad \text{for all } N, C^k, S$$

Note that Dummy and Additivity together imply a generalization of the Constant Returns property (2.2). If *C* is linear, $C(x) = \sum c_i x_i$, the method simply “separates” costs:

$$\varphi_i(N, C, S) = c_i \cdot x_i \quad \text{where } x_i = 1 \text{ if } i \in S, x_i = 0 \text{ if } i \notin S$$

We denote $\mathcal{C}(DUM, ADD)$ the family of cost sharing methods meeting Dummy and Additivity. These two axioms place no restriction on the method across different populations *N* and N': therefore Proposition 3.1 describes this family in the *fixed population* context, where *N* is fixed and *S* varies (note that most of the literature, only looks at the case $S = N$). Next we introduce a mild consistency requirement linking the solutions across variable populations; in turn, the corresponding methods take a natural structure: Theorem 3.1.

Given *N*, an *incremental cost sharing method* specifies for each nonempty subset *S* of *N* (including *N* itself) an ordering $\sigma(S)=(\sigma_1,\ldots,\sigma_s)$ where $s=\#(S)$. The cost shares $y=\varphi^{\sigma}(N,C,S)$ are computed as follows:

$$y_i = 0 \text{ if } i \notin S$$
$$y_{\sigma_1(S)} = C(\{\sigma_1(S)\}); y_{\sigma_2(S)} = \partial_{\sigma_2(S)}C(\{\sigma_1(S),\sigma_2(S)\}) = C(\{\sigma_1(S),\sigma_2(S)\}) - C(\{\sigma_1(S)\});\ldots \quad (3.2)$$
$$y_{\sigma_k(S)} = \partial_{\sigma_k(S)}C(\{\sigma_1(S),\ldots,\sigma_k(S)\}) \text{ for all } k=1,\ldots,s$$

A *random order value* is a convex combination of incremental methods where the weights of the combination are independent of *C*. Denoting by $\mathcal{S}(S)$ the set of permutations of *S*, a random order value is written as

$$y=\varphi(N,C,S)=\sum_{\sigma(S)\in\mathcal{S}(S)} \lambda_{\sigma(S)}\,\varphi^{\sigma(S)}(N,C,S) \quad \text{for all } S \quad (3.3)$$

Note that we can choose an arbitrary set of convex coefficients $\lambda_{\sigma(S)}$ for each coalition *S*. For instance in $S=\{1,2,3\}$ we may choose the incremental method with ordering 2, 1, 3 and in $S'=\{1,2,4\}$ we may choose that with ordering 1, 2, 4.

Finally we need an equity property to state Shapley's original characterization. If two agents affect the cost function symmetrically, we require that they receive the same share

Equal Treatment of Equals (ETE)

$$\{C(T\cup i)=C(T\cup j) \text{ for all } T \text{ such that } i,j\notin T\} \Rightarrow$$
$$\{\varphi_i(N,C,S)=\varphi_j(N,C,S) \text{ for all } S, S\subseteq N\} \text{ for all } C,i,j$$

Proposition 3.1. *(Fixed population, Weber [1988])*

The set of random order values coincides with the set $\mathcal{C}(DUM, ADD)$ *of the cost sharing methods meeting the Dummy and Additivity axioms.*

Corollary to Proposition 3.1. *(Shapley [1953])*

The three axioms Dummy, Additivity and Equal Treatment of Equals characterize a single method namely the Shapley value; that is, the set $\mathcal{C}(DUM, ADD, ETE)$ *contains a single method:*

$$\varphi_i(N,C,S)=\sum_{t=0}^{s-1}\frac{t!(s-t-1)!}{s!}\sum_{\substack{T:T\subset S\setminus i\\ \#(T)=t}}\partial_i C(T\cup i) \text{ for all } i\in S$$
$$\varphi_j(N,C,S)=0 \text{ if } j\notin S \qquad (3.4)$$

Incremental methods ((3.2)) and random order values ((3.3)) defined in the fixed population context, may allocate priorities (or weigh the various priority orderings) inconsistently when *S* changes. In order to avoid this unpalatable feature, we must switch to the *variable population* context and impose a mild consistency requirement. We denote by $\mathcal{N}$ the maximal set from which agents can be drawn (a finite or infinite set) and by σ a priority ordering of $\mathcal{N}$. On each finite set *S*, this ordering induces an ordering denoted $\sigma(S)$, and the corresponding formula (3.2) defines the σ*-incremental cost sharing method*. Similarly, a *consistent random order value* is a convex combination of the σ-incremental methods, where σ varies over all orderings of N and the coefficients are independent of *N*, *C* and *S*:

$$\varphi(N,C,S)=\sum_{\sigma\in S(N)}\lambda_\sigma\varphi^{\sigma(S)}(N,C,S) \quad \text{for all } N,C,S \qquad (3.5)$$

The following axiom corresponds to the zero-consistency property for rationing methods ((1.6)): a dummy agent can be removed without affecting the distribution of costs among the rest of the agents:

Dummy-Consistency (DCY)

$$\{\partial_i C(T)=0 \text{ for all } T\subseteq N\}\Rightarrow\{\varphi(N,C,S)_{[N\setminus i]}=\varphi(N\setminus i,C,S\setminus i) \text{ for all } S\} \text{ for all } N,i, \text{ and } C$$

(where the restriction of *C* to $N\setminus i$ is denoted *C* as well).

Proposition 3.2. *(Variable population)*

The set of consistent random order values coincides with the set $\mathcal{C}(DUM,DCY,ADD)$ *of the cost sharing methods meeting Dummy, Dummy-Consistency and Additivity.*

Several alternative characterizations of the Shapley value and the random order values have been proposed in the literature. They replace the Additivity axiom by another powerful requirement; the two most striking results rely on the property of marginalism and the notion of potential. We describe these two results in the fixed population context.

In a random order value, the cost share of an agent only depends upon his marginal costs $\partial_i C(T)$ for the various coalitions containing *i*. This property, called *Marginalism*, is defined as:

$$\{\partial_i C^1(T) = \partial_i C^2(T) \text{ for all } T \subseteq S\} \Rightarrow \{\varphi_i(N, C^1, S) = \varphi_i(N, C^2, S)\} \text{ for all } N, C^k, S \text{ and } i \qquad (3.6)$$

Loehman and Whinston [1974] and Young [1985a] show that the Shapley value is characterized by Marginalism and Equal Treatment of Equals. Khmelnitskaya [1999] shows that the combination of Marginalism, Dummy, and an axiom called Monotonicity characterizes the random order values when N contains three agents or more. The monotonicity requirement is as follows:

$$\{C^1(T) = C^2(T) \text{ for all } T \subseteq N, T \neq S \text{ and } C^1(S) \leq C^2(S)\}$$
$$\Rightarrow \{\varphi_i(N, C^1, N) \leq \varphi_i(N, C^2, N) \text{ for all } i \in S\} \text{ for all } N, C^k, \text{ and } S$$

If we add Dummy-Consistency to this list of requirements, we characterize the family of consistent random order values.

The second characterization result concerns the Shapley value alone. Consider the following *potential function*:

$$P(N, C) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} C(S) \quad \text{where } n = \#(N), s = \#(S) \qquad (3.7)$$

The Shapley value ((3.4)) can be equivalently written as:

$$\varphi_i(N, C, S) = \partial_i P(S, C) = P(S, C) - P(S \setminus i, C) \qquad (3.8)$$

Thus agent i's share is simply the i-th derivative of the potential function. As second derivatives commute, this implies for all i, j in N:

$$\partial_j \varphi_i(N, C, S) = \partial_i \varphi_j(N, C, S) \Leftrightarrow \varphi_i(N, C, S) - \varphi_i(N, C, S \setminus j) = \varphi_j(N, C, S) - \varphi_j(N, C, S \setminus i) \quad (3.9)$$

The effect on i's share of removing j is the same as that on j's share of removing i.

Hart and Mas Colell [1989] show that the Shapley value is fully characterized by the existence of *some* potential function P of which derivatives deliver the individual cost shares as in (3.8) and such that $P(\varnothing, C) = 0$. Equivalently, property (3.9) alone is enough to characterize the Shapley value.

Remark 3.1. A third characterization (also due to Hart and Mas-Colell [1989]) of the Shapley value follows from a Consistency axiom. Consider a cost sharing method φ. For any cost function C and agent i, we define a new function as follows:

$$C^{i,\varphi}(T) = C(T \cup i) - \varphi_i(N, C, T \cup i) \qquad (3.10)$$

Note that this function depends on N as well, but under no charge for null demands this plays no role.

The idea is that agent i wants "service" but is leaving the jurisdiction of the division problem; he will pay whatever cost share is recommended by the method, whatever is the set of other agents who want service as well. The Consistency axiom says that this move should not affect the allocation of cost shares among the remaining agents:

Consistency

$$\varphi(N, C, S)_{[N \setminus i]} = \varphi(N \setminus i, C^{i,\varphi}, S \setminus i) \text{ for all } N, C, S \text{ and } i$$

The result is that if we choose the standard value (i.e., the Shapley value) for two person problems, the Consistency axiom forces the Shapley value for every set N.

There are two problems with the above axiom. First of all the definition of the reduced cost function $C^{i,\varphi}$ does not preserve the monotonicity of costs: we may have $S \subset T$ and $C^{i,\varphi}(S) > C^{i,\varphi}(T)$. In fact this inequality does occur even if φ is the Shapley value itself. For an example, take $N = \{1,2,3\}$ and the cost function

$$C(N) = C(13) = C(23) = 1 \quad ; \quad C(12) = C(i) = 0 \text{ for } i = 1,2,3$$

Compute

$$C^{3,\varphi}(1) = C(13) - \varphi_3(C, \{13\}) = 1 - \frac{1}{2} = \frac{1}{2} \text{ and } C^{3,\varphi}(12) = C(123) - \varphi_3(C, \{123\}) = 1 - \frac{2}{3} = \frac{1}{3}$$

Thus the *only* way to make sense of the axiom is to allow for decreasing cost functions. But in this enlarged domain, the very foundations of our model must be revised: allowing negative cost shares is compelling (think of an agent whose presence eliminates all costs); the axioms of Demand monotonicity, and the Upper and Lower Bounds below must be abandoned, and so on.

The second difficulty is the interpretation of the axiom. It does not represent a clear-cut reduction of the allocation problem to the subset $N \setminus i$ of agents, because the agent i must still be ready to pay a different share when the set of other agents who want service changes. This is in sharp contrast with the Consistency property in the rationing (or surplus sharing) problem; where agent i puts his money on the table (or takes his share of the output) and departs without leaving an address: the remaining division problem can be conducted entirely without him.

The above difficulty applies to all forms of Consistency for the binary model (the classical cooperative game framework) such as the concept due to Davis and Maschler [1965] used to characterize the nucleolus by Sobolev [1975]. See Section 3.7. It also applies to the Consistency for general cost functions proposed by Friedman [1999].

3.3. Variable demands of indivisible goods: the *dr*-model

Each agent *i* demands a number x_i of units of the idiosyncratic good *i*, $x_i \in \{0,1,2,\ldots,X_i\}$ where X_i is the capacity of this good. For the main representation theorem below, we must assume that X_i is finite, although the model makes sense for $X_i = +\infty$ as well.

A cost function *C* is now a mapping from $[0, X_{[N]}]$ (the Cartesian product of the integer intervals $[0, X_i]$) into $\mathbb{R}_+$ such that

$$C(0) = 0 \text{ and } x \leq x' \Rightarrow C(x) \leq C(x')$$

As in Section 2, the vector *x* is called the demand profile, $x \in [0, X_{[N]}]$. The definitions of a cost sharing problem (N, C, x) is now complete. A solution is a vector *y* in $\mathbb{R}^N$ such that

$$y \geq 0 \quad \sum_N y_i = C(x) \qquad (3.11)$$

As usual a cost sharing method φ associates a solution to each problem. This model generalizes the binary model of the previous subSection where we had $x_i = 1$ for all *i*.

Our first task is to generalize Propositions 3.1 and 3.2 to the variable demand context. As in Section 3.2, we give first the fixed population version of the result, based on the two axioms Dummy and Additivity: Theorem 3.1. Next we give the variable population result, with the help of the additional axiom Dummy-Consistency: corollary to Theorem 3.1.

Dummy (DUM)

where the notation , and , stands for the marginal cost when *i* raises her demand from $x_i - 1$ to x_i. The interpretation is as in Section 3.2: the demand of a dummy agent, no matter how large, does not cause any extra cost, hence this agent should never be charged.

Additivity (ADD)

The *incremental cost sharing methods* of the binary model (Section 3.2) as well as of the model with homogeneous goods (Section 2.4, see (2.18)) generalize. We fix an ordering s of *N* (recall that *N* is fixed for the time being) and define the s -*incremental* method (or method with priority ordering s) as follows:

(3.12)

where $(x_{[T]},0)$ denotes the vector with the same projection as *x* on *T*, and zero on .

The incremental methods obviously meet *DUM* and *ADD* and so do their convex combinations. Yet there are many more methods in $C\ (DUM\ ,ADD)$. We construct a family of such methods, called the *path-generated* methods: these methods are the key to the representation results below.

Pick a monotonic rationing method *r* for indivisible goods: their set is denoted R_{dd} and they are discussed in Section 1.9. The society *N* is fixed for the time being so we write instead of $r(N,t,x)$, where .

Recall that the path $t ® r(t,x)$ is equivalently described by a sequence where agent *i* appears exactly x_i times. To each rationing method *r* in R_{dd}, or equivalently to each family of sequences $s(x)$ (one for each *x* in $[0,X_{[N]}]$) we associate the following cost sharing method :

(3.13)

where $dr_i(t,x) = 1 \text{ if } i = i_t$ is the *t*-th element of the sequence $s(x)$, and $dr_i(t,x) = 0$ otherwise.

The cost sharing method (3.13) is called *path generated* because for each *x* the cost shares are computed along the path $t ® r(t,x)$ i.e., along the sequence $s(N,x)$ as follows: $C(r(1,x))$ is charged to agent i_1, is charged to agent i_2, and so on. The definition (3.13) makes very clear that this method satisfies the two axioms *ADD* and *DUM*. As convex combinations respect these three properties, we find that any convex combination of path generated methods do, too. In fact there are no other methods.

Theorem 3.1. *(Fixed population) (Wang [1999])*

Every cost sharing method meeting Dummy and Additivity is a convex combination of path generated methods (where the coefficients may depend on N and x but not on C). No other cost sharing method meets these two properties. Identifying a rationing method with the path-generated cost sharing method (3.13) we write this result as follows:

$$C \qquad [R_{dd}] \qquad \textit{(DUM, ADD)}$$

(where denotes the set of extreme points of *Z*, and CO the convex hull)

We turn to the variable population framework of the result. As in the binary model (Section3.2) we require that dropping a dummy agent from the society be of no consequence.

Dummy Consistency (DCY)

The following fact is easy to prove: a path-generated method is dummy consistent if and only if the corresponding rationing method is consistent. It is easy to check that Consistency of *r* amounts to the following property of the generating sequences $s(N, x)$: the sequence obtains from $s(N, x)$ by removing all occurrences of *i*. Thus we call a method *generated by consistent paths* if it is derived from a consistent rationing method via (3.13).

Corollary 1 to Theorem 3.1. (*Variable population)*

Every cost sharing method in C (D U M , D C Y , A D D) *is a convex combination of methods generated by consistent paths. There are no other methods in* C *(DUM, DCY, ADD):*

C R dd (C S Y)] Û R dd (C S Y) = C&& *(DUM, DCY, ADD)*

Note that a convex combination of paths can be interpreted as one of the probabilistic rationing methods discussed in Section 1.9. This interpretation is especially useful for Examples 3.2 and 3.3 below.

We illustrate the results by three crucial symmetric cost sharing methods.

Example 3.1. The Shapley-Shubik method

The arithmetic average of all incremental methods is also called the Shapley-Shubik method (Shubik [1962]) namely

(3.14)

This method is not path-generated; it is a proper convex combination of path-generated methods, namely the incremental methods. Contrast this with the Shapley-Shubik method for homogeneous goods (2.19), that *is* path-generated (like all methods in M (C R , A D D) : see the discussion in Section 2.4).

Remarkably, the Shapley-Shubik method can be characterized by one single additional axiom within C (D U M , A D D) , namely a lower bound on individual cost shares that depends only on *N*, *C* and x_i :

Lower Bound

(3.15)

This bound generalizes to the heterogeneous goods context the universal lower bound (2.14) of Part 2. In the homogeneous good context, the Lower Bound is met by many costs sharing methods, such as serial cost sharing, Shapley-Shubik and more. Its impact in the current model is much more dramatic.

Corollary 2 to Theorem 3.1. *The Shapley-Shubik method is the only method in* C (D U M , A D D) *meeting the Lower bound axiom.*

Example 3.2. The Aumann-Shapley method (*dr* model) (Moulin [1995a], de Nouweland et al. [1995])

This is the discrete version of the Aumann-Shapley cost sharing method for divisible goods that plays a major role in the next subpart.

For a given problem (N, C, x), the method is the uniform average of all path-generated methods, in other words it gives an equal weight to each path between 0 and *x*. Hence the Aumann-Shapley method corresponds, in the representation (3.13), to the random proportional rationing method described in Section 1.9.

Straightforward computations give the formulas of the *AS* cost shares. For any vector *t* in $\mathbb{N}^N$, we use the notation

that is, the number of monotonic paths from 0 to *t* in $[0, t]$. Then we have

$$ \tag{3.16} $$

(with the convention $l(0) = 0$)

An important feature of the *AS* method is to coincide with average cost sharing when the goods are homogeneous, that is to say when *C* takes the form . Sprumont and Wang [1998] argue that the *AS* method is the most natural extension of proportional cost sharing to the context of heterogeneous goods.

Example 3.3. Serial cost sharing (*dr* model) (Moulin [1995a])

In the homogeneous good model (Part 2), serial cost sharing is associated with the uniform gains rationing method via equation (2.16). Similarly, in the case of heterogeneous goods serial cost sharing is associated, via equation (3.13), with the Fair Queuing (probabilistic) method described in Section 1.9. The corresponding cost shares are as follows in the case of two agents:

3.4. Demand Monotonicity in the dr-model

Demand monotonicity

This Demand Monotonicity can be viewed as a mild incentive compatibility requirement. Absent *DM*, the cost sharing method is vulnerable to "sabotage" when the output goods are freely disposable: an agent can raise artificially her demand, throw away the excess good and receive a smaller bill!

The *fixed-path cost sharing* methods are derived from the fixed path rationing methods (Section1.8) via formula (3.13). Recall that for a given society *N* and a capacity x_i for each agent *i*, a fixed path rationing method is defined by a single path $g(N)$ from 0 to :

The corresponding cost sharing method reads as follows, where the variable *N* is omitted in g for simplicity:

This formula makes clear that a fixed path cost sharing method is demand monotonic: increasing x_i to x_i' enlarges the set of indices *t* at which the marginal cost $\partial_i C(g(t) \wedge x)$ is charged to agent *i*, moreover for all *t* in the initial set.

Because *DM* is preserved by convex combinations, every convex combination of fixed path cost sharing methods (where the coefficients are independent of *C* and *x*) meets *DM* as well. Examples include the Shapley-Shubik and serial methods. Theorem 3.3 below states the converse statement. Before writing this important result, we note that the Aumann-Shapley method (Example 3.2) is *not* demand monotonic.

Proposition 3.3. *(Moulin [1995a])*

Let be a cost sharing method in $\mathcal{C}(DUM, ADD)$ *that coincides with average cost sharing when the goods are homogeneous, i.e., when the cost function takes the form* .
*Then is **not** demand monotonic.*

Theorem 3.2. *(Fixed population)*

i) A path-generated cost sharing method is demand monotonic if and only if it is a fixed path method.

ii) A cost sharing method meets Dummy, Additivity and Demand Monotonicity if and only if it is a convex combination of fixed path methods (where the coefficients depend on N but not on C or x):

$$C \qquad [R_{dd} \text{ (fixed paths)}]$$

In the variable population formulation of theorem 3.2, the extreme points of the relevant set of methods are the *consistent fixed path* cost sharing methods, for which the rationing method defined by the fixed paths $N \to g(N)$ is consistent. As noticed in Part 1 (see (1.21)), this means that g commutes with the projection over subsets, namely

(3.17)

Corollary to Theorem 3.2. *(Variable population)*

Every cost sharing method in C *is a convex combination of consistent fixed path methods. No other method meets these four axioms:*

$$C \qquad [R_{dd}$$

Remark 3.2. Two characterizations of the random order values

The random order values are the convex combinations of incremental cost sharing methods (3.12). They are an important subset of $C(DUM, ADD, DM)$ because of their simplicity. Two characterizations of this subset within $C(DUM, ADD)$ have been proposed in the *dr*-model. The first one (Wang [2000]) uses one additional axiom, a *dr*-counterpart of the Unit Invariance axiom of the *rr*-model described below in Section 3.6. The second one (Sprumont [2000]) relies on Demand Monotonicity and the property Informational Coherence.

Unit Invariance is defined below by equation (3.26), with the operator given by (3.24) and (3.25). Because , the axiom is written equivalently as

In the *dr*-model, the domain of C is N^N, so if l is an integer, the function is well defined over N^N (see (3.25)). The *dr*-version of Unit Invariance (that Sprumont calls

Measurement Invariance) requires the above equality to hold for every *N*, *C*, *x*, *i* and every integer λ.

In order to define Informational Coherence, consider the following reformulation of Unit Invariance in the *rr*-model. For every *vector* , we write $\mu \otimes x$ their coordinatewise product: $(\mu \otimes x)_i = \mu_i x_i$.

$$C^{\mu}(\mu \otimes x) = C(x) \quad \text{for all } x \qquad (3.18)$$

(where μ_i is positive for all *i*). Unit Invariance (3.26) is equivalent to the following property

(3.19)

In the *dr*-model, the rescaling vector μ is integer valued, hence for a given *C* and a given μ, property (3.18) does not define a unique cost function : it only specifies its value on a subset of $\mathbb{N}^N$. The axiom *Informational Coherence* requires that for all problems (*N*, *C, x*) and all (integer valued) rescaling vector μ, there exists *at least* one cost function meeting properties (3.18) and (3.19).

Ironically, the *dr*-version of the Aumann-Shapley method fails both Informational Coherence and Unit Invariance (*dr*-version); by contrast in the *rr*-model, Unit Invariance is one of the keys to the characterization of the Aumann-Shapley method (corollary 2 to Theorem 3.4). See Sprumont [2000] for more comments and interpretations.

3.5. Variable demands of divisible goods: the *rr*-model

The only change from the model in the previous subpart is that the output goods are perfectly divisible and the demand profile *x* is in $\mathbb{R}^N_+$. The cost function *C* remains nonincreasing and $C(0) = 0$. However we must now assume that *C* is sufficiently regular to apply the general results about linear operators in functional spaces.

We assume throughout that the cost function *C* is twice continuously differentiable. This is not the only conceivable regularity assumption on *C*.

All the *dr*-axioms discussed in Sections 3.3, 3.4 (with the exception of Remark 3.2) are defined in exactly the same way in the *rr*-model. For instance agent *i* is a dummy in *C iff* the partial derivative $\partial_i C$ is identically zero, which amounts to saying that *C* is independent of the variable x_i, as in the *dr*-model.

Much of the theory in the *rr*-model parallels that in the *dr*-model: this is true for the representation of the sets $\mathcal{C}(DUM, ADD)$ and $\mathcal{C}$ (Theorem 3.3) and for the impact of *DM* in these two sets (Theorem 3.4). The only new feature is the Unit Invariance axiom that has no counterpart in the *dr*-model (see, however, Remark 3.2). In this subpart we state Theorem 3.3 and present the main examples. The discussion of Demand Monotonicity and Unit Invariance are the subject of Section 3.5.

As in the *dr*-model, the key to the representation of $\mathcal{C}(DUM, ADD)$ is the path-generated methods. We denote by $\mathcal{R}_{rr}$ the set of monotonic rationing methods (with divisible claims and shares) studied in Sections 1.2 to 1.8. Recall that a method *r* in this set is a mapping $y = r(N, t, x)$ where t, x_i, y_i vary in $\mathbb{R}_+$, where $y_N = t$ and r_i is non decreasing in *t*. These assumptions imply that r_i is continuous in *t* as well. The *path-generated* cost sharing method φ associated with $\mathcal{R}_{rr}$ is defined as follows:

$$\tag{3.20}$$

where the integral is the Stieljes integral of a continuous function with respect to a monotonic function, and dr_i represents the derivative of $r_i(t, x)$ with respect to *t*. This definition corresponds to (3.13) in the *dr*-model.

Theorem 3.3. (Fixed population), Friedman [1999], Haimanko [1998]

A cost sharing method satisfies Dummy and Additivity if and only if it is an (infinite) convex combination of path-generated methods:

$$C \qquad [\mathcal{R}_{rr}] \Leftrightarrow \mathcal{R}_{rr} \qquad \mathcal{C}(DUM, ADD)$$

In the above statement, a convex combination may be infinite; in other words it is a positive probability measure over the (infinite dimensional) set of monotonic rationing methods (see Friedman [1999] for details). By contrast, in the *dr*-model the set $\mathcal{R}_{dd}$ is finite. This difference notwithstanding, Theorem 3.3 is the exact counterpart of Theorem 3.1.

In the variable population context, the extreme points of the relevant set of methods are the *methods generated by consistent paths*: they are derived via formula (3.20) from a *consistent* rationing method; equivalently, the paths $g(N,x)$ commute with the projection operator

Hence the counterpart of Corollary 1 to Theorem 3.1.

Corollary 1 to Theorem 3.3. *(Variable population)*

i) A path-generated cost sharing method is Dummy-Consistent if and only if it is generated by a consistent path.

ii) A cost sharing method meets Dummy, Dummy-Consistency and Additivity if and only if it is an (infinite) convex combination of methods generated by consistent paths.

We can now attach a cost sharing method to any rationing method discussed in Part 1. For instance, the contested garment method (Section 1.4) yields the following (path-generated) cost sharing method among two agents:

(and a symmetrical formula if $x_2 \leq x_1$).

From the rich family of monotonic rationing methods, only three generate a cost sharing method that has received some independent attention in the literature. These are proportional rationing, uniform gains rationing and priority rationing (and convex combinations of the latter).

We start by the family of random order values, and their symmetric member the Shapley-Shubik method. They are defined exactly as before: formulas (3.12) and (3.14) are unchanged. So is the characterization of the Shapley-Shubik method by the Lower Bound axiom, Dummy and Additivity: Corollary 2 to Theorem 3.1 remains true, word for word.

Next we turn to the method generated by proportional rationing.

Example 3.4. The Aumann-Shapley method (*rr*-model) (Aumann and Shapley [1974])

This method has been the subject of the most voluminous research and its characterization by Billera and Heath [1982] and Mirman and Tauman [1982] initiated the literature on cost sharing with variable demands.

The *AS* method is path-generated: its path is the simplest path between 0 and *x*, namely the straight line; equivalently, the *AS* method is derived from the proportional rationing method via formula (3.20). Thus it generalizes average cost sharing for homogeneous goods (Part 2) to the heterogeneous goods model. The method is written as follows:

(3.21)

It can be shown that the above *AS* method is the limit of the *AS* methods in the *dr*-model (Example 3.2). Given *N* and *x*, we approximate each interval by a sequence of increasingly fine discrete grids, and define a sequence of *dr*-cost sharing problems by simply restricting the initial cost function to the profiles of the grid. Then the *AS* cost shares (3.18) of the discrete approximating problems converge to the cost shares (3.21).

Example 3.5. Serial cost sharing (*rr*-model) (Friedman and Moulin [1999])

As in the homogeneous goods model, serial cost sharing is generated by the uniform gains rationing method. Thus it is path-generated. Using the notations and $e = (1,1,\ldots,1)$, the uniform gains method writes

Change the variable *t* in (3.13) to λ : the latter varies from 0 to $\max_i x_i$ when *t* varies from 0 to x_N ; moreover $\partial_i C(\lambda e \wedge x) = 0$ whenever . Hence the integral formula for serial cost sharing:

(3.22)

For instance, take $N = \{1,2\}$ and *x* such that $x_1 \leq x_2$:

The conscientious reader will check that (3.22) yields the cost sharing formula (2.13) if the cost function *C* takes the form .

We look now at some examples of the cost function *C* for which our three methods are both easy to compute and interestingly different.

Consider first a Cobb-Douglas cost function. Fix some positive numbers α_i and define:

The Shapley-Shubik method splits total cost equally between all users; similarly, the Aumann-Shapley method assigns the fraction a_i/a_N to user *i*. Thus both methods disregard the relative size of individual demands, which creates a strong “tragedy of the commons” effect. The serial method yields a more appealing set of cost shares, notwithstanding the fact that it relies on the interpersonal comparison of individual demands. For instance,

Consider next the cost function of an excludable public good: . Note that this function is not continuously differentiable, so we need to approximate the given function by regular cost functions and take the limit. This is straightforward. Examples include the cost sharing of a capacity, as in the celebrated airport landing game (Littlechild and Owen [1973]) where each user requests a certain length of runway suitable for his own airplanes.

The Shapley-Shubik and serial cost sharing methods coincide for his own airplanes. Assume $x_1 \le x_2 \le \dots \le x_n$, then the cost shares are

(3.23)

These cost shares make good sense: they separate total cost into *n* components and split the cost of each component between the agents who “use” this component.

By contrast, the Aumann-Shapley method recommends an unpalatable division of costs:

The agent with the largest demand bears the full cost: this follows from (3.21) because on the straight line $[0, x]$ the only nonzero partial derivative is . These cost shares change discontinuously whenever the identity of the largest demander changes; they make it extremely costly to raise one's demand by a very small amount.

The next example of the function *C* is easier to interpret as a production function, i.e., in the surplus sharing model, so we write it as . The inputs of the various agents are perfect complements in the production of the single output (as in several coordination games of a macroeconomic inspiration; see, e.g., Bryant [1983]). The Shapley-Shubik and serial cost shares again coincide: they simply split total output equally for all *i*. This time the Aumann-Shapley method gives the full surplus to the agents with the *smallest* input contribution!

Thus agent 1's share drops to zero when he raises his input contribution above the next smallest contribution, in sharp contradiction of Demand Monotonicity.

The above two examples make a powerful point against the Aumann-Shapley cost sharing: it makes little sense when the cost or production function exhibits strong complementarities. In axiomatic form, this critique of the AS-method hinges on two axioms: Demand Monotonicity and Ranking

Ranking

Ranking is a strengthening of Equal Treatment of Equals. When all goods affect the cost function symmetrically, their quantities are structurally comparable and Ranking is a compelling equity requirement.

Serial cost sharing and the Shapley-Shubik methods both meet Ranking and Demand Monotonicity. In the case of the function $F(\min_i x_i)$, the Aumann-Shapley method (strongly) violates both axioms. Note that the *dr*-version of the Aumann-Shapley method (Example 3.2) also violates Ranking.

3.6. Unit invariance and Demand Monotonicity in the *rr*-model

We explore the impact in the set $C(DUM, ADD)$ (and in C) of two important requirements: Demand Monotonicity and the new axiom Unit Invariance. Theorem 3.4 gives a complete answer to both questions, and leads to three characterization results of, respectively, the Shapley-Shubik, Aumann-Shapley and serial methods.

Given an agent *i* and a positive number l, we denote by the l-rescaling of the *i*-th coordinate:

(3.24)

Given a cost function *C*, the l-rescaling of x_i transforms *C* into :

$$t^i_l(C)(x) = C(t^i_{1/l}(x)) \text{ for all } x \qquad (3.25)$$

Unit Invariance (UI)

(3.26)

Unit Invariance says that changing the unit in which a particular good is measured would not affect cost shares: whether I measure a quantity of corn in kilos, pounds or bushels should not matter. This is compelling when the different goods entering the cost function are not only different but also noncomparable (e.g., corn and fruits). Less so if the goods are of the same nature, but do not enter symmetrically in the cost function. Even less so if the goods affect *C* symmetrically.

Unit Invariance is violated by serial cost sharing, as its generating path (namely the uniform gains path) along the diagonal is not invariant by rescaling.

As in the *dr*-model, it is enough to analyze the impact of the two axioms *DM* and *UI* on the extreme points of the (convex) sets $C(DUM, ADD)$ and $C(DUM, ADD, DM)$, namely the path-generated methods and the methods generated by fixed paths: Theorem 3.4 says that there are no other extreme points in the relevant set of methods.

A path-generated method is demand monotonic if and only if it is a *fixed path* method, namely a method derived via formula (3.20) from a fixed path rationing method (Section 1.8). Recall that such a rationing method is constructed, via formula 1.19, from a family of paths $g(N)$ joining 0 to , where x_i is agent *i*'s capacity, $x_i \leq +\infty$. Hence the corresponding cost sharing method is written as follows:

(3.27)

Naturally, a fixed path cost sharing method is Dummy consistent if and only if the corresponding fixed path rationing method is consistent, or equivalently if the fixed paths N ® g(N) commute with the projection operator: property (3.17).

A path-generated method is unit invariant if and only if the corresponding family of paths are such that:

or equivalently, the corresponding rationing method is such that

In this case we speak of a unit invariant path generated method.

Theorem 3.4. (Friedman [1999])

i) A cost sharing method satisfies Dummy, Additivity and Demand Monotonicity if and only if it is an (infinite) convex combination of fixed path methods:

[*(fixed path)*]

ii) A cost sharing method satisfies Dummy, Additivity and Unit Invariance if and only if it is an (infinite) convex combination of unit invariant path methods.

C (DUM, ADD, SI) [*(SI)*]

iii) C [R rr (consistent fixed path)]

iv) C [R rr *(CSY, SI)*]

As in the *dr*-model, the *random order values* are the convex combinations of the incremental cost sharing methods (3.12) (where the coefficients are independent of *C* and *x*). They have a very simple characterization.

Corollary 1 to Theorem 3.4. (Friedman and Moulin [1999])

A cost sharing method satisfies Dummy, Additivity, Demand Monotonicity and Unit Invariance if and only if it is a random order value.

If we add Dummy-Consistency to the list of requirements, we characterize the set of consistent random order values: these methods are described as in the binary model (see (3.5)). Similarly, if we add Equal Treatment of Equals to the list of axioms in Corollary 1, we characterize the Shapley-Shubik method.

Remark 3.2 in Section 3.4 describes two characterizations of the random order values in the *dr*-model; they are the counterpart of Corollary 1 in the case of indivisible units of demand.

Remark 3.3. Wang [2000] offers a characterization of the random order values and the Shapley-Shubik method where the two axioms *DM* and *UI* are replaced by a single property called ordinality. The latter strengthens *UI* by requiring the invariance of cost shares when we change the measurement of any one good in a nonlinear way. Let l denote an increasing (and differentiable) one-to-one mapping of R_+ into itself. We generalize the definition of l -rescaling ((3.24) (3.25)) as follows

(3.28)

The ordinality axiom is defined exactly as Unit Invariance, with l now varying over all increasing bijections of R_+.

The set of random order values is characterized by the combination of Dummy, Additivity and Ordinality. Adding Equal Treatment of Equals singles out the Shapley-Shubik method.

We turn to the celebrated characterization of the Aumann-Shapley method.

Corollary 2 to Theorem 3.4. (Billera and Heath [1982], Mirman and Tauman [1982])

i) The Aumann-Shapley cost sharing method is characterized by the four properties Dummy, Additivity, Unit Invariance and the following:

Average Cost for Homogeneous Good (ACH)

—— (3.29)

ii) There is no cost sharing method in $C(DUM, ADD)$ *meeting Average Cost for Homogeneous Goods and* ***either one*** *of Demand Monotonicity or Ranking.*

Note the tension between the axioms *UI* and *ACH*, the former bearing on the case of heterogeneous, noncomparable goods, the latter on the case of identical goods.

Statement *ii)* in Corollary 2 includes the *rr*-version of Proposition 3.3. In fact, the incompatibility of *ACH* and *RKG* (in $C(DUM, ADD)$) holds true in the *dr*-model as well. The proof uses exactly the same numerical example as in the proof of Proposition 3.3.

Remark 3.4. Young [1985b] offers an alternative characterization of the Aumann-Shapley method using neither Dummy nor Additivity. Instead, he proposes the Strong Aggregation Invariance (*SAI*) axiom that considerably strengthens *ACH*, and a Symmetric Monotonicity (*SM*) axiom combining the intuition of Marginalism ((3.6)) with the interpersonal comparison of marginal costs.

Corollary 3 to Theorem 3.4

i) The serial cost sharing method is characterized by the four properties Dummy, Additivity, Demand Monotonicity and Upper Bound.

Upper Bound (UB)

(3.30)

ii) The serial cost sharing method is characterized by the four properties Dummy, Additivity, Demand Monotonicity and

Serial cost shares for Homogeneous Good (SCH)

(3.31)

iii) The Shapley-Shubik method is the only method in $C(DUM, ADD)$ *meeting Lower Bound ((3.15)).*

Note that statement *i)* still holds if we weaken *UB* by restricting attention to homogeneous cost functions (i.e., we only require the inequality when *C* takes the form). Statements *i)* and *iii)* have direct counterparts in the *dr*-model (see, respectively, the discussion after the corollary to Theorem 3.2 and Corollary 2 to Theorem 3.1). Statement *ii)* can be adapted to that model as well, by adding Equal Treatment of Equals to the requirements.

3.7. Nonadditive cost sharing methods

a) The Stand Alone core approach

In the theory of classical cooperative games (namely the model in Section 3.2) surplus sharing methods that are not additive in the production (or cost) function have been proposed for three decades (Schmeidler [1969]). For simplicity we fix the population *N* throughout this Section.

Consider the binary problem (N, C, N), denoted *(N, C)* for simplicity, and suppose that the cost function *C* is *subadditive*, namely:

This property is plausible in every problem where “serving coalition S” is orthogonal to “serving coalition *T* ”: the costs incurred by *T* (resp. *S*) are the same whether or not *S* (resp. *T*) is served.

Under subadditivity of costs, a natural equity requirement on the solution *y* to the problem (N, C) is the *Stand Alone core* property:

(3.32)

If this property fails, the coalition of agents *S* has an objection to *y*: if *S* were standing alone it would benefit from a reduced cost share, whereas the cost share of would increase.

In view of budget-balance, $y_N = C(N)$, the above system of inequalities is equivalent to

This is the *No Subsidy principle*: *S* should pay at least the incremental cost of service (if this inequality fails, coalition has a Stand Alone objection).

The following fact is well known: subadditivity of the cost function *C* is *not* sufficient to guarantee the existence of a solution *y* in the Stand Alone core (i.e., meeting (3.32)). The characterization of those cost functions generating a nonempty core is one central theme of the theory of cooperative games, and is well understood: see Owen [1982] or Moulin [1988], [1995b] for textbook presentations.

The Stand Alone core property for a cost sharing method j takes the following form (for a given population *N* and demand *N*):

(3.33)

An interesting result connects the Stand Alone core to the incremental cost shares (3.2): the Stand Alone core is *contained* in the convex hull of all incremental cost shares when s varies over all permutations of *N* (Weber [1988]). In other words *any* allocation *y* in the Stand Alone core is achieved by at least one random order value. However, the choice of a particular random order value (the choice of the convex coefficients over the methods j^s) depends upon the particular cost function *C*. If we pick the same random value j for all cost functions *C*, it must be the case that for some choices of the subadditive cost function *C*, the Stand Alone core is nonempty and does not contain the solution (this is an easy consequence of Theorem 1 in Young [1985a]). In view of Theorem 3.1, there is no additive cost sharing method meeting the Stand Alone core property (3.33).

Thus if we are committed to the Stand Alone core property, a nonadditive method must be used. The abundant literature on classical cooperative games has proposed a number of such methods; the most prominent among these is the nucleolus (Schmeidler [1969]), selecting a central point in the Stand Alone core by equalizing as much as possible the cost savings $C(S) - y_S$ across all coalitions of *N*. See Owen [1982] or Moulin [1988] for a textbook presentation.

The Stand Alone core approach is easily adapted to the model with variable demands. Given a profile of demands *x*, the Stand Alone cost of coalition ; inequality (3.32) becomes and the subadditivity of the cost function *C* writes $C(x + x') \le C(x) + C(x')$.

An important exception to the incompatibility of the Stand Alone core and Additivity is when the cost function *C* is *submodular,* a considerably more demanding requirement than subadditivity:

binary model:

variable demands model:

(where

If *C* is submodular, the inclusion of the Stand Alone core in the range of the random order values becomes an equality (Ichiishi [1981]):

(in the binary and variable demands model respectively). In particular any random order value meets the Stand Alone core property and the latter property is thus always true for additive cost sharing methods in the binary model. The same holds true in the variable demands model. The Stand Alone requirement comes for free in the world of additive methods. An instance of this general fact is statement *ii*, in the Corollary to Theorem 2.2: with a homogeneous good and , submodularity of *C* is equivalent to the concavity of .

b) Extending homogeneous good methods

One of the natural requirements in the heterogeneous good model is that the cost sharing method coincides with a given method whenever the goods are actually homogeneous. In other words, we wish to impose the solutions $\varphi(N, C, x)$ whenever *C* takes the form .

The *ACH* axiom (3.29) and the *SCH* axiom (3.31) are the key to characterize respectively the Aumann-Shapley and serial methods (Corollaries 2 and 3 to Theorem 3.4). Note that in the case of the *AS* method, we do not know of any characterization result that would dispense with *ACH*.

Among additive methods, the two properties *ACH* and *SCH* also lead to severe impossibility results. Within the set $C(DUM, ADD)$ of additive methods, each one of the following pairs of requirements are incompatible:

i) ACH and Demand Monotonicity

ii) ACH and Ranking

iii) SCH and Unit Invariance

iv) ACH and Serial for excludable public good, namely for all x } Þ { y given by (3.23)}

v) ACH and Ordinality (Remark 3.3)

vi) SCH and Ordinality

The first two incompatibilities are statement *ii)* in Corollary 2 to Theorem 3.4. The next two are also easy to derive from Theorem 3.4 and the last two follow Sprumont's result in Remark 3.3.

I regard each one of these impossibility results as a strong argument against the Additivity requirement. Each pair of axioms is normatively meaningful, whereas Additivity is only a structural invariance property.

Sprumont [1998] proposes a handful of nonadditive methods for which the axioms listed above *are* compatible. For instance, in the two agents case he constructs a method satisfying *ACH*, *DM* and Ordinality as follows. We are given a problem (N, C, x), and assume that all partial derivatives of *C* are strictly positive and bounded away from zero. We say that two problems are ordinally equivalent if there are rescaling functions l_i, one for each , such that:

Given the problem (C, x) one shows that there exists a unique problem such that

Then the *ordinally proportional rule* is defined as

This definition has been generalized to an arbitrary number of agents: Wang and Zhu [2000] and Wang [2001]. It is also limited by restrictive regularity conditions. Nonetheless the combined properties of *ACH*, *DM* and Ordinality are a remarkable achievement.

At least two ordinal extensions of serial cost sharing have been proposed. We describe one of them, called the Moulin-Shenker rule by Sprumont [1998] who was the first to analyze it formally. See also Koster [1988a], [1988b] and Koster, Tijs, and Borm [1998]. We are given a problem (C, x) where we assume, again, that all partial derivatives are positive and bounded away from zero. The number of agents is arbitrary. The following ordinary differential equation has a unique solution $t \to r(t, x)$, namely a monotonic rationing path from 0 to *x*. This path depends on the cost function *C* itself, but for simplicity we omit this from the notation:

(3.34)

By definition of the counting operation $a(t, x)$ we have

hence $t \to r(t, x)$ is indeed a rationing path. Then we define the Moulin-Shenker cost sharing method:

The intuition behind this method is simple: as long as his demand is not met, the "active" agent *i* is served at a speed that equalizes the marginal cost of service among all active agents:

Sprumont [1998a] proposes another ordinal extension of serial cost sharing in the vein of the ordinal extension of average cost sharing described above. He gives axiomatic characterizations for both extensions. Koster [1998a] offers a related characterization of the Moulin-Shenker rule.

c) More nonadditive methods and an open question

The study of nonadditive cost sharing methods has just begun and it shows great potential. In the homogeneous good model, Hougaard and Thorlund-Petersen [2001] propose a nonadditive method mixing the increasing and decreasing versions of serial cost sharing: see Section 2.5 point b). In the same homogeneous good model, Tijs and Koster [1998] propose a very natural nonadditive generalization of incremental cost sharing. Fix an ordering , say . Denote by *L* the Lebesgue measure of a measurable set in $\mathbb{R}_+$. The method in question charges to agent 1 the cheapest marginal costs in $[0, x_N]$:

(3.35)

Finally, a largely unexplored model is the cost sharing problem where several outputs are jointly produced and each agent demands some amount of every good. Kolpin [1996] extends to that context the incompatibility of Additivity, *SCH* and Unit Invariance; see also Téjédo and Truchon [1999]. McLean and Sharkey [1996], [1998] adapt the Aumann-Shapley method to that context and extend the classical characterization result (Corollary 2 to Theorem 3.4).

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